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Four classes of Rogers–Ramanujan identities with quintuple products

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Abstract. Combining the finite form of Jacobi's triple product identity with the q-Gauss summation theorem, we present a new and unified proof for the two transformation lemmas due to Andrews (1981). The same approach is then utilized to establish two further transformations from unilateral to bilateral series. They are employed to review forty identities of Rogers–Ramanujan type with quintuple products.

1. Introduction and Motivation

For two indeterminate q and x, the shifted factorial of x with base q is defined by

$$(x;q)_0 = 1 \qquad \text{and} \qquad (x;q)_n = (1-x)(1-qx)\dots(1-q^{n-1}x) \qquad \text{for } n \in \mathbf{N}.$$

When $|q| < 1$, we have two well-defined infinite products

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x) \qquad \text{and} \qquad (x;q)_n = (x;q)_\infty/(q^n x;q)_\infty.$$

The product and fraction of shifted factorials are abbreviated respectively to

$$[\alpha,\beta,\dots,\gamma;q]_n = (\alpha;q)_n(\beta;q)_n\dots(\gamma;q)_n,$$
$$\left[\begin{matrix}\alpha,\beta,\dots,\gamma\\ A,B,\dots,C\end{matrix}\middle| q\right]_n = \frac{(\alpha;q)_n(\beta;q)_n\dots(\gamma;q)_n}{(A;q)_n(B;q)_n\dots(C;q)_n}.$$

Following Bailey [2], Gasper–Rahman [12] and Slater [19], the basic hypergeometric series reads as

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$$
{}_{1+r}\phi_s\left[\begin{matrix} a_0, a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}\middle| q; z\right] = \sum_{n=0}^{\infty}\{(-1)^n q^{\binom{n}{2}}\}^{s-r}\left[\begin{matrix} a_0, a_1, \dots, a_r \\ q, b_1, \dots, b_s \end{matrix}\middle| q\right]_n z^n,
$$

$$
{}_r\psi_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}\middle| q; z\right] = \sum_{n=-\infty}^{+\infty}\{(-1)^n q^{\binom{n}{2}}\}^{s-r}\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}\middle| q\right]_n z^n;
$$

where the base q will be restricted to $|q| < 1$ for nonterminating q-series.

There are numerous identities expressing infinite series in terms of infinite products, usually called identities of Rogers–Ramanujan type. Comprehensive accounts on them as well as different proving techniques can be found in Sills [17], McLaughlin–Sills–Zimmer [16] and Chu–Zhang [9].

In 1981, Andrews [1, Lemmas 1 and 2] found two useful q-series transformations in his study of ϑ-function expansions. By combining the finite form of Jacobi's triple product identity with the q-Gauss summation theorem, we shall present a new and unified proof for Andrews' lemmas. The same approach will lead us to two further transformations from unilateral to bilateral series, analogous to Andrews' ones. Their specializations will then be employed to review forty identities of Rogers–Ramanujan type whose product expressions are quintuple moduli "$6+12$", "$9+18$", "$12+24$" and "$18+36$", mainly covered in the papers by Slater [18], McLaughlin–Sills [15] and Chu–Zhang [9].

The paper will be divided into three sections. The next section will show three transformations from unilateral to bilateral series, including one due to Andrews. Then in the third section, we shall display forty identities of Rogers–Ramanujan type with quintuple products. What is remarkable is that all these forty identities are deduced from only three transformation theorems.

2. Three transformation formulae

In this section, we shall first give a new and unified proof of Andrews' transformations (Theorem 1 below) and then establish two variants (Theorems 2 and 3). They will be employed in the next section to derive identities of Rogers–Ramanujan type.

In order to present the new proof just mentioned, we need the following two basic formulae. The first one is called the finite form of Jacobi's triple product identity

$$
(\xi; q)_n (q/\xi; q)_m = \sum_{k=-m}^{n} (-1)^k \begin{bmatrix} m+n \\ m+k \end{bmatrix} q^{\binom{k}{2}} \xi^k. \tag{1}
$$

It is originally due to Cauchy and Gauss. Their proofs can be found in Chu [7]. The second one reads as (under the convergence condition $|q^{1+\delta}/xy| < 1$)

$$\sum_{n=-\infty}^{+\infty} \frac{[x, y; q]_n (q^{1+\delta}/xy)^n}{(q;q)_{n-k}(q;q)_{n+k+\delta}}$$
$$= \begin{bmatrix} x, & y \\ q^{1+\delta}/x, q^{1+\delta}/y \end{bmatrix} q \Bigg]_k \begin{bmatrix} q^{1+\delta}/x, q^{1+\delta}/y \\ q, q^{1+\delta}/xy \end{bmatrix} q \Bigg]_\infty \left(\frac{q^{1+\delta}}{xy}\right)^k \tag{2}$$

