Hiroshima Math. J.
40 (2010), 133–183

A new description of convex bases of PBW type for untwisted quantum affine algebras

Ken Ito

(Received September 19, 2008)
(Revised April 21, 2009)

Abstract. In [8], we classified all "convex orders" on the positive root system Δ_+ of an arbitrary untwisted affine Lie algebra $\mathfrak{g}$ and gave a concrete method of constructing all convex orders on Δ_+. The aim of this paper is to give a new description of "convex bases" of PBW type of the positive subalgebra U^+ of the quantum affine algebra $U = U_q(\mathfrak{g})$ by using the concrete method of constructing all convex orders on Δ_+. Applying convexity properties of the convex bases of U^+, for each convex order on Δ_+, we construct a pair of dual bases of U^+ and the negative subalgebra U^- with respect to a q-analogue of the Killing form, and then present the multiplicative formula for the universal R-matrix of U.

1. Introduction

In the theory of quantum algebras, it is an important problem to construct the dual bases of the positive subalgebra U^+ and the negative subalgebra U^- of the quantum algebra $U = U_q$ with respect to the q-analogue of the Killing form which is defined in [12] and [15]. For example, the dual bases of U^+ and U^- were applied to express the universal R-matrix and the extremal projector of the quantum algebra U in an explicit formula ([12], [13]), and it is known that the dual bases are related to the canonical bases of U^+ or the global crystal bases of U^- ([3]). The positive and negative parts of the dual bases used to be constructed as a kind of Poincaré-Birkhoff-Witt (PBW) type bases of U^+ and U^- respectively, and the both parts have several convexity properties concerning the q-commutator and the coproduct of U. We would like to emphasize that the convexity properties are useful for calculating values of the q-Killing form, so we call the positive or negative parts of the dual bases *convex bases* of U^+ or U^- respectively.

By the way, each convex basis of U^+ is formed by monomials in certain q-root vectors E_α with α positive roots, which are multiplied in a predetermined total order on the positive root system Δ_+ of the underlying Lie algebra $\mathfrak{g}$.

2000 *Mathematics Subject Classification*. Primary 17B37, 17B67; Secondary 20F55.
Key words and phrases. quantum algebra, convex basis, convex order.

Because the total order on Δ_+ has several convexity properties, we call a such total order on Δ_+ "convex order" on Δ_+.

In the case where $\mathfrak{g}$ is an arbitrary finite dimensional simple Lie algebra, there is a natural bijective mapping between the set of the convex orders on Δ_+ and the set of the reduced expressions of the longest element of the Weyl group, and G. Lusztig constructed convex bases of U^+ associated with all reduced expressions of the longest element of the Weyl group by using a braid group action on $U = U_q(\mathfrak{g})$ ([14]). Therefore all convex bases of U^+ had been constructed in the finite case.

In the case where $\mathfrak{g}$ is an arbitrary untwisted affine Lie algebra, in [2], J. Beck constructed convex bases of U^+ associated with convex orders on Δ_+ of a special type. On the other hand, in [8], we classified all convex orders on Δ_+, and we found out that there exist new types of convex orders on Δ_+ which was not used in the Beck's construction, and then we gave a concrete method of constructing all convex orders on Δ_+ for the untwisted affine case. So we think that it is natural to extend the Beck's construction of convex bases of U^+ by using the new knowledge about convex orders on Δ_+.

In this article, we give a new description of convex bases of U^+ for the quantum affine algebra $U_q(\mathfrak{g})$ in the case where $\mathfrak{g}$ is an arbitrary untwisted affine Lie algebra, i.e., the affine Lie algebra of type $X_r^{(1)}$, where $X = A, B, C, D, E, F, G$. More precisely, we construct convex bases of U^+ by using the concrete method of constructing all convex orders on Δ_+ introduced in the paper [8]. Theorem 3.2 is a summary of the results of the paper [8]. Then the main results of this paper are Theorem 8.4 and Theorem 8.6, which are presented by using the notation of Theorem 3.2. In Theorem 8.4 and Theorem 8.6, we will use some parameter $\mathbf{J}$, which is an arbitrary non-empty subset of $\mathring{\mathbf{I}} := \{1, \dots, r\}$. Here, the set $\mathring{\mathbf{I}}$ is the index set of the simple root system of the underlying finite dimensional simple Lie algebra $\mathring{\mathfrak{g}}$ of type X_r introduced in the book [11]. We note that the algebras $U_{\mathbf{J}}$ and $U_{\mathbf{J}}^+$ are subalgebras of U and U^+ respectively and that $U_{\mathbf{J}}^+$ is nothing but U^+ in the case where $\mathbf{J} = \mathring{\mathbf{I}}$.

This paper is organized as follows. The notations of this paper are basically the same as the notations of the papers [7] and [8]. So, by referring to the papers [7] and [8], we omit the description of the notations in this paper. In section 2, we give notations and preliminary results on the root system of the untwisted affine Lie algebra $\mathfrak{g}$. In section 3, we give notations and preliminary results on reduced words of the Coxeter group $(W_{\mathbf{J}}, S_{\mathbf{J}})$ and convex orders on the positive root system $\Delta_{\mathbf{J}+}$. In section 4, we give notations and preliminary results on the quantum algebra $U = U_q(\mathfrak{g})$. In section 5, we construct the subalgebra $U_{\mathbf{J}}$ of U associated with $\Delta_{\mathbf{J}}$ and the braid group action on it. In section 6, we define imaginary root vectors of $U_{\mathbf{J}}^+$. In section 7, we give several tensor product decompositions of the positive subalgebra $U_{\mathbf{J}}^{+}$ of $U_{\mathbf{J}}$. In section 8, we give a concrete method of constructing convex bases of $U_{\mathbf{J}}^{+}$ associated with convex orders on $\Delta_{\mathbf{J}+}$. In section 9, we construct the dual convex bases of U^{+} and U^{-} with respect to the q-Killing form, and then present the multiplicative formula for the universal R-matrix of U associated with an arbitrary convex order on Δ_{+}.

2. Preliminary results on the untwisted affine root systems

First of all, we would like to mention that the notations of this paper for the root system of $\mathfrak{g}$ follow that in [7] and [8], where $\mathfrak{g}$ is the untwisted affine Lie algebra of type $X_r^{(1)}$ with $X = A, B, \dots, G$ and $r \in \mathbf{N}$ the rank of the underlying finite dimensional simple Lie algebra of type X_r. So we will omit the description of the notations. However, for writing this paper, we will make a few changes in the notations and give some additional notations. In this paper, let us denote by $\mathrm{A} = [\mathrm{A}_{ij}]_{i,j \in \mathbf{I}}$ the generalized Cartan matrix of the type $X_r^{(1)}$ with $\mathbf{I} = \{0, 1, \dots, r\}$. In addition, we assume that $[\mathrm{A}_{ij}]_{i,j \in \mathring{\mathbf{I}}}$ is the Cartan matrix of the type X_r with $\mathring{\mathbf{I}} = \{1, \dots, r\}$.

For each $i \in \mathring{\mathbf{I}}$, let ε_i be a unique element of $\mathring{\mathfrak{h}}^*$ such that $(\varepsilon_i \,|\, \alpha_j) = \delta_{ij}$ for all $j \in \mathring{\mathbf{I}}$. For each non-empty subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, we set

$$\mathring{P}_{\mathbf{J}}^{\vee} := \bigoplus_{j \in \mathbf{J}} \mathbf{Z}\varepsilon_j, \quad \hat{T}_{\mathbf{J}} := \{t_\lambda \,|\, \lambda \in \mathring{P}_{\mathbf{J}}^{\vee}\}, \quad \hat{W}_{\mathbf{J}} := \hat{T}_{\mathbf{J}} \rtimes \mathring{W}_{\mathbf{J}} \subset \mathrm{GL}(\mathfrak{h}_{\mathbf{J}}^{*\prime})$$

and set

$$\mathrm{Aut}(\Delta_{\mathbf{J}}) := \{\phi \in \mathrm{GL}(\mathfrak{h}_{\mathbf{J}}^{*\prime}) \,|\, \phi(\Delta_{\mathbf{J}}) = \Delta_{\mathbf{J}},\ (\phi(\lambda) \,|\, \phi(\mu)) = (\lambda \,|\, \mu)\,(\forall \lambda, \mu \in \mathfrak{h}_{\mathbf{J}}^{*\prime})\},$$

$$\mathrm{Aut}(\Delta_{\mathbf{J}}, \Pi_{\mathbf{J}}) := \{\phi \in \mathrm{Aut}(\Delta_{\mathbf{J}}) \,|\, \phi(\Pi_{\mathbf{J}}) = \Pi_{\mathbf{J}}\}, \qquad \Omega_{\mathbf{J}} := \hat{W}_{\mathbf{J}} \cap \mathrm{Aut}(\Delta_{\mathbf{J}}, \Pi_{\mathbf{J}}).$$

Then $\hat{W}_{\mathbf{J}} = W_{\mathbf{J}} \rtimes \Omega_{\mathbf{J}} \subset \mathrm{Aut}(\Delta_{\mathbf{J}})$. For each $\mathbf{K} \subset \mathbf{J}$, we set

$$\mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}} := \{\phi \in \mathrm{Aut}(\Delta_{\mathbf{J}}) \,|\, \phi(\mathring{\Pi}_{\mathbf{K}}) \subset \Delta_{\mathbf{J}+}\},$$

$$W_{\mathbf{J}}^{\mathbf{K}} := W_{\mathbf{J}} \cap \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}}, \qquad \hat{W}_{\mathbf{J}}^{\mathbf{K}} := \hat{W}_{\mathbf{J}} \cap \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}}.$$

Note that $\mathrm{Aut}(\Delta_{\mathbf{J}}, \Pi_{\mathbf{J}}) \subset \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{J}}$. Let $\ell_{\mathbf{J}} : \hat{W}_{\mathbf{J}} \to \mathbf{Z}_{+}$ be the extended length function defined by setting $\ell_{\mathbf{J}}(x\rho) := \ell_{\mathbf{J}}(x)$ for each $x \in W_{\mathbf{J}}$ and $\rho \in \Omega_{\mathbf{J}}$. We note that $\ell_{\mathbf{J}}(y) = \#\Phi_{\mathbf{J}}(y)$ for all $y \in \hat{W}_{\mathbf{J}}$.