where $\delta = 0$ or 1 and $k \in \mathbf{Z}$. This can be considered as a variation of the q-Gauss summation theorem. In fact for $m \in \mathbf{Z}$ with $m < 0$, recalling that $1/(q;q)_m = 0$, the last sum with respect to n starts effectively with $\max\{k, -k-\delta\}$. When $k \geq 0$, we can reformulate, by letting $n \to n+k$, the sum displayed in (2) as

$$\sum_{n=k}^{\infty} \frac{[x, y; q]_n (q^{1+\delta}/xy)^n}{(q;q)_{n-k}(q;q)_{n+k+\delta}} = \sum_{n=0}^{\infty} \frac{[x, y; q]_{n+k} (q^{1+\delta}/xy)^{n+k}}{(q;q)_n (q;q)_{n+2k+\delta}}$$
$$= \frac{[x, y; q]_k}{(q;q)_{2k+\delta}} \left(\frac{q^{1+\delta}}{xy}\right)^k \sum_{n=0}^{\infty} \begin{bmatrix} q^k x, & q^k y \\ q, & q^{1+2k+\delta} \end{bmatrix} q \Bigg]_n \left(\frac{q^{1+\delta}}{xy}\right)^n.$$

Evaluating the last sum by the q-Gauss summation theorem (cf. [12, II-8])

$${}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix} \,\Big|\, q; \frac{c}{ab}\right] = \begin{bmatrix} c/a, c/b \\ c, c/ab \end{bmatrix} q \Bigg]_\infty \qquad \text{where } |c/ab| < 1$$

we find that the product expression is equivalent to the right-hand side of (2). Instead, when $k < 0$, replacing k by $-\ell - \delta \geq 0$ will lead the sum in (2) to the same one as the case $k \geq 0$ just treated

$$\sum_{n=-k-\delta}^{\infty} \frac{[x, y; q]_n (q^{1+\delta}/xy)^n}{(q;q)_{n-k}(q;q)_{n+k+\delta}} = \sum_{n=\ell}^{\infty} \frac{[x, y; q]_n (q^{1+\delta}/xy)^n}{(q;q)_{n-\ell}(q;q)_{n+\ell+\delta}}. \qquad \square$$

Now we are ready to present a new proof for the following transformation.

Theorem 1 (Andrews [1, Lemmas 1 and 2]: $\delta = 0, 1$).

$$\sum_{n=0}^{\infty} \frac{(\xi;q)_n (q/\xi;q)_{n+\delta}}{(q;q)_{2n+\delta}} [x, y; q]_n \left(\frac{q^{1+\delta}}{xy}\right)^n$$
$$= \begin{bmatrix} q^{1+\delta}/x, q^{1+\delta}/y \\ q, q^{1+\delta}/xy \end{bmatrix} q \Bigg]_\infty \sum_{k=-\infty}^{+\infty} (-\xi)^k q^{\binom{k}{2}} \begin{bmatrix} x, & y \\ q^{1+\delta}/x, q^{1+\delta}/y \end{bmatrix} q \Bigg]_k \left(\frac{q^{1+\delta}}{xy}\right)^k.$$

Proof. According to (1), writing the unilateral sum with respect to n as a double sum and then interchanging the summation order, we have the following expression

$$\sum_{n\geq 0}\frac{(\xi;q)_n(q/\xi;q)_{n+\delta}}{(q;q)_{2n+\delta}}[x,y;q]_n\left(\frac{q^{1+\delta}}{xy}\right)^n$$
$$=\sum_{n\geq 0}\frac{[x,y;q]_n}{(q;q)_{2n+\delta}}\left(\frac{q^{1+\delta}}{xy}\right)^n\sum_{k=-n-\delta}^{n}(-1)^k\begin{bmatrix}2n+\delta\\ n-k\end{bmatrix}q^{\binom{k}{2}}\xi^k$$
$$=\sum_{k=-\infty}^{+\infty}(-\xi)^k q^{\binom{k}{2}}\sum_{n=\max\{k,-k-\delta\}}^{\infty}\frac{[x,y;q]_n(q^{1+\delta}/xy)^n}{(q;q)_{n-k}(q;q)_{n+k+\delta}}.$$

Evaluating the last inner sum by (2) yields the transformation in Theorem 1. □

By means of linear combinations of (1), we can establish two further transformation theorems from unilateral to bilateral series, similar to Theorem 1.