Proposition 2.1 ([9]). *For each connected subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, the assignment*

$$j \mapsto \rho_{\mathbf{J}j} := t_{\varepsilon_j} w_{\circ j} w_{\circ} \tag{2.1}$$

defines a bijective mapping from the set $\mathbf{J}_* := \{j \in \mathbf{J} \,|\, (\varepsilon_j|\theta_{\mathbf{J}}) = 1\}$ to $\Omega_{\mathbf{J}} \backslash \{1\}$. Here, $w_\circ$ and $w_{\circ j}$ are the longest elements of $\mathring{W}_{\mathbf{J}}$ and $\mathring{W}_{\mathbf{J}\backslash\{j\}}$, respectively. Moreover, the condition that $\rho(\delta - \theta_{\mathbf{J}}) = \alpha_j$ for $\rho \in \Omega_{\mathbf{J}} \backslash \{1\}$ and $j \in \mathbf{J}$ is equivalent to the condition that $\rho = \rho_{\mathbf{J}j}$ with $j \in \mathbf{J}_*$.

Proof. Although the setting of Proposition 1.18 in [9] is different from that of this case, the proof can be applied to this case with suitable modification.

Lemma 2.2. *Let* $\mathbf{J}$ *be an arbitrary connected subset of* $\mathring{\mathbf{I}}$, *and* $\mathbf{K}$ *an arbitrary subset of* $\mathbf{J}$. *Then each* $\phi \in \mathrm{Aut}(\Delta_{\mathbf{J}})$ *can be uniquely written as* $\phi = \phi^{\mathbf{K}}\phi_{\mathbf{K}}$ *with* $\phi^{\mathbf{K}} \in \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}}$ *and* $\phi_{\mathbf{K}} \in \mathring{W}_{\mathbf{K}}$.

Proof. We first prove the uniqueness. Suppose that $\phi = a^{\mathbf{K}}a_{\mathbf{K}} = b^{\mathbf{K}}b_{\mathbf{K}}$ with $a^{\mathbf{K}}, b^{\mathbf{K}} \in \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}}$ and $a_{\mathbf{K}}, b_{\mathbf{K}} \in \mathring{W}_{\mathbf{K}}$. Then $a^{\mathbf{K}} = b^{\mathbf{K}}b_{\mathbf{K}}a_{\mathbf{K}}^{-1}$. Since $a^{\mathbf{K}} \in \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}}$ and $b_{\mathbf{K}}a_{\mathbf{K}}^{-1} \in \mathring{W}_{\mathbf{J}}$, we have $b_{\mathbf{K}}a_{\mathbf{K}}^{-1} = 1$, hence $b_{\mathbf{K}} = a_{\mathbf{K}}$ and $b^{\mathbf{K}} = a^{\mathbf{K}}$. We next prove the existence. By Corollary 3.10 in [11], the automorphism ϕ can be uniquely written as $\sigma\rho z$ with $\sigma = \pm 1$, $\rho \in \mathrm{Aut}(\Delta_{\mathbf{J}}, \Pi_{\mathbf{J}})$, and $z \in W_{\mathbf{J}}$. Moreover, we see that z can be uniquely written as xy with $x \in W_{\mathbf{J}}^{\mathbf{K}}$ and $y \in \mathring{W}_{\mathbf{K}}$. Hence $\phi = \sigma\rho xy$. In the case where $\sigma = 1$, put $\phi^{\mathbf{K}} = \rho x$ and $\phi_{\mathbf{K}} = y$. Then $\phi = \phi^{\mathbf{K}}\phi_{\mathbf{K}}$ with $\phi^{\mathbf{K}} \in \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}}$ and $\phi_{\mathbf{K}} \in \mathring{W}_{\mathbf{K}}$. In the case where $\sigma = -1$, put $\phi^{\mathbf{K}} = -\rho x w_\circ$ and $\phi_{\mathbf{K}} = w_\circ y$, where $w_\circ$ is the longest element of $\mathring{W}_{\mathbf{K}}$. Then $\phi = \phi^{\mathbf{K}}\phi_{\mathbf{K}}$ with $\phi^{\mathbf{K}} \in \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}}$ and $\phi_{\mathbf{K}} \in \mathring{W}_{\mathbf{K}}$. □

Lemma 2.3. *Let* $\mathbf{J}$ *and* $\mathbf{J}'$ *be connected subsets of* $\mathring{\mathbf{I}}$ *which are disjoint from each other.*

(1) *For each* $j \in \mathbf{J}_*$, *there exists a unique element* $w_{\mathbf{J}j} \in \mathring{W}_{\mathbf{J}}$ *such that*

$$\text{(i)} \quad t_{\varepsilon_j}|_{\mathfrak{h}_{\mathbf{J}}^{*\prime}} = \rho_{\mathbf{J}j}w_{\mathbf{J}j}. \tag{2.2}$$

Moreover, the following equalities hold:

$$\text{(ii)} \quad \rho_{\mathbf{J}j} = (t_{\varepsilon_j})^{\mathbf{J}}|_{\mathfrak{h}_{\mathbf{J}}^{*\prime}}, \qquad \text{(iii)} \quad w_{\mathbf{J}j} = (t_{\varepsilon_j})_{\mathbf{J}} = w_\circ w_{\circ j}. \tag{2.3}$$

Here, $(t_{\varepsilon_j})^{\mathbf{J}} \in \hat{W}^{\mathbf{J}}$ *and* $(t_{\varepsilon_j})_{\mathbf{J}} \in \mathring{W}_{\mathbf{J}}$ *are unique elements such that* $t_{\varepsilon_j} = (t_{\varepsilon_j})^{\mathbf{J}}(t_{\varepsilon_j})_{\mathbf{J}}$, *and* $w_\circ$ *and* $w_{\circ j}$ *are the longest elements of* $\mathring{W}_{\mathbf{J}}$ *and* $\mathring{W}_{\mathbf{J}\setminus\{j\}}$ *respectively.*

(2) *For each* $i \in \mathring{\mathbf{I}}\setminus\mathbf{J}$, $z \in \mathring{W}_{\mathbf{J}'}$, *and* $j' \in \mathbf{J}'_*$, *the following equalities hold:*

$$\text{(i)} \quad [(t_{\varepsilon_j})^{\mathbf{J}}, t_{\varepsilon_i}] = 0, \qquad \text{(ii)} \quad [(t_{\varepsilon_j})^{\mathbf{J}}, z] = 0, \qquad \text{(iii)} \quad [(t_{\varepsilon_j})^{\mathbf{J}}, (t_{\varepsilon_{j'}})^{\mathbf{J}'}] = 0.$$

Here, $[\,,]$ *is the commutator, i.e.,* $[a,b] = ab - ba$. *Moreover,*

$$\text{(iv)} \quad \ell((t_{\varepsilon_j})^{\mathbf{J}}t_{\varepsilon_i}) = \ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell(t_{\varepsilon_i}), \qquad \text{(v)} \quad \ell((t_{\varepsilon_j})^{\mathbf{J}}z) = \ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell(z),$$

$$\text{(vi)} \quad \ell((t_{\varepsilon_j})^{\mathbf{J}}(t_{\varepsilon_{j'}})^{\mathbf{J}'}) = \ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell((t_{\varepsilon_{j'}})^{\mathbf{J}'}).$$

(3) *For each* $j \in \mathbf{J}_*$ *and* $\beta \in \Delta_{\mathbf{J}'}$, *the equality* $(t_{\varepsilon_j})^{\mathbf{J}}(\beta) = \beta$ *holds.*

(4) *For each* $j \in \mathbf{J}_*$, *the element* $(t_{\varepsilon_j})^{\mathbf{J}}$ *satisfies the following equalities:*

(i) $\Phi((t_{\varepsilon_j})^{\mathbf{J}}) \subset \Delta^{\mathbf{J}}(1,-)$, (ii) $\Phi(t_{\varepsilon_j}) \cap \Delta_{\mathbf{J}+} = (t_{\varepsilon_j})^{\mathbf{J}}\Phi(w_{\mathbf{J}_j})$,

(iii) $(t_{\varepsilon_j})^{\mathbf{J}}\Delta^{\mathbf{J}}(1,-) \subset \Delta^{\mathbf{J}}(1,-)$.

Moreover, $\ell((t_{\varepsilon_j})^{\mathbf{J}}) = 0$ *if and only if* $\mathbf{J} = \mathring{\mathbf{I}}$.

(5) *For each* $j \in \mathbf{J}_*$, *there exists a unique element* $j^- \in \mathbf{J}_*$ *such that*

$$\text{(i)}\quad \rho_{\mathbf{J}j}(\alpha_{j^-}) = \delta - \theta_{\mathbf{J}}, \qquad \text{(ii)}\quad \rho_{\mathbf{J}j^-} = (\rho_{\mathbf{J}j})^{-1}.$$

In addition, $(\rho_{\mathbf{J}j})^2 = 1$ *if and only if* $j^- = j$. *Moreover,*

(iii) $(t_{\varepsilon_j})^{\mathbf{J}}s_{j^-}(t_{\varepsilon_{j^-}})^{\mathbf{J}}|_{\mathfrak{h}_{\mathbf{J}}^{*\prime}} = s_{\delta-\theta_{\mathbf{J}}}$, (iv) $\Phi((t_{\varepsilon_j})^{\mathbf{J}}s_{j^-}(t_{\varepsilon_{j^-}})^{\mathbf{J}}) \cap \Delta_{\mathbf{J}+} = \{\delta - \theta_{\mathbf{J}}\}$,

(v) $\Phi((t_{\varepsilon_j})^{\mathbf{J}}s_{j^-}(t_{\varepsilon_{j^-}})^{\mathbf{J}}) \backslash \{\delta - \theta_{\mathbf{J}}\} \subset \Delta^{\mathbf{J}}(1,-)$,

(vi) $\ell((t_{\varepsilon_j})^{\mathbf{J}}s_{j^-}(t_{\varepsilon_{j^-}})^{\mathbf{J}}) = \ell((t_{\varepsilon_j})^{\mathbf{J}}) + 1 + \ell((t_{\varepsilon_{j^-}})^{\mathbf{J}})$.

Proof. (1) Set $w_{\mathbf{J}j} := w_\circ w_{\circ j}$. Then $w_{\mathbf{J}j} \in \mathring{W}_{\mathbf{J}}$. By Proposition 2.1, we have $t_{\varepsilon_j}|_{\mathfrak{h}_{\mathbf{J}}^{*\prime}} = \rho_{\mathbf{J}j}w_{\mathbf{J}j}$. On the other hand, by Lemma 2.2, we have $t_{\varepsilon_j} = (t_{\varepsilon_j})^{\mathbf{J}}(t_{\varepsilon_j})_{\mathbf{J}}$ with $(t_{\varepsilon_j})^{\mathbf{J}} \in \hat{W}^{\mathbf{J}}$ and $(t_{\varepsilon_j})_{\mathbf{J}} \in \mathring{W}_{\mathbf{J}}$. It follows that $t_{\varepsilon_j}|_{\mathfrak{h}_{\mathbf{J}}^{*\prime}} = (t_{\varepsilon_j})^{\mathbf{J}}|_{\mathfrak{h}_{\mathbf{J}}^{*\prime}}(t_{\varepsilon_j})_{\mathbf{J}}$. Hence (2.3) follows from Lemma 2.2. The uniqueness of the decomposition (2.2) follows from (2.3).

(2) By the part (1), we have $(t_{\varepsilon_j})^{\mathbf{J}} = t_{\varepsilon_j}w_{\mathbf{J}j}^{-1}$. It is clear that $[t_{\varepsilon_j}, t_{\varepsilon_i}] = [w_{\mathbf{J}j}^{-1}, t_{\varepsilon_i}] = 0$, which implies (i). Since $(\varepsilon_j|\alpha) = 0$ for all $\alpha \in \mathring{\Pi}_{\mathbf{J}'}$, we have $[t_{\varepsilon_j}, s_\alpha] = 0$. Since $w_{\mathbf{J}j} \in \mathring{W}_{\mathbf{J}}$, we see that $[w_{\mathbf{J}j}^{-1}, s_\alpha] = 0$ for all $\alpha \in \mathring{\Pi}_{\mathbf{J}'}$. Thus we get $[(t_{\varepsilon_j})^{\mathbf{J}}, s_\alpha] = 0$ for all $\alpha \in \mathring{\Pi}_{\mathbf{J}'}$, which implies (ii). Since $(t_{\varepsilon_{j'}})^{\mathbf{J}'} = t_{\varepsilon_{j'}}w_{\mathbf{J}'j'}^{-1}$ with $w_{\mathbf{J}'j'} \in \mathring{W}_{\mathbf{J}'}$, (iii) follows from (i) and (ii).

It is clear that $\ell((t_{\varepsilon_j})^{\mathbf{J}}t_{\varepsilon_i}) \leq \ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell(t_{\varepsilon_i})$. Since $[w_{\mathbf{J}j}, t_{\varepsilon_i}] = 0$ we have $t_{\varepsilon_j}t_{\varepsilon_i} = (t_{\varepsilon_j})^{\mathbf{J}}t_{\varepsilon_i}w_{\mathbf{J}j}$, and hence $\ell(t_{\varepsilon_j}t_{\varepsilon_i}) \leq \ell((t_{\varepsilon_j})^{\mathbf{J}}t_{\varepsilon_i}) + \ell(w_{\mathbf{J}j})$. On the other hand, we have

$$\ell(t_{\varepsilon_j}t_{\varepsilon_i}) = \ell(t_{\varepsilon_j}) + \ell(t_{\varepsilon_i}) = \{\ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell(w_{\mathbf{J}j})\} + \ell(t_{\varepsilon_i}).$$

Thus we get $\ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell(t_{\varepsilon_i}) \leq \ell((t_{\varepsilon_j})^{\mathbf{J}}t_{\varepsilon_i})$, which implies (iv). The assertion (v) is clear. It is easy to see that $\ell((t_{\varepsilon_j})^{\mathbf{J}}t_{\varepsilon_{j'}}) \leq \ell((t_{\varepsilon_j})^{\mathbf{J}}(t_{\varepsilon_{j'}})^{\mathbf{J}'}) + \ell(w_{\mathbf{J}'j'})$. From (iv) and (v), it follows that

$$\ell((t_{\varepsilon_j})^{\mathbf{J}}t_{\varepsilon_{j'}}) = \ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell((t_{\varepsilon_{j'}})^{\mathbf{J}'}) + \ell(w_{\mathbf{J}'j'}).$$

Thus we get that $\ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell((t_{\varepsilon_{j'}})^{\mathbf{J}'}) \leq \ell((t_{\varepsilon_j})^{\mathbf{J}}(t_{\varepsilon_{j'}})^{\mathbf{J}'})$, which implies (vi), since $\ell((t_{\varepsilon_j})^{\mathbf{J}}(t_{\varepsilon_{j'}})^{\mathbf{J}'}) \leq \ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell((t_{\varepsilon_{j'}})^{\mathbf{J}'})$.

(3) Since $t_{\varepsilon_j}(\beta) = \beta$ and $w_{\mathbf{J}j}(\beta) = \beta$, we have $(t_{\varepsilon_j})^{\mathbf{J}}(\beta) = t_{\varepsilon_j}w_{\mathbf{J}j}^{-1}(\beta) = \beta$.

(4) It is easy to see that

$$\Phi(t_{\varepsilon_j}) = \Phi((t_{\varepsilon_j})^{\mathbf{J}}) \amalg (t_{\varepsilon_j})^{\mathbf{J}}\Phi(w_{\mathbf{J}j}) \subset \Delta(1,-). \tag{2.4}$$

By (1)(ii), we have $\Phi((t_{\varepsilon_j})^{\mathbf{J}}) \cap \Delta_{\mathbf{J}+} = \emptyset$, hence we see that (i) and (ii) follow from (2.4). Since both t_{ε_j} and $w_{\mathbf{J}j}^{-1}$ stabilize $\Delta^{\mathbf{J}}(1,-)$, the equality $(t_{\varepsilon_j})^{\mathbf{J}} = t_{\varepsilon_j} w_{\mathbf{J}j}^{-1}$ implies (iii). We see that $\Phi(t_{\varepsilon_j}) \cap \Delta^{\mathbf{J}}(1,-) = \emptyset$ if and only if $\mathbf{J} = \mathring{\mathbf{I}}$, hence the second assertion follows from (i)(ii) and (2.4).

(5) By Proposition 2.1, there exists a unique element $j^- \in \mathbf{J}_*$ satisfying (i)(ii). Suppose that $(\rho_{\mathbf{J}j})^2 = 1$, i.e., $(\rho_{\mathbf{J}j})^{-1} = \rho_{\mathbf{J}j}$. By (ii) and Proposition 2.1 we get $j^- = j$. Suppose that $j^- = j$. Then, by (ii) we get $\rho_{\mathbf{J}j} = (\rho_{\mathbf{J}j})^{-1}$, i.e., $(\rho_{\mathbf{J}j})^2 = 1$. By (1)(ii) and (ii), we see that

$$(t_{\varepsilon_j})^{\mathbf{J}} s_{j^-} (t_{\varepsilon_{j^-}})^{\mathbf{J}}|_{\mathfrak{h}_{\mathbf{J}}^{*\prime}} = \rho_{\mathbf{J}j} s_{j^-} (\rho_{\mathbf{J}j})^{-1}. \tag{2.5}$$

Since $\delta - \theta_{\mathbf{J}} = \rho_{\mathbf{J}j}(\alpha_{j^-})$, we have $\rho_{\mathbf{J}j} s_{j^-} (\rho_{\mathbf{J}j})^{-1}(\delta - \theta_{\mathbf{J}}) = -(\delta - \theta_{\mathbf{J}})$. Since $(\alpha_{j^-} \,|\, \alpha_{j^-}) = (\delta - \theta_{\mathbf{J}} \,|\, \delta - \theta_{\mathbf{J}})$ and $((\rho_{\mathbf{J}j})^{-1}(\alpha_i) \,|\, \alpha_{j^-}) = (\alpha_i \,|\, \delta - \theta_{\mathbf{J}})$ for all $i \in \mathbf{J}$, we have $s_{j^-}(\rho_{\mathbf{J}j})^{-1}(\alpha_i) = (\rho_{\mathbf{J}j})^{-1}(\alpha_i) - \frac{2(\alpha_i \,|\, \delta - \theta_{\mathbf{J}})}{(\delta - \theta_{\mathbf{J}} \,|\, \delta - \theta_{\mathbf{J}})} \alpha_{j^-}$, which implies that

$$\rho_{\mathbf{J}j} s_{j^-} (\rho_{\mathbf{J}j})^{-1}(\alpha_i) = \alpha_i - \frac{2(\alpha_i \,|\, \delta - \theta_{\mathbf{J}})}{(\delta - \theta_{\mathbf{J}} \,|\, \delta - \theta_{\mathbf{J}})}(\delta - \theta_{\mathbf{J}}) = s_{\delta - \theta_{\mathbf{J}}}(\alpha_i).$$