Theorem 2 (Transformation formula: $\delta = 0, 1$).

$$\sum_{n=0}^{\infty}\frac{[\xi,q^\delta/\xi,x,y;q]_n}{(q;q)_{2n+\delta}}\left(\frac{q^{1+\delta}}{xy}\right)^n$$
$$=\left[\begin{matrix}q^{1+\delta}/x,q^{1+\delta}/y\\ q,q^{1+\delta}/xy\end{matrix}\,\middle|\,q\right]_\infty\sum_{k=-\infty}^{+\infty}(-\xi)^k\frac{\xi+q^{k+\delta}}{\xi+(-q/\xi)^\delta}q^{\binom{k}{2}}$$
$$\times\left[\begin{matrix}x, & y\\ q^{1+\delta}/x, & q^{1+\delta}/y\end{matrix}\,\middle|\,q\right]_k\left(\frac{q^{1+\delta}}{xy}\right)^k.$$

Proof. From (1), it is trivial to check

$$[\xi,q^\delta/\xi;q]_n=\sum_{k=-n-\delta}^{n}(-1)^k\frac{\xi+q^{k+\delta}}{\xi+(-q/\xi)^\delta}\begin{bmatrix}2n+\delta\\ n-k\end{bmatrix}q^{\binom{k}{2}}\xi^k.$$

Then the following sum can analogously be treated as

$$\sum_{n\geq 0}\frac{[\xi,q^\delta/\xi,x,y;q]_n}{(q;q)_{2n+\delta}}\left(\frac{q^{1+\delta}}{xy}\right)^n$$
$$=\sum_{n\geq 0}\frac{[x,y;q]_n}{(q;q)_{2n+\delta}}\left(\frac{q^{1+\delta}}{xy}\right)^n\sum_{k=-n-\delta}^{n}(-1)^k\frac{\xi+q^{k+\delta}}{\xi+(-q/\xi)^\delta}\begin{bmatrix}2n+\delta\\ n-k\end{bmatrix}q^{\binom{k}{2}}\xi^k$$
$$=\sum_{k=-\infty}^{+\infty}(-\xi)^k\frac{\xi+q^{k+\delta}}{\xi+(-q/\xi)^\delta}q^{\binom{k}{2}}\sum_{n=\max\{k,-k-\delta\}}^{\infty}\frac{[x,y;q]_n(q^{1+\delta}/xy)^n}{(q;q)_{n-k}(q;q)_{n+k+\delta}}.$$

Evaluating the last inner sum by (2) leads to the transformation in Theorem 2. □

Finally consider another linear combination of (1)

$$[\xi, q^\delta/\xi; q]_n q^n = \sum_{k=-n-\delta}^{n} (-1)^k \frac{1+q^k\xi}{\xi+(-q/\xi)^\delta} \begin{bmatrix} 2n+\delta \\ n-k \end{bmatrix} q^{\binom{k}{2}} \xi^k.$$

Substituting it into the following unilateral sum

$$\begin{aligned} &\sum_{n\geq 0} \frac{[\xi, q^\delta/\xi, x, y; q]_n}{(q;q)_{2n+\delta}} \left(\frac{q^{2+\delta}}{xy}\right)^n \\ &= \sum_{n\geq 0} \frac{[x,y;q]_n}{(q;q)_{2n+\delta}} \left(\frac{q^{1+\delta}}{xy}\right)^n \sum_{k=-n-\delta}^{n} (-1)^k \frac{1+q^k\xi}{\xi+(-q/\xi)^\delta} \begin{bmatrix} 2n+\delta \\ n-k \end{bmatrix} q^{\binom{k}{2}} \xi^k \\ &= \sum_{k=-\infty}^{+\infty} (-\xi)^k \frac{1+q^k\xi}{\xi+(-q/\xi)^\delta} q^{\binom{k}{2}} \sum_{n=\max\{k,-k-\delta\}}^{\infty} \frac{[x,y;q]_n (q^{1+\delta}/xy)^n}{(q;q)_{n-k}(q;q)_{n+k+\delta}} \end{aligned}$$

and then evaluating the last inner sum by (2), we get the following third transformation from unilateral to bilateral series.

Theorem 3 (Transformation formula: $\delta = 0, 1$).

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{[\xi, q^\delta/\xi, x, y; q]_n}{(q;q)_{2n+\delta}} \left(\frac{q^{2+\delta}}{xy}\right)^n \\ &= \left[\begin{matrix} q^{1+\delta}/x, q^{1+\delta}/y \\ q, q^{1+\delta}/xy \end{matrix} \middle| q\right]_\infty \sum_{k=-\infty}^{+\infty} (-\xi)^k \frac{1+q^k\xi}{\xi+(-q/\xi)^\delta} q^{\binom{k}{2}} \\ &\times \left[\begin{matrix} x, & y \\ q^{1+\delta}/x, & q^{1+\delta}/y \end{matrix} \middle| q\right]_k \left(\frac{q^{1+\delta}}{xy}\right)^k. \end{aligned}$$