Therefore (iii) follows from (2.5). By (4)(i), we see that

$$s_{j^-}\Phi((t_{\varepsilon_{j^-}})^{\mathbf{J}}) \subset \Delta^{\mathbf{J}}(1,-), \qquad \Phi(s_{j^-}(t_{\varepsilon_{j^-}})^{\mathbf{J}}) = \{\alpha_{j^-}\} \amalg s_{j^-}\Phi((t_{\varepsilon_{j^-}})^{\mathbf{J}}), \tag{2.6}$$

since $s_{j^-}\Delta^{\mathbf{J}}(1,-) \subset \Delta^{\mathbf{J}}(1,-)$. By (4)(iii) and the left equality in (2.6), we have

$$(t_{\varepsilon_j})^{\mathbf{J}} s_{j^-} \Phi((t_{\varepsilon_{j^-}})^{\mathbf{J}}) \subset \Delta^{\mathbf{J}}(1,-). \tag{2.7}$$

By (2.6) and the equality $(t_{\varepsilon_j})^{\mathbf{J}}(\alpha_{j^-}) = \delta - \theta_{\mathbf{J}}$, we have

$$\Phi((t_{\varepsilon_j})^{\mathbf{J}} s_{j^-} (t_{\varepsilon_{j^-}})^{\mathbf{J}}) = \Phi((t_{\varepsilon_j})^{\mathbf{J}}) \amalg \{\delta - \theta_{\mathbf{J}}\} \amalg (t_{\varepsilon_j})^{\mathbf{J}} s_{j^-} \Phi((t_{\varepsilon_{j^-}})^{\mathbf{J}}). \tag{2.8}$$

Therefore (iv), (v), and (vi) follow from (4)(i) and (2.7)(2.8). □

Lemma 2.4. *Let us use the notations as in Proposition* 2.1. *Assume that* $\mathbf{J}$ *is a connected subset of* $\mathring{\mathbf{I}}$ *with* $\#\mathbf{J} \geq 2$ *and that an element* $j \in \mathbf{J}_*$ *satisfies* $(\rho_{\mathbf{J}j})^2 = 1$. *Then there exist distinct elements* $i, i' \in \mathbf{I}$ *and an element* $z \in W$ *satisfying* $\Phi(z) \subset \Delta(1,-)$, $\alpha_j = z(\alpha_i)$, *and* $\delta - \theta_{\mathbf{J}} = z(\alpha_{i'})$.

Proof. Let B be the subset of $\Delta(1,-)$ consisting of all β such that

$$\beta + \alpha_{i_1} + \cdots + \alpha_{i_n} = \delta - \theta_{\mathbf{J}} \tag{2.9}$$

for some sequence $(i_1, \ldots, i_n)$ consisting of elements of $\mathring{\mathbf{I}}$ with $n \in \mathbf{N}$. Then, it is easy to see that both B and $B' = B \amalg \{\delta - \theta_{\mathbf{J}}\}$ are finite biconvex sets. Hence, there exist unique $z \in W$ and $i' \in \mathbf{I}$ such that $B = \Phi(z)$ and $\delta - \theta_{\mathbf{J}} = z(\alpha_{i'})$ by Theorem 2.4. We next show that

$$s_j(\delta - \theta_{\mathbf{J}}) = \delta - \theta_{\mathbf{J}}, \qquad s_j(B) = B. \tag{2.10}$$

By the assumption of the Lemma and the extended Dynkin diagram of $\mathring{\Delta}_{\mathbf{J}}$, we see that $(\delta - \theta_{\mathbf{J}} \mid \alpha_j) = 0$, which implies the left equality in (2.10). Let β be an arbitrary element of B. To prove the right equality in (2.10), it suffices to show that B includes the α_j-string through β. Since α_j is not a short root, we see that the length of the α_j-string through β is less than 2. If the length is 1, there is nothing to prove. Suppose that the length is 2. In the case where $s_j(\beta) = \beta - \alpha_j$, we see that $s_j(\beta) \in \Delta(1,-)$ and

$$s_j(\beta) + \alpha_j + (\alpha_{i_1} + \cdots + \alpha_{i_n}) = \delta - \theta_{\mathbf{J}},$$

which implies $s_j(\beta) \in B$. In the case where $s_j(\beta) = \beta + \alpha_j$, we see that $j = i_k$ for some $1 \leq k \leq n$. Indeed, if $j \neq i_k$ for all $1 \leq k \leq n$, then we see that $\beta + (\alpha_{i_1} + \cdots + \alpha_{i_n}) + m\alpha_j = \delta - \theta_{\mathbf{J}}$ for some $m \geq 1$ by applying s_j to the equality (2.9). Here we use the left equality in (2.10). This contradicts (2.9). Hence, $j = i_k$ for some $1 \leq k \leq n$. Thus we see that

$$s_j(\beta) + (\alpha_{i_1} + \cdots + \alpha_{i_{k-1}} + \alpha_{i_{k+1}} + \cdots + \alpha_{i_n}) = \delta - \theta_{\mathbf{J}}$$

with $s_j(\beta) \in \Delta(1,-)$. Here we have $n \geq 2$. Indeed, if $n = 1$ then $s_j(\beta) = \delta - \theta_{\mathbf{J}}$, which contradicts the left equation in (2.10). Thus we get $s_j(\beta) \in B$.

By the right equality in (2.10) and the equality $B = \Phi(z)$, we see that $\Phi(s_j z) = \{\alpha_j\} \amalg s_j \Phi(z) = \{\alpha_j\} \amalg \Phi(z)$. On the other hand, since $\Phi(z) \subset \Phi(s_j z)$ and $\#\{\Phi(s_j z) \backslash \Phi(z)\} = 1$, we see that $\Phi(s_j z) = \Phi(z) \amalg z\{\alpha_i\} = \Phi(z s_i)$ for some unique $i \in \mathbf{I}$. Thus we get $\alpha_j = z(\alpha_i)$. □

3. Preliminary results on reduced words and convex orders

We denote by $\mathbf{N}_n$ the set $\{m \in \mathbf{N} \mid m \leq n\}$ for each $n \in \mathbf{N}$, and set $\mathbf{N}_\infty := \mathbf{N}$ and $\mathbf{N}_* := \mathbf{N} \amalg \{\infty\}$, where ∞ is a symbol. We extend the usual order $\leq$ on $\mathbf{N}$ to a total order on $\mathbf{N}_*$ by setting $n < \infty$ for each $n \in \mathbf{N}$. We also set $\infty + n = n + \infty = \infty n = n\infty = \infty$ for each $n \in \mathbf{N}_*$.

For each non-empty subset $\mathbf{J}$ of $\mathring{\mathbf{I}}$, we set $\hat{S}_{\mathbf{J}} := S_{\mathbf{J}} \amalg (\Omega_{\mathbf{J}} \backslash \{1\}) = \coprod_{c=1}^{\mathrm{C}(\mathbf{J})} (S_{\mathbf{J}_c} \amalg \Omega_{\mathbf{J}_c} \backslash \{1\})$. For each $n \in \mathbf{N}_*$, we denote a sequence consisting of elements $\boldsymbol{s}(p) \in \hat{S}_{\mathbf{J}}$ with $p \in \mathbf{N}_n$ by $\boldsymbol{s} = (\boldsymbol{s}(p))_{p \in \mathbf{N}_n}$, and denote the set of all such sequences by $\hat{S}_{\mathbf{J}}^{\mathbf{N}_n}$. In addition, let us denote by $S_{\mathbf{J}}^{\mathbf{N}_n}$ the set of all sequences $\boldsymbol{s} \in \hat{S}_{\mathbf{J}}^{\mathbf{N}_n}$ such that $\boldsymbol{s}(p) \in S_{\mathbf{J}}$ for all $p \in \mathbf{N}_n$. Several operations (initial p-sections $\boldsymbol{s}|_p$, products $\boldsymbol{s}\boldsymbol{s}'$, limits $\lim_{p \to \infty} \boldsymbol{s}_p$, etc.) for the elements of $S_{\mathbf{J}}^{\mathbf{N}_n}$ are defined in [7], and the same operations can be defined for the elements of $\hat{S}_{\mathbf{J}}^{\mathbf{N}_n}$ in the same manner.

For each $\boldsymbol{s} \in \hat{S}_{\mathbf{J}}^{\mathbf{N}_n}$ with $n < \infty$, we define an element $[\boldsymbol{s}]$ of $\hat{W}_{\mathbf{J}}$ by setting $[\boldsymbol{s}] := \boldsymbol{s}(1)\boldsymbol{s}(2)\cdots\boldsymbol{s}(n)$. For each $n \in \mathbf{N}_*$, we call an element $\boldsymbol{s} \in \hat{S}_{\mathbf{J}}^{\mathbf{N}_n}$ a *reduced word* of $(\hat{W}_{\mathbf{J}}, \hat{S}_{\mathbf{J}})$ if $\ell_{\mathbf{J}}([\boldsymbol{s}|_{p-1}]) \leq \ell_{\mathbf{J}}([\boldsymbol{s}|_p])$ for all $p \in \mathbf{N}_n$. Here, $\ell_{\mathbf{J}} : \hat{W}_{\mathbf{J}} \to \mathbf{Z}_+$ is the extended length function.

For each reduced word $s = (s(p))_{p \in \mathbf{N}_n}$ of $(\hat{W}_{\mathbf{J}}, \hat{S}_{\mathbf{J}})$ with $n \in \mathbf{N}_*$, we set $s^{-1}(S_{\mathbf{J}}) := \{p \in \mathbf{N}_n \mid s(p) \in S_{\mathbf{J}}\}$ and $\ell_{\mathbf{J}}(s) := \# s^{-1}(S_{\mathbf{J}})$, and call the non-negative integer $\ell_{\mathbf{J}}(s)$ *the length of* s. We denote by $\hat{\mathscr{W}}_{\mathbf{J}}^n$ the set of all reduced words with length n and set $\hat{\mathscr{W}}_{\mathbf{J}} := \coprod_{n \in \mathbf{N}} \hat{\mathscr{W}}_{\mathbf{J}}^n$ and $\hat{\mathscr{W}}_{\mathbf{J}}^* := \hat{\mathscr{W}}_{\mathbf{J}} \amalg \hat{\mathscr{W}}_{\mathbf{J}}^{\infty}$. We call an element of $\hat{\mathscr{W}}_{\mathbf{J}}$ (resp. $\hat{\mathscr{W}}_{\mathbf{J}}^{\infty}$) *a finite reduced word* (resp. *an infinite reduced word*) of $(\hat{W}_{\mathbf{J}}, \hat{S}_{\mathbf{J}})$. For each $n \in \mathbf{N}_*$, we denote by $\mathscr{W}_{\mathbf{J}}^n$ the subset of $\hat{\mathscr{W}}_{\mathbf{J}}^n$ which consists of elements $s \in \hat{\mathscr{W}}_{\mathbf{J}}^n$ such that $s(p) \in S_{\mathbf{J}}$ for all $p \in \mathbf{N}_n$, and call an element $s \in \mathscr{W}_{\mathbf{J}}^n$ a reduced word of $(W_{\mathbf{J}}, S_{\mathbf{J}})$. We set $\mathscr{W}_{\mathbf{J}} := \coprod_{n \in \mathbf{N}} \mathscr{W}_{\mathbf{J}}^n$ and $\mathscr{W}_{\mathbf{J}}^* := \mathscr{W}_{\mathbf{J}} \amalg \mathscr{W}_{\mathbf{J}}^{\infty}$, and call an element of $\mathscr{W}_{\mathbf{J}}$ (resp. $\mathscr{W}_{\mathbf{J}}^{\infty}$) a finite reduced word (resp. an infinite reduced word) of $(W_{\mathbf{J}}, S_{\mathbf{J}})$.

For each reduced word $s \in \hat{\mathscr{W}}_{\mathbf{J}}^*$, an injective mapping $\phi_s : \mathbf{N}_{\ell(s)} \to \Delta_{\mathbf{J}+}^{re}$ is defined by setting $\phi_s(p) := [s|_{\kappa(p)-1}](\alpha_{s(\kappa(p))})$ for each $p \in \mathbf{N}$, where the κ is a unique strictly increasing function $\kappa : \mathbf{N}_{\ell(s)} \to \mathbf{N}$ such that the image of κ equals to $s^{-1}(S_{\mathbf{J}})$, i.e., $\mathrm{Im}(\kappa) = s^{-1}(S_{\mathbf{J}})$. We denote by $\Phi_{\mathbf{J}}^{\ell(s)}(s)$ the image of the injective mapping ϕ_s. Note that if $\ell(s) < \infty$ then $\Phi_{\mathbf{J}}^{\ell(s)}(s) = \Phi_{\mathbf{J}}([s])$.

For a pair (s, s') of elements of $\hat{\mathscr{W}}_{\mathbf{J}}^{\infty}$, we write $s \sim s'$ if for each $(p, q) \in \mathbf{N}^2$ there exists $(p_0, q_0) \in \mathbf{Z}_{\geq p} \times \mathbf{Z}_{\geq q}$ such that $\ell_{\mathbf{J}}([s|_p]^{-1}[s'|_{p_0}]) = p_0 - p$ and $\ell_{\mathbf{J}}([s'|_q]^{-1}[s|_{q_0}]) = q_0 - q$. Then we see that $\sim$ is an equivalence relation on $\hat{\mathscr{W}}_{\mathbf{J}}^{\infty}$ (cf. [7]). We denote by $\hat{W}_{\mathbf{J}}^{\infty}$ the quotient set of $\hat{\mathscr{W}}_{\mathbf{J}}^{\infty}$ relative to the equivalence relation $\sim$, and by $[s]$ the coset containing $s \in \hat{\mathscr{W}}_{\mathbf{J}}^{\infty}$. Let $W_{\mathbf{J}}^{\infty}$ be the image of $\mathscr{W}_{\mathbf{J}}^{\infty}$ by the canonical mapping $\hat{\mathscr{W}}_{\mathbf{J}}^{\infty} \to \hat{W}_{\mathbf{J}}^{\infty}$. Then we can easily show that $W_{\mathbf{J}}^{\infty} = \hat{W}_{\mathbf{J}}^{\infty}$. Moreover, we see that $s \sim s'$ if and only if $\Phi_{\mathbf{J}}^{\infty}(s) = \Phi_{\mathbf{J}}^{\infty}(s')$ (cf. [7]). Hence we may denote by $\Phi_{\mathbf{J}}^{\infty}([s])$ the set $\Phi_{\mathbf{J}}^{\infty}(s)$.

Therefore we can set $\Phi_{\mathbf{J}}^*([s]) := \Phi_{\mathbf{J}}^{\ell(s)}(s)$ for each $s \in \hat{\mathscr{W}}_{\mathbf{J}}^*$.

In the case where $\mathbf{J} = \mathring{\mathbf{I}}$, we will denote the symbols above more simply by removing $\mathbf{J}$ from them.

Definition 3.1. Let $\preceq$ be a total order on a subset B of $\Delta_{\mathbf{J}+}$. We say that $\preceq$ is a *convex order* on B if it satisfies the following conditions:

$$\mathrm{CO(i)} \quad (\beta, \gamma) \in B^2 \backslash (\Delta_+^{im})^2,\ \beta \prec \gamma,\ \beta + \gamma \in B \Rightarrow \beta \prec \beta + \gamma \prec \gamma;$$

$$\mathrm{CO(ii)} \quad \beta \in B,\ \gamma \in \Delta_{\mathbf{J}+} \backslash B,\ \beta + \gamma \in B \Rightarrow \beta \prec \beta + \gamma.$$

Here we write $\beta \prec \gamma$ if $\beta \preceq \gamma$ and $\beta \neq \gamma$. We denote by $\preceq^{op}$ the total order on B defined by setting $\beta \preceq^{op} \gamma \Leftrightarrow \beta \succeq \gamma$ for each pair $(\beta, \gamma) \in B^2$, and call $\preceq^{op}$ the opposite of $\preceq$. We also say that $\preceq$ is an *opposite convex order* if the opposite $\preceq^{op}$ is a convex order. For subsets C and D of B, we write $C \prec D$ if $c \prec d$ for all pair $(c, d) \in C \times D$.

For each non-empty subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, we set

$$\mathscr{C}_n\mathbf{J} := \{\mathbf{K}_\bullet = (\mathbf{K}_0, \mathbf{K}_1, \ldots, \mathbf{K}_n) \mid \mathbf{J} = \mathbf{K}_0 \supsetneq \mathbf{K}_1 \supsetneq \cdots \supsetneq \mathbf{K}_n = \varnothing\}.$$

We note that if $n > \#\mathbf{J}$ then $\mathscr{C}_n\mathbf{J} = \varnothing$, and set $\mathscr{C}\mathbf{J} := \coprod_{n=1}^{\#\mathbf{J}} \mathscr{C}_n\mathbf{J}$. For each $n \in \mathbf{N}_{\#\mathbf{J}}$ and $\mathbf{K}_\bullet \in \mathscr{C}_n\mathbf{J}$, we set

$$W_{\mathbf{K}_\bullet} := W_{\mathbf{K}_1} \times \cdots \times W_{\mathbf{K}_n}, \qquad \mathscr{W}_{\mathbf{K}_\bullet}^{\infty} := \mathscr{W}_{\mathbf{K}_0}^{\infty} \times \cdots \times \mathscr{W}_{\mathbf{K}_{n-1}}^{\infty}.$$

Denote an element $(y_1, \ldots, y_n) \in W_{\mathbf{K}_\bullet}$ by $y_\bullet$, and an element $(\boldsymbol{s}_0, \ldots, \boldsymbol{s}_{n-1}) \in \mathscr{W}_{\mathbf{K}_\bullet}^{\infty}$ by $\boldsymbol{s}_\bullet$. Note that $W_{\mathbf{K}_n} = \{1\}$ and $y_n = 1$ for each $\mathbf{K}_\bullet \in \mathscr{C}_n\mathbf{J}$ and $y_\bullet \in W_{\mathbf{K}_\bullet}$.

Theorem 3.2 ([8]). *Let* $\mathbf{J}$ *be an arbitrary non-empty subset of* $\mathring{\mathbf{I}}$, *and* w *an arbitrary element of* $\mathring{W}_{\mathbf{J}}$.