3. Identities of the Rogers–Ramanujan type

Specializing the transformations established in the last section so that the bilateral series on the right-hand side can be factorized into infinite products, we shall derive several identities of Rogers–Ramanujan type. The factorization process requires the triple product identity due to Jacobi [8, 14]

$$\sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{k}{2}} x^k = [q, x, q/x; q]_\infty \tag{3}$$

and the quintuple product identity [6, 10, 20]

$$\sum_{k=-\infty}^{+\infty} q^{3\binom{k}{2}}\{1-zq^k\}(qz^3)^k = [q, z, q/z; q]_\infty \times [qz^2, q/z^2; q^2]_\infty \tag{4}$$

as well as the following relation between quintuple and triple products

$$[q, z, q/z; q]_\infty [qz^2, q/z^2; q^2]_\infty = [q^3, -qz^3, -q^2/z^3; q^3]_\infty - z[q^3, -q^2z^3, -q/z^3; q^3]_\infty \tag{5}$$

where the last one will occasionally be utilized to derive a third identity by linearly combining two identities of Rogers–Ramanujan type.

For instance, letting $x, y \to \infty$ in Theorem 1 and then factorizing the corresponding bilateral sum with respect to k via Jacobi's triple product identity

$$\sum_{k=-\infty}^{+\infty} (-1)^k q^{3\binom{k}{2}+k(1+\delta)}\xi^k = [q^3, q^{1+\delta}\xi, q^{2-\delta}/\xi; q^3]_\infty$$

we recover the following interesting identity.

Example 1 (Ismail–Stanton [13, Propositions 6B and 6C]).

$$\sum_{m=0}^{\infty} \frac{(\xi; q)_m (q/\xi; q)_{m+\delta}}{(q; q)_{2m+\delta}} q^{m(m+\delta)} = \frac{[q^3, q^{1+\delta}\xi, q^{2-\delta}/\xi; q^3]_\infty}{(q; q)_\infty}.$$

This example shows that the derivation of identities of the Rogers–Ramanujan type from Theorems 1, 2 and 3 is entirely routine. Therefore it is not necessary to produce the details. However, the parameter settings for the transformation formulae to be utilized and eventual references for the identities will be highlighted in the headers for all the examples.

It is not our intention to exhibit all the identities of Rogers–Ramanujan type deduced from the three theorems. Instead, we shall emphasize on four classes of identities of Rogers–Ramanujan type whose product expressions are quintuple moduli "$6+12$", "$9+18$", "$12+24$" and "$18+36$". A few identities of Rogers–Ramanujan type with reduced triple products will also be displayed.

For the sake of brevity, we fix the notation $\omega := e^{2\pi i/3}$ as the cubic root of unity and abbreviate "T" for "Theorem", "E" for "Example" as well as "$\pm$" for "Linear combination" in the headers of examples.

§ 3.1. Quintuple products moduli "$6+12$"

EXAMPLE 2 (T1: $\delta=1$, $\xi=-q\omega$, $x=-q$, $y\to\infty$; [9, No. 161], [15, Eq. 1.24]).

$$\sum_{n=0}^{\infty}\frac{(-q^3;q^3)_n}{(q;q)_{2n+1}}q^{\binom{n+1}{2}}=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,q^2,q^4;q^6]_\infty[q^{10},q^2;q^{12}]_\infty.$$

EXAMPLE 3 (T1: $\delta=1$, $\xi=q\omega$, $x=-q$, $y\to\infty$; [3], [15, Eq. 1.29], [18, Eq. 77]).

$$\sum_{n=0}^{\infty}\frac{(-q;q)_n(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+1}}q^{\binom{n+1}{2}}=\frac{(-q;q)_\infty}{(q;q)_\infty}(q^6;q^6)_\infty.$$

EXAMPLE 4 (T3: $\delta=0$, $\xi=-\omega$, $x=-q$, $y\to\infty$; [9, No. 158], [15, Eq. 1.22]).

$$\sum_{n=0}^{\infty}\frac{(-q;q)_n(-1;q^3)_n}{(-1;q)_n(q;q)_{2n}}q^{\binom{n+1}{2}}=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,q,q^5;q^6]_\infty[q^8,q^4;q^{12}]_\infty.$$

EXAMPLE 5 (T3: $\delta=0$, $\xi=\omega$, $x=-q$, $y\to\infty$; [15, Eq. 1.27 corrected]).

$$1+3\sum_{n=1}^{\infty}\frac{(-q;q)_n(q^3;q^3)_{n-1}}{(q;q)_{n-1}(q;q)_{2n}}q^{\binom{n+1}{2}}=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,-q,-q^5;q^6]_\infty[q^8,q^4;q^{12}]_\infty.$$

EXAMPLE 6 (T3: $\delta=1$, $\xi=-\sqrt{q}\omega$, $x=-q$, $y=-q^{3/2}\,|\,q\to q^2$; [9, No. 159]).

$$\sum_{n=0}^{\infty}\frac{(-q^3;q^6)_nq^n}{(-q;q^2)_n(q;q)_{2n+1}}=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,q^2,q^4;q^6]_\infty[q^{10},q^2;q^{12}]_\infty.$$

EXAMPLE 7 (T3: $\delta=1$, $\xi=\sqrt{q}\omega$, $x=-q$, $y=-q^{3/2}\,|\,q\to q^2$; [9, No. 138]).

$$\sum_{n=0}^{\infty}\frac{(q^3;q^6)_nq^n}{(q;q^2)_n(q;q)_{2n+1}}=\frac{(-q;q)_\infty}{(q;q)_\infty}(q^6;q^6)_\infty.$$

EXAMPLE 8 (T3: $\delta=0$, $\xi=-\omega$, $x=-\sqrt{q}$, $y=-q\,|\,q\to q^2$; [9, No. 156]).

$$\sum_{n=0}^{\infty}\frac{(-1;q^6)_nq^n}{(-1;q^2)_n(q;q)_{2n}}=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,q,q^5;q^6]_\infty[q^8,q^4;q^{12}]_\infty.$$

EXAMPLE 9 (T3: $\delta=0$, $\xi=\omega$, $x=-\sqrt{q}$, $y=-q\,|\,q\to q^2$).

$$1+3\sum_{n=1}^{\infty}\frac{(q^6;q^6)_{n-1}q^n}{(q^2;q^2)_{n-1}(q;q)_{2n}}=\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,-q,-q^5;q^6]_\infty[q^8,q^4;q^{12}]_\infty.$$

§ 3.2. Quintuple products moduli "$9+18$"

EXAMPLE 10 (T1: $\underset{x,y\to\infty}{\delta=1,\xi=-q\omega}$; [5, Eq. 3.35], [9, No. 172], [15, Eq. 1.5]).

$$\sum_{n=0}^{\infty}\frac{(-q^3;q^3)_n q^{n^2+n}}{(-q;q)_n(q;q)_{2n+1}}=\frac{[q^9,q^3,q^6;q^9]_\infty}{(q;q)_\infty}[q^{15},q^3;q^{18}]_\infty.$$

EXAMPLE 11 (T1: $\underset{x,y\to\infty}{\delta=1,\xi=q\omega}$; [3, 4], [9, No. 147], [15, Eq. 1.9], [18, Eq. 92]).

$$\sum_{n=0}^{\infty}\frac{(q^3;q^3)_n q^{n^2+n}}{(q;q)_n(q;q)_{2n+1}}=\frac{(q^9;q^9)_\infty}{(q;q)_\infty}=\frac{[q^9,-q^3,-q^6;q^9]_\infty}{(q;q)_\infty}[q^{15},q^3;q^{18}]_\infty.$$

EXAMPLE 12 (T2: $\delta=0$, $\xi=-\omega$, $x,y\to\infty$; [9, No. 171], [15, Eq. 1.4]).

$$\sum_{n=0}^{\infty}\frac{(-1;q^3)_n q^{n^2}}{(-1;q)_n(q;q)_{2n}}=\frac{[q^9,q^2,q^7;q^9]_\infty}{(q;q)_\infty}[q^{13},q^5;q^{18}]_\infty.$$

EXAMPLE 13 (T2: $\delta=0$, $\xi=\omega$, $x,y\to\infty$; [3], [9, No. 146], [18, Eq. 91]).

$$\sum_{n=0}^{\infty}\frac{(q^3;q^3)_n q^{n^2+2n}}{(q;q)_n(q;q)_{2n+2}}=\frac{[q^{27},q^6,q^{21};q^{27}]_\infty}{(q;q)_\infty}.$$

EXAMPLE 14 (T3: $\delta=0$, $\xi=-\omega$, $x,y\to\infty$; [9, No. 170], [15, Eq. 1.3]).

$$\sum_{n=0}^{\infty}\frac{(-1;q^3)_n q^{n^2+n}}{(-1;q)_n(q;q)_{2n}}=\frac{[q^9,q,q^8;q^9]_\infty}{(q;q)_\infty}[q^{11},q^7;q^{18}]_\infty.$$

EXAMPLE 15 (T3: $\delta=0$, $\xi=\omega$, $x,y\to\infty$; [3], [9, No. 145], [11], [18, Eq. 90]).

$$\sum_{n=0}^{\infty}\frac{(q^3;q^3)_n q^{n^2+3n}}{(q;q)_n(q;q)_{2n+2}}=\frac{[q^{27},q^3,q^{24};q^{27}]_\infty}{(q;q)_\infty}.$$

EXAMPLE 16 (E12 $\pm$ E14; [9, No. 173], [15, Eq. 1.6]).