(1) *Let* $\preceq_-$ *be an arbitrary convex order on* $\Delta_{\mathbf{J}}(w,-)$, $\preceq_0$ *an arbitrary total order on* Δ_+^{im}, *and* $\preceq_+$ *an arbitrary opposite convex order on* $\Delta_{\mathbf{J}}(w,+)$. *We can define a convex order* $\preceq$ *on* $\Delta_{\mathbf{J}+}$ *by extending* $\preceq_-$, $\preceq_0$, $\preceq_+$ *to* $\Delta_{\mathbf{J}+} = \Delta_{\mathbf{J}}(w,-) \amalg \Delta_+^{im} \amalg \Delta_{\mathbf{J}}(w,+)$ *in such a way that* $\Delta_{\mathbf{J}}(w,-) \prec \Delta_+^{im} \prec \Delta_{\mathbf{J}}(w,+)$. *Moreover, we can obtain every convex order on* $\Delta_{\mathbf{J}+}$ *by applying the procedure above.*

(2) *For each* $n \in \mathbf{N}_{\#\mathbf{J}}$ *and* $\mathbf{K}_\bullet \in \mathscr{C}_n\mathbf{J}$, *there exists* $(y_\bullet, \boldsymbol{s}_\bullet) \in W_{\mathbf{K}_\bullet} \times \mathscr{W}_{\mathbf{K}_\bullet}^{\infty}$ *such that*

$$\Delta_{\mathbf{J}}(w,-) = \coprod_{i=1}^{n} w^{\mathbf{K}_{i-1}} y_{i-1} \Phi_{\mathbf{K}_{i-1}}^{\infty}([\boldsymbol{s}_{i-1}]), \tag{3.1}$$

$$C_i := \coprod_{j=1}^{i} w^{\mathbf{K}_{j-1}} y_{j-1} \Phi_{\mathbf{K}_{j-1}}^{\infty}([\boldsymbol{s}_{j-1}]) \in \mathfrak{B}_{\mathbf{J}}^{\infty} \qquad \text{for each } 1 \leqq i \leqq n, \tag{3.2}$$

where $y_0 := 1$. *Then we can define a convex order* $\preceq$ *on* $\Delta_{\mathbf{J}}(w,-)$ *by applying the following procedure* Steps 1, 2.

Step 1. *For each* $i = 1, \ldots, n$, *define a total order* $\preceq_i$ *on the set* $R_i := w^{\mathbf{K}_{i-1}} y_{i-1} \Phi_{\mathbf{K}_{i-1}}^{\infty}([\boldsymbol{s}_{i-1}])$ *by setting*

$$w^{\mathbf{K}_{i-1}} y_{i-1} \phi_{\boldsymbol{s}_{i-1}}(p) \preceq_i w^{\mathbf{K}_{i-1}} y_{i-1} \phi_{\boldsymbol{s}_{i-1}}(q) \qquad \text{for each } p \leq q.$$

Step 2. *Define* $\preceq$ *by extending* $\preceq_1, \ldots, \preceq_n$ *to* $\Delta_{\mathbf{J}}(w,-) = \coprod_{i=1}^{n} R_i$ *in such a way that* $R_i \prec R_{i'}$ *for each* $i < i'$. *Moreover, we can obtain every convex orders on* $\Delta_{\mathbf{J}}(w,-)$ *by applying the procedure above.*

Remark 3.3. (1) *Theorem* 3.2 *gives a concrete method of constructing all convex orders on* $\Delta_{\mathbf{J}+}$, *since* $\Delta_{\mathbf{J}}(w,+) = \Delta_{\mathbf{J}}(ww_\circ,-)$ *with* $w_\circ$ *the longest element of* $\mathring{W}_{\mathbf{J}}$. (2) *For each* $n \in \mathbf{N}_{\#\mathbf{J}}$, *we call the convex order on* $\Delta_{\mathbf{J}}(w,-)$ *described above that of* n*-row type.*

Definition 3.4. Let us use the notations as in Proposition 2.1 and Lemma 2.3(5). From now on, we often denote the translation t_{ε_j} $(j \in \mathring{\mathbf{I}})$ simply by ε_j if there is no fear of misunderstanding. Let $\mathbf{J}$ be an arbitrary non-empty subset of $\mathring{\mathbf{I}}$. For each $s \in \hat{S}_{\mathbf{J}}$, we define an element $\tilde{s} \in \hat{W}$ by setting

$$\tilde{s} := \begin{cases} (\varepsilon_j)^{\mathbf{J}_c} & \text{if } s = \rho_{\mathbf{J}_c j} \text{ with } c = 1, \ldots, \mathrm{C}(\mathbf{J}) \text{ and } j \in \mathbf{J}_{c*}, \\ s_j & \text{if } s = s_j \text{ with } j \in \mathbf{J}, \\ (\varepsilon_{j_c})^{\mathbf{J}_c} s_{j_c^-} (\varepsilon_{j_c^-})^{\mathbf{J}_c} & \text{if } s = s_{\delta - \theta_{\mathbf{J}_c}} \text{ with } c = 1, \ldots, \mathrm{C}(\mathbf{J}), \end{cases} \tag{3.3}$$

where we fix an element $j_c \in \mathbf{J}_{c*}$ for each $c = 1, \dots, \mathrm{C}(\mathbf{J})$. For each $\rho \in \Omega_{\mathbf{J}}$, we define an element $\tilde{\rho} \in \hat{W}$ by setting $\tilde{\rho} := \prod_{c=1}^{\mathrm{C}(\mathbf{J})} \widetilde{\rho_c}$, where $\rho = \prod_{c=1}^{\mathrm{C}(\mathbf{J})} \rho_c$ with $\rho_c \in \Omega_{\mathbf{J}_c}$, and if $\rho_c = 1$ then we set $\widetilde{\rho_c} := 1$.

For each $s \in \hat{S}_{\mathbf{J}}$, we fix a finite reduced word $\boldsymbol{s}'_s = (\boldsymbol{s}'_s(p))_{p \in \mathbf{N}_{N_s}} \in \hat{\mathscr{W}}$ such that $[\boldsymbol{s}'_s] = \tilde{s}$, where $\boldsymbol{s}'_s(p) \in \hat{S}$ for all $p \in \mathbf{N}_{N_s}$. For each $\boldsymbol{s} = (\boldsymbol{s}(p))_{p \in \mathbf{N}_n} \in \hat{\mathscr{W}}_{\mathbf{J}}^*$ with $n \in \mathbf{N}_*$, we set $s_p := \boldsymbol{s}(p)$ for each $p \in \mathbf{N}_n$, and define a sequence $\tilde{\boldsymbol{s}} = (\tilde{\boldsymbol{s}}(p))_{p \in \mathbf{N}_{\tilde{n}}} \in \hat{S}^{\mathbf{N}_{\tilde{n}}}$ with $\tilde{n} \in \mathbf{N}_*$ by setting

$$\tilde{\boldsymbol{s}} := \begin{cases} \boldsymbol{s}'_{s_1} \boldsymbol{s}'_{s_2} \dots \boldsymbol{s}'_{s_n} & \text{if } n < \infty, \\ \lim_{p \to \infty} \boldsymbol{s}'_{s_1} \boldsymbol{s}'_{s_2} \dots \boldsymbol{s}'_{s_p} & \text{if } n = \infty, \end{cases} \tag{3.4}$$

where $\tilde{n} := N_{s_1} + N_{s_2} + \cdots + N_{s_n}$ if $n < \infty$, and $\tilde{n} := \infty$ if $n = \infty$. For the definitions of the product $\boldsymbol{s}'_{s_1} \boldsymbol{s}'_{s_2} \dots \boldsymbol{s}'_{s_n}$ and the limit $\lim_{p \to \infty} \boldsymbol{s}'_{s_1} \boldsymbol{s}'_{s_2} \dots \boldsymbol{s}'_{s_p}$, the reader is referred to the paper [7]. Note that for each $p \in \mathbf{N}_n$,

$$\widetilde{[\boldsymbol{s}|_p]} = \widetilde{\boldsymbol{s}(1)} \widetilde{\boldsymbol{s}(2)} \cdots \widetilde{\boldsymbol{s}(p)}. \tag{3.5}$$

Lemma 3.5. (1) *The sequence* $\tilde{\boldsymbol{s}} = (\tilde{\boldsymbol{s}}(p))_{p \in \mathbf{N}_{\tilde{n}}}$ *defined in Definition* 3.4 *is an element of* $\hat{\mathscr{W}}^*$ *such that* $\phi_{\tilde{\boldsymbol{s}}} \circ f = \phi_{\boldsymbol{s}}$ *for some unique strictly increasing function* $f : \mathbf{N}_{\ell(\boldsymbol{s})} \to \mathbf{N}$. *In particular,* $\tilde{\boldsymbol{s}} \in \hat{\mathscr{W}}^\infty$ *if and only if* $\boldsymbol{s} \in \hat{\mathscr{W}}_{\mathbf{J}}^\infty$. *Moreover, the following equalities hold:*

$$\text{(i)} \quad \widetilde{[\boldsymbol{s}|_p]}|_{\mathfrak{h}_{\mathbf{J}}^{*\prime}} = [\boldsymbol{s}|_p], \qquad \text{(ii)} \quad \widetilde{[\boldsymbol{s}|_p]} \Delta^{\mathbf{J}}(1,-) \subset \Delta^{\mathbf{J}}(1,-),$$

$$\text{(iii)} \quad \Phi(\widetilde{[\boldsymbol{s}|_p]}) \cap \Delta_{\mathbf{J}+} = \Phi_{\mathbf{J}}([\boldsymbol{s}|_p]), \qquad \text{(iv)} \quad \Phi(\widetilde{[\boldsymbol{s}|_p]}) \backslash \Phi_{\mathbf{J}}([\boldsymbol{s}|_p]) \subset \Delta^{\mathbf{J}}(1,-),$$

$$\text{(v)} \quad \ell(\widetilde{[\boldsymbol{s}|_p]}) = \sum_{k=1}^{p} \ell(\widetilde{\boldsymbol{s}(k)})$$

for all $p \in \mathbf{N}_n$. *In particular,*

$$\text{(vi)} \quad \Phi^*([\tilde{\boldsymbol{s}}]) \cap \Delta_{\mathbf{J}+} = \Phi_{\mathbf{J}}^*([\boldsymbol{s}]), \qquad \text{(vii)} \quad \Phi^*([\tilde{\boldsymbol{s}}]) \backslash \Phi_{\mathbf{J}}^*([\boldsymbol{s}]) \subset \Delta^{\mathbf{J}}(1,-).$$