$$\sum_{n=0}^{\infty}\frac{(-q^3;q^3)_n q^{n^2+2n}}{(-q;q)_{n+1}(q;q)_{2n+1}}=\frac{[q^9,q^4,q^5;q^9]_\infty}{(q;q)_\infty}[q^{17},q;q^{18}]_\infty.$$

In order to illustrate the *linear combination method*, a detailed proof of this example is included below. Dividing by q the difference of the two equations displayed in Examples 12 and 14 and then replacing the summation index n by $n+1$, we get the equality

$$\sum_{n=0}^{\infty}\frac{(-q^3;q^3)_n q^{n^2+2n}}{(-q;q)_{n+1}(q;q)_{2n+1}}=\frac{q^{-1}}{(q;q)_\infty}\left\{\begin{array}{l}[q^9,q^2,q^7;q^9]_\infty[q^{13},q^5;q^{18}]_\infty\\ -[q^9,q,q^8;q^9]_\infty[q^{11},q^7;q^{18}]_\infty\end{array}\right\}.$$

Denote by $\mathscr{D}$ the difference inside the braces on the right hand side of the last equation. According to the quintuple product identity (4), there is the following infinite series expression

$$\begin{aligned}\mathscr{D}&=\sum_{k=-\infty}^{+\infty}q^{27\binom{k}{2}+15k}(1-q^{2+9k})-\sum_{k=-\infty}^{+\infty}q^{27\binom{k}{2}+12k}(1-q^{1+9k})\\&=\sum_{k=-\infty}^{+\infty}q^{27\binom{k}{2}+21k+1}-\sum_{k=-\infty}^{+\infty}q^{27\binom{k}{2}+24k+2}\end{aligned}$$

where two terms have been canceled by inverting the summation index $k\to-k$. Reformulating the last sum by making the replacement $k\to-k-1$ and then combining it with the first one, we can factorize $\mathscr{D}$ as follows

$$\begin{aligned}\mathscr{D}&=\sum_{k=-\infty}^{+\infty}q^{27\binom{k}{2}+21k+1}-\sum_{k=-\infty}^{+\infty}q^{27\binom{k}{2}+30k+5}\\&=q\sum_{k=-\infty}^{+\infty}q^{27\binom{k}{2}+21k}(1-q^{4+9k})\\&=q[q^9,q^4,q^5;q^9]_\infty[q^{17},q;q^{18}]_\infty\end{aligned}$$

where we have appealed again to (4) with $q\to q^9$ and $z\to q^4$. This confirms the identity stated in Example 16.

Linear combinations can analogously be employed to derive further identities.

Example 17 (E13 $\pm$ E15; [3], [9, No. 148], [18, Eq. 93]).

$$1+\sum_{n=1}^{\infty}\frac{(q^3;q^3)_{n-1}q^{n^2}}{(q;q)_n(q;q)_{2n-1}}=\frac{[q^{27},q^{12},q^{15};q^{27}]_\infty}{(q;q)_\infty}.$$

Example 18 (E13 $\pm$ E15; [15, Eq. 1.10]).

$$\sum_{n=0}^{\infty}\frac{(q^3;q^3)_n q^{n^2+2n}}{(q;q)_n^2(q^{n+2};q)_{n+1}}=\frac{[q^9,-q^4,-q^5;q^9]_\infty}{(q;q)_\infty}[q^{17},q;q^{18}]_\infty.$$

Example 19 (E13 $\pm$ E17; [15, Eq. 1.7]).

$$1+\sum_{n=1}^{\infty}\frac{(q^3;q^3)_{n-1}(2+q^n)}{(q;q)_{n-1}(q;q)_{2n}}q^{n^2}=\frac{[q^9,-q,-q^8;q^9]_\infty}{(q;q)_\infty}[q^{11},q^7;q^{18}]_\infty.$$

Example 20 (E15 $\pm$ E17; [15, Eq. 1.8]).

$$1+\sum_{n=1}^{\infty}\frac{(q^3;q^3)_{n-1}(1+2q^n)}{(q;q)_{n-1}(q;q)_{2n}}q^{n^2}=\frac{[q^9,-q^2,-q^7;q^9]_\infty}{(q;q)_\infty}[q^{13},q^5;q^{18}]_\infty.$$

§ 3.3. Quintuple products moduli "$12+24$"

Example 21 (T3: $\substack{\delta=0,\,\xi=-\omega\\ x=-\sqrt{q},\,y\to\infty}\Big|\, q\to q^2$; [9, No. 182], [15, Eq. 1.12]).

$$\sum_{n=0}^{\infty}\frac{(-1;q^6)_n(-q;q^2)_n}{(-1;q^2)_n(q^2;q^2)_{2n}}q^{n^2+2n}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},q,q^{11};q^{12}]_\infty[q^{14},q^{10};q^{24}]_\infty.$$

Example 22 (T1: $\substack{\delta=0,\,\xi=-\sqrt{q}\omega\\ x=-\sqrt{q},\,y\to\infty}\Big|\, q\to q^2$; [9, No. 183], [15, Eq. 1.11]).

$$\sum_{n=0}^{\infty}\frac{(-q^3;q^6)_n}{(q^2;q^2)_{2n}}q^{n^2}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},q^2,q^{10};q^{12}]_\infty[q^{16},q^8;q^{24}]_\infty.$$

Example 23 (T1: $\substack{\delta=0,\,\xi=\sqrt{q}\omega\\ x=-\sqrt{q},\,y\to\infty}\Big|\, q\to q^2$; [9, No. 184], [15, Eq. 1.16], [17, Eq. 109a]).

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n(q^3;q^6)_n}{(q;q^2)_n(q^2;q^2)_{2n}}q^{n^2}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},-q^2,-q^{10};q^{12}]_\infty[q^{16},q^8;q^{24}]_\infty.$$

Example 24 (T2: $\substack{\delta=0,\,\xi=-\omega\\ x=-\sqrt{q},\,y\to\infty}\Big|\, q\to q^2$; [9, No. 186], [15, Eq. 1.13]).

$$\sum_{n=0}^{\infty}\frac{(-1;q^6)_n(-q;q^2)_n}{(-1;q^2)_n(q^2;q^2)_{2n}}q^{n^2}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},q^3,q^9;q^{12}]_\infty[q^{18},q^6;q^{24}]_\infty.$$

Example 25 (T3: $\substack{\delta=1,\,\xi=-\sqrt{q}\omega\\ x=-q^{3/2},\,y\to\infty}\Big|\, q\to q^2$; [9, No. 187], [15, Eq. 1.14]).

$$\sum_{n=0}^{\infty}\frac{(-q^3;q^6)_nq^{n^2+2n}}{(-q;q)_{2n}(q;q)_{2n+1}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},q^4,q^8;q^{12}]_\infty[q^{20},q^4;q^{24}]_\infty.$$

EXAMPLE 26 (T3: $\substack{\delta=1,\,\xi=\sqrt{q}\omega\\ x=-q^{3/2},\,y\to\infty}\,\big|\, q\to q^2$; [9, No. 151], [15, Eq. 1.19], [18, Eq. 110]).

$$\sum_{n=0}^{\infty}\frac{(q^3;q^6)_n(-q;q^2)_{n+1}}{(q;q^2)_n(q^2;q^2)_{2n+1}}q^{n^2+2n}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^{12};q^{12})_\infty.$$

EXAMPLE 27 (T3: $\substack{\delta=0,\,\xi=\omega\\ x=-\sqrt{q},\,y\to\infty}\,\big|\, q\to q^2$; [3], [9, No. 149], [18, Eq. 116]).

$$\sum_{n=0}^{\infty}\frac{(q^6;q^6)_nq^{n^2+4n}}{(q;q)_{2n+1}(q^4;q^4)_{n+1}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{36},q^3,q^{33};q^{36}]_\infty.$$

EXAMPLE 28 (T2: $\substack{\delta=0,\,\xi=\omega\\ x=-\sqrt{q},\,y\to\infty}\,\big|\, q\to q^2$; [3], [9, No. 150], [18, Eq. 115]).

$$\sum_{n=0}^{\infty}\frac{(q^6;q^6)_nq^{n^2+2n}}{(q;q)_{2n+1}(q^4;q^4)_{n+1}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{36},q^9,q^{27};q^{36}]_\infty.$$

EXAMPLE 29 (E27 $\pm$ E28; [3], [9, No. 152], [18, Eq. 114]).

$$1+\sum_{n=1}^{\infty}\frac{(-q;q^2)_n(q^6;q^6)_{n-1}}{(q^2;q^2)_n(q^2;q^2)_{2n-1}}q^{n^2}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{36},q^{15},q^{21};q^{36}]_\infty.$$

EXAMPLE 30 (E21 $\pm$ E24; [9, No. 188], [15, Eq. 1.15]).

$$\sum_{n=0}^{\infty}\frac{(-q^6;q^6)_n(-q;q^2)_{n+1}}{(-q^2;q^2)_{n+1}(q^2;q^2)_{2n+1}}q^{n^2+2n}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},q^5,q^7;q^{12}]_\infty[q^{22},q^2;q^{24}]_\infty.$$

EXAMPLE 31 (E27 $\pm$ E28; [15, Eq. 1.20]).

$$\sum_{n=0}^{\infty}\frac{(q^6;q^6)_n(-q;q^2)_{n+1}}{(q^2;q^2)_n^2(q^{2n+4};q^2)_{n+1}}q^{n^2+2n}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},-q^5,-q^7;q^{12}]_\infty[q^{22},q^2;q^{24}]_\infty.$$

EXAMPLE 32 (E28 $\pm$ E29; [15, Eq. 1.17]).