(2) *If* $\boldsymbol{s} \in \hat{\mathscr{W}}_{\mathbf{J}}^\infty$, *then* $\tilde{\boldsymbol{s}} \in \hat{\mathscr{W}}^\infty$ *with the following equality:*

$$\Phi^\infty([\tilde{\boldsymbol{s}}]) = \Phi_{\mathbf{J}}^\infty([\boldsymbol{s}]) \amalg \Delta^{\mathbf{J}}(1,-). \tag{3.6}$$

Proof. (1) The assertion (i) follows from Lemma 2.3(5)(iii). By Lemma 2.3(4)(iii), we see that $\widetilde{\boldsymbol{s}(k)} \Delta^{\mathbf{J}}(1,-) \subset \Delta^{\mathbf{J}}(1,-)$ for all $k \in \mathbf{N}_p$. Thus, by (3.5) we get (ii). By Lemma 2.3(4)(ii) and (5)(iv)(v), we have

$$\Phi(\widetilde{\boldsymbol{s}(k)}) \cap \Delta_{\mathbf{J}+} = \Phi_{\mathbf{J}}(\boldsymbol{s}(k)), \qquad \Phi(\widetilde{\boldsymbol{s}(k)}) \backslash \Phi_{\mathbf{J}}(\boldsymbol{s}(k)) \subset \Delta^{\mathbf{J}}(1,-). \tag{3.7}$$

By (i), (ii), and (3.5)(3.7), we have

$$\Phi(\widetilde{[\boldsymbol{s}|_p]}) = \coprod_{k=1}^{p} \widetilde{[\boldsymbol{s}|_{k-1}]} \Phi(\widetilde{\boldsymbol{s}(k)}), \tag{3.8}$$

where $\widetilde{[s|_0]} = 1$. Therefore we see that (v) holds and the sequence $\tilde{s}$ is an element of $\hat{\mathscr{W}}^*$ satisfying (iii) and (iv). It is easy to see that $\Phi^*([\tilde{s}]) = \bigcup_{p \in \mathbf{N}_n} \Phi(\widetilde{[s|_p]})$. Hence (iii) and (iv) imply (vi) and (vii). By (1)(i), (3.8), and the left equality of (3.7), we see that there exists a unique strictly increasing function $f : \mathbf{N}_{\ell(s)} \to \mathbf{N}$ such that $\phi_{\tilde{s}} \circ f = \phi_s$.

(2) By the part (1), the sequence $\tilde{s}$ is an element of $\hat{\mathscr{W}}^\infty$. In the case where $\mathbf{J} = \mathring{\mathbf{I}}$, we see that $\Delta^{\mathbf{J}}(1,-) = \varnothing$ and $\tilde{s} = s$, and hence the equality (3.6) is valid. Suppose that $\mathbf{J}$ is a proper subset of $\mathring{\mathbf{I}}$. By (1)(vi) and (1)(vii), and Theorem 7.4 in [7], we see that the set $\Phi^\infty([\tilde{s}])$ is an infinite real biconvex set such that $\Phi^\infty([\tilde{s}]) \cap \Delta_{\mathbf{J}+} = \Phi^\infty_{\mathbf{J}}([s])$ and the set $\Phi^\infty([\tilde{s}]) \backslash \Phi^\infty_{\mathbf{J}}([s])$ is an infinite subset of $\Delta^{\mathbf{J}}(1,-)$, which implies the equality (3.6) by Theorem 6.7 in [7]. □

Recall that $\mathfrak{B}^*_{\mathbf{J}}$ is the set of all real biconvex sets in $\Delta_{\mathbf{J}+}$ (see [7]). For each $B \in \mathfrak{B}^*_{\mathbf{J}}$, we set $\hat{W}_{\mathbf{J}}(B) := \{y \in \hat{W}_{\mathbf{J}} \mid \Phi_{\mathbf{J}}(y) \subset B\}$ and $W_{\mathbf{J}}(B) := \hat{W}_{\mathbf{J}}(B) \cap W_{\mathbf{J}}$.

Lemma 3.6. (1) *Let B be a real biconvex set in $\Delta_{\mathbf{J}+}$. Then, for each pair $(y_1, y_2) \in \hat{W}_{\mathbf{J}}(B)^2$, there exists an element $y_3 \in \hat{W}_{\mathbf{J}}(B)$ such that $\Phi_{\mathbf{J}}(y_1) \cup \Phi_{\mathbf{J}}(y_2) \subset \Phi_{\mathbf{J}}(y_3)$.*

(2) *Suppose that a subset $Y \subset \hat{W}_{\mathbf{J}}$ satisfies the following condition: For each pair $(y_1, y_2) \in Y^2$, there exists an element $y_3 \in Y$ such that $\Phi_{\mathbf{J}}(y_1) \cup \Phi_{\mathbf{J}}(y_2) \subset \Phi_{\mathbf{J}}(y_3)$. Then the set $\Phi_{\mathbf{J}}(Y) = \bigcup_{y \in Y} \Phi_{\mathbf{J}}(y)$ is a real biconvex set in $\Delta_{\mathbf{J}+}$.*

Proof. (1) By Corollary 7.6 in [7], we have $B = \Phi^*_{\mathbf{J}}([s])$ for some $s \in \mathscr{W}^*_{\mathbf{J}}$, hence $B = \bigcup_{p=1}^{\ell(s)} \Phi_{\mathbf{J}}([s|_p])$. Since $\Phi_{\mathbf{J}}(y_1) \cup \Phi_{\mathbf{J}}(y_2)$ is a finite set and $\Phi_{\mathbf{J}}([s|_p]) \subsetneq \Phi_{\mathbf{J}}([s|_{p'}])$ for $p < p'$, we see that $\Phi_{\mathbf{J}}(y_1) \cup \Phi_{\mathbf{J}}(y_2) \subset \Phi_{\mathbf{J}}([s|_{p_0}])$ for some $p_0 \in \mathbf{N}_{\ell(s)}$.

(2) Suppose that $\beta, \gamma \in \Phi_{\mathbf{J}}(Y)$ satisfy $\beta + \gamma \in \Delta_{\mathbf{J}+}$. By the assumption on Y, we may assume that $\beta, \gamma \in \Phi_{\mathbf{J}}(y)$ for some $y \in Y$. Then $\beta + \gamma \in \Phi_{\mathbf{J}}(y)$, hence $\beta + \gamma \in \Phi_{\mathbf{J}}(Y)$. It is clear that $\Delta_{\mathbf{J}} \backslash \Phi_{\mathbf{J}}(Y) = \bigcap_{y \in Y} \{\Delta_{\mathbf{J}} \backslash \Phi_{\mathbf{J}}(y)\}$. Suppose that $\beta, \gamma \in \Delta_{\mathbf{J}} \backslash \Phi_{\mathbf{J}}(Y)$ satisfy $\beta + \gamma \in \Delta_{\mathbf{J}+}$. Then $\beta, \gamma \in \Delta_{\mathbf{J}} \backslash \Phi_{\mathbf{J}}(y)$ for all $y \in Y$. It follows that $\beta + \gamma \in \Delta_{\mathbf{J}} \backslash \Phi_{\mathbf{J}}(y)$ for all $y \in Y$, hence $\beta + \gamma \in \Delta_{\mathbf{J}} \backslash \Phi_{\mathbf{J}}(Y)$. □

Proposition 3.7. *Let $(\mathbf{J}, u, y)$ be an arbitrary element of $\mathscr{P}$ (see* [7]*). Suppose that $\varepsilon \in \mathring{P}^\vee$ satisfies $(\varepsilon|\alpha_i) > 0$ for all $i \in \mathring{\mathbf{I}} \backslash \mathbf{J}$ and $(\varepsilon|\alpha_j) = 0$ for all $j \in \mathbf{J}$ and that $s \in \mathscr{W}_{\mathbf{J}}$ satisfies $[s] = y$. Then $\nabla(\mathbf{J}, u, y) = \bigcup_{n \geq 0} \Phi(u[\tilde{s}]t^n_\varepsilon)$.*

Proof. Set $B = \bigcup_{n \geq 0} \Phi([\tilde{s}]t^n_\varepsilon)$. By the assumption on ε, we see that $\bigcup_{n \geq 0} \Phi(t^n_\varepsilon) = \Delta^{\mathbf{J}}(1,-)$. Hence, by Lemma 3.5(1)(iii) and Lemma 2.3(2) in [7], we see that $\Phi([\tilde{s}]t^n_\varepsilon) = \Phi([\tilde{s}]) \amalg [\tilde{s}]\Phi(t^n_\varepsilon)$ for all $n \geq 0$. Thus, by Lemma 3.6(2), we see that B is an infinite real biconvex set such that $B = \Phi([\tilde{s}]) \amalg [\tilde{s}]\Delta^{\mathbf{J}}(1,-)$. Hence, by Lemma 3.5(1)(ii)(iii)(iv) we have $B \cap \Delta_{\mathbf{J}+} = \Phi_{\mathbf{J}}(y)$ and $B \backslash \Phi_{\mathbf{J}}(y) \subset \Delta^{\mathbf{J}}(1,-)$. Since $\Delta^{\mathbf{J}}(1,-) \backslash [\tilde{s}]\Delta^{\mathbf{J}}(1,-)$ is a finite set, we see that $\Delta^{\mathbf{J}}(1,-)\backslash B$ is a finite set. By Theorem 6.7 in [7], we get $B = \Phi_{\mathbf{J}}(y) \amalg \Delta^{\mathbf{J}}(1,-)$. Since $u \in \mathring{W}^{\mathbf{J}}$ we see that $\Phi(u[\tilde{s}]t_{\varepsilon}^{n}) = \Phi(u) \amalg u\Phi([\tilde{s}]t_{\varepsilon}^{n})$ for all $n \geq 0$, which implies that

$$\bigcup_{n \geq 0} \Phi(u[\tilde{s}]t_{\varepsilon}^{n}) = \Phi(u) \amalg u \bigcup_{n \geq 0} \Phi([\tilde{s}]t_{\varepsilon}^{n})$$
$$= \Phi(u) \amalg uB = u\Phi_{\mathbf{J}}(y) \amalg \Delta^{\mathbf{J}}(u,-) = \nabla(\mathbf{J}, u, y). \qquad \square$$