$$1+\sum_{n=1}^{\infty}\frac{(q^6;q^6)_{n-1}(2+q^{2n})}{(q;q)_{2n-1}(q^4;q^4)_n}q^{n^2}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},-q,-q^{11};q^{12}]_\infty[q^{14},q^{10};q^{24}]_\infty.$$

EXAMPLE 33 (E27 $\pm$ E29; [15, Eq. 1.18]).

$$1+\sum_{n=1}^{\infty}\frac{(q^6;q^6)_{n-1}(1+2q^{2n})}{(q;q)_{2n-1}(q^4;q^4)_n}q^{n^2}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},-q^3,-q^9;q^{12}]_\infty[q^{18},q^6;q^{24}]_\infty.$$

Example 34 (T2: $\substack{\delta=1,\,\xi=\sqrt{q}\omega \\ x=-q,\,y\to\infty} \,\big|\, q\to q^2$; [9, No. 190], [15, Eq. 1.23], [18, Eq. 107]).

$$\sum_{n=0}^{\infty}\frac{(q^3;q^6)_n q^{n^2+n}}{(q;q)_{2n}(q^2;q^4)_{n+1}}=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},-q^3,-q^9;q^{12}]_\infty[q^{18},q^6;q^{24}]_\infty.$$

Example 35 (T3: $\substack{\delta=1,\,\xi=\sqrt{q}\omega \\ x=-q,\,y\to\infty} \,\big|\, q\to q^2$; [9, No. 191], [15, Eq. 1.25]).

$$\sum_{n=0}^{\infty}\frac{(q^3;q^6)_n q^{n^2+3n}}{(q;q)_{2n}(q^2;q^4)_{n+1}}=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},-q^5,-q^7;q^{12}]_\infty[q^{22},q^2;q^{24}]_\infty.$$

Example 36 (E34 $\pm$ E35; [9, No. 189], [15, Eq. 1.21]).

$$\sum_{n=0}^{\infty}\frac{(q^3;q^6)_n q^{n^2+n}}{(q^2;q^4)_n(q;q)_{2n+1}}=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},-q,-q^{11};q^{12}]_\infty[q^{14},q^{10};q^{24}]_\infty.$$

§ 3.4. Quintuple products moduli "18 + 36"

Example 37 (T1: $\delta=0,\ \xi=\sqrt{q}\omega,\ x,y\to\infty\,|\,q\to q^2$; [9, No. 198]).

$$\sum_{n=0}^{\infty}\frac{(q^3;q^6)_n q^{2n^2}}{(q;q^2)_n(q^2;q^2)_{2n}}=\frac{[q^{18},-q^3,-q^{15};q^{18}]_\infty}{(q^2;q^2)_\infty}[q^{24},q^{12};q^{36}]_\infty.$$

Example 38 (T3: $\delta=1,\ \xi=\sqrt{q}\omega,\ x,y\to\infty\,|\,q\to q^2$; [9, No. 200], [18, Eq. 125]).

$$\sum_{n=0}^{\infty}\frac{(q^3;q^6)_n q^{2n^2+4n}}{(q;q^2)_n(q^2;q^2)_{2n+1}}=\frac{[q^{18},-q^7,-q^{11};q^{18}]_\infty}{(q^2;q^2)_\infty}[q^{32},q^4;q^{36}]_\infty.$$

Example 39 (T2: $\delta=1,\ \xi=\sqrt{q}\omega,\ x,y\to\infty\,|\,q\to q^2$; [9, No. 199], [18, Eq. 124]).

$$\sum_{n=0}^{\infty}\frac{(q^3;q^6)_n q^{2n^2+2n}}{(q;q^2)_n(q^2;q^2)_{2n+1}}=\frac{[q^{18},-q^5,-q^{13};q^{18}]_\infty}{(q^2;q^2)_\infty}[q^{28},q^8;q^{36}]_\infty.$$

Example 40 (E38 $\pm$ E39; [9, No. 197]).

$$\sum_{n=0}^{\infty}\frac{(q^3;q^6)_n q^{2n^2+2n}}{(q;q^2)_{n+1}(q^2;q^2)_{2n}}=\frac{[q^{18},-q,-q^{17};q^{18}]_\infty}{(q^2;q^2)_\infty}[q^{20},q^{16};q^{36}]_\infty.$$

Before concluding the paper, we remark that the three simple products expressions in Examples 3, 7 and 26 can also be written formally in terms of quintuple products, i.e., the both right members displayed in Examples 3 and 7 are equal to

$$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,-q^2,-q^4;q^6]_\infty[q^{10},q^2;q^{12}]_\infty$$

while that of Example 26 has the following equivalent product expression

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},-q^4,-q^8;q^{12}]_\infty[q^{20},q^4;q^{24}]_\infty.$$

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