Lemma 3.8. *Let* $\mathbf{J}$ *and* $\mathbf{K}$ *be connected subsets of* $\mathring{\mathbf{I}}$ *such that* $\mathbf{K} \subset \mathbf{J}$, *and* k *an element of* $\mathbf{K}_{*}$. *Suppose that* $[s] = t_{\varepsilon_k}$ *with* $s \in \hat{\mathscr{W}}_{\mathbf{J}}$, *and write the elements* $[\tilde{s}]$ *and* t_{ε_k} *of* $\hat{W}$ *uniquely as* $[\tilde{s}] = [\tilde{s}]^{\mathbf{K}}[\tilde{s}]_{\mathbf{K}}$ *and* $t_{\varepsilon_k} = (t_{\varepsilon_k})^{\mathbf{K}}(t_{\varepsilon_k})_{\mathbf{K}}$ *with* $[\tilde{s}]^{\mathbf{K}} \in \hat{W}^{\mathbf{K}}$, $(t_{\varepsilon_k})^{\mathbf{K}} \in \hat{W}^{\mathbf{K}}$, $[\tilde{s}]_{\mathbf{K}} \in \mathring{W}_{\mathbf{K}}$, *and* $(t_{\varepsilon_k})_{\mathbf{K}} \in \mathring{W}_{\mathbf{K}}$. *Then*

$$\text{(i)} \quad [\tilde{s}]^{\mathbf{K}}|_{\mathfrak{h}_{\mathbf{K}}^{*\prime}} = (t_{\varepsilon_k})^{\mathbf{K}}|_{\mathfrak{h}_{\mathbf{K}}^{*\prime}}, \qquad \text{(ii)} \quad [\tilde{s}]_{\mathbf{K}} = (t_{\varepsilon_k})_{\mathbf{K}}.$$

Proof. By Lemma 3.5(1)(i), we have $[\tilde{s}]|_{\mathfrak{h}_{\mathbf{J}}^{*\prime}} = t_{\varepsilon_k}|_{\mathfrak{h}_{\mathbf{J}}^{*\prime}}$, hence $[\tilde{s}]|_{\mathfrak{h}_{\mathbf{K}}^{*\prime}} = t_{\varepsilon_k}|_{\mathfrak{h}_{\mathbf{K}}^{*\prime}}$ since $\mathfrak{h}_{\mathbf{K}}^{*\prime} \subset \mathfrak{h}_{\mathbf{J}}^{*\prime}$. On the other hand, we see that $[\tilde{s}]|_{\mathfrak{h}_{\mathbf{K}}^{*\prime}} = [\tilde{s}]^{\mathbf{K}}|_{\mathfrak{h}_{\mathbf{K}}^{*\prime}}[\tilde{s}]_{\mathbf{K}}$ and $t_{\varepsilon_k}|_{\mathfrak{h}_{\mathbf{K}}^{*\prime}} = (t_{\varepsilon_k})^{\mathbf{K}}|_{\mathfrak{h}_{\mathbf{K}}^{*\prime}}(t_{\varepsilon_k})_{\mathbf{K}}$. Thus the assertions (i)(ii) follow from Lemma 2.2.

4. Notations and preliminary results on U_q

For each $n \in \mathbf{N}$, we define $[n]_t, [n]_t!, (n)_t, (n)_t! \in \mathbf{Z}[t, t^{-1}]$ by setting

$$[n]_t := \frac{t^n - t^{-n}}{t - t^{-1}}, \qquad [n]_t! := \prod_{k=1}^{n} [k]_t, \qquad (n)_t := \frac{t^{2n} - 1}{t^2 - 1}, \qquad (n)_t! := \prod_{k=1}^{n} (k)_t,$$

and set $[0]_t = (0)_t = [0]_t! = (0)_t! := 1$.

We assume that q is an indeterminate over $\mathbf{Q}$. Let $\mathbf{Q}(q)$ be the field of rational functions of q with coefficients in $\mathbf{Q}$. Let P be the weight lattice of $\mathfrak{g}$, i.e., $P = \{\lambda \in \mathfrak{h}^* \mid \langle \alpha_i^\vee, \lambda \rangle \in \mathbf{Z} \ (\forall i \in \mathbf{I})\}$, and Q the root lattice of $\mathfrak{g}$. Let $U = U_q(\mathfrak{g})$ be the quantized enveloping algebra over $\mathbf{Q}(q)$ of the untwisted affine Lie algebra $\mathfrak{g}$ of type $X_r^{(1)}$, that is, the associative $\mathbf{Q}(q)$-algebra U with the unit 1 defined by the generators $\{E_i, F_i \mid i \in \mathbf{I}\} \amalg \{K_\lambda \mid \lambda \in P\}$ and the following relations:

$$K_\lambda K_\mu = K_{\lambda+\mu}, \qquad K_0 = 1,$$

$$K_\lambda E_i K_\lambda^{-1} = q^{(\alpha_i|\lambda)} E_i, \qquad K_\lambda F_i K_\lambda^{-1} = q^{-(\alpha_i|\lambda)} F_i,$$

$$[E_i, F_j] = \delta_{ij}(K_i - K_i^{-1})/(q_i - q_i^{-1}),$$

$$\sum_{k=0}^{1-\mathrm{A}_{ij}} (-1)^k X_i^{(1-\mathrm{A}_{ij}-k)} X_j X_i^{(k)} = 0 \qquad (\text{with } i \neq j \text{ for each } X = E, F),$$

where $q_i := q^{d_i}$, $K_i := K_{\alpha_i}$, and $X_i^{(k)} = X_i^k/[k]_{q_i}!$. The last relations are called the quantum Serre relations. Let U' be the $\mathbf{Q}(q)$-subalgebra of U generated by $\{E_i, F_i, K_i^{\pm 1} \mid i \in \mathbf{I}\}$, U^+ the $\mathbf{Q}(q)$-subalgebra of U generated by $\{E_i \mid i \in \mathbf{I}\}$, U^- the $\mathbf{Q}(q)$-subalgebra of U generated by $\{F_i \mid i \in \mathbf{I}\}$, and U^0 the $\mathbf{Q}(q)$-subalgebra of U generated by $\{K_\lambda \mid \lambda \in P\}$. The multiplication $x \otimes y \otimes z \mapsto xyz$ defines the following isomorphism of $\mathbf{Q}(q)$-vector spaces: $U^+ \otimes U^0 \otimes U^- \xrightarrow{\sim} U$, which is called the triangular decomposition. Let $U^{\geq 0}$ and $U^{\leq 0}$ be the images of $U^+ \otimes U^0$ and $U^0 \otimes U^-$ by the triangular decomposition, respectively. Let $\Omega : U \to U$ be the $\mathbf{Q}$-algebra anti-automorphism such that $\Omega(E_i) = F_i$, $\Omega(F_i) = E_i$, $\Omega(K_\lambda) = K_\lambda^{-1}$, and $\Omega(q) = q^{-1}$. Let $\Psi : U \to U$ be the $\mathbf{Q}(q)$-algebra anti-automorphism such that $\Psi(E_i) = E_i$, $\Psi(F_i) = F_i$, and $\Psi(K_\lambda) = K_\lambda^{-1}$.

For each $\mu \in Q$, let U_μ be the weight space of U with weight μ. Then $U = \bigoplus_{\mu \in Q} U_\mu$. We call a non-zero element u of U_μ a weight vector with weight μ and set $\mathrm{wt}(u) := \mu$. If a subspace V of U is stable under the conjugate action of U^0 on U, then $V = \bigoplus_{\mu \in Q} V_\mu$, where $V_\mu := U_\mu \cap V$. For each $\mu, \nu \in Q$, $u \in U_\mu$, $v \in U_\nu$, we set $[u, v]_q := uv - q^{(\mu|\nu)} vu$, and define a $\mathbf{Q}(q)$-bilinear mapping $[\,,]_q : U \times U \to U$ by setting $(x, y) \mapsto [x, y]_q := \sum_{\mu, \nu \in Q} [x_\mu, y_\nu]_q$, where $x = \sum_{\mu \in Q} x_\mu$ $(x_\mu \in U_\mu)$, $y = \sum_{\nu \in Q} y_\nu$ $(y_\nu \in U_\nu)$. The mapping $[\,,]_q$ is called the q-commutator or the q-bracket. For each $x \in U$, we define a $\mathbf{Q}(q)$-linear mapping $ad_q x : U \to U$ by setting $(ad_q x).y := [x, y]_q$. For each $\alpha, \beta \in \Delta$ $(\alpha \neq \beta)$, $x \in U_\alpha$, $y \in U_\beta$, and $n \in \mathbf{Z}_+$, we see that $\frac{1}{[n]_{q_\alpha}!}(ad_q x)^n.y = \sum_{k=0}^n (-1)^k q_\alpha^{k(n-1+\mathrm{A}_{\alpha\beta})} x^{(n-k)} y x^{(k)}$, where $\mathrm{A}_{\alpha\beta} := \frac{2(\alpha|\beta)}{(\alpha|\alpha)} \in \mathbf{Z}$, $q_\alpha := q^{(\alpha|\alpha)/2}$, and $x^{(k)} := x^k/[k]_{q_\alpha}!$. In addition, we set $(ad_q x)^{(n)}.y := \frac{1}{[n]_{q_\alpha}!}(ad_q x)^n.y$. Then the quantum Serre relations can be written as $(ad_q E_i)^{(1-\mathrm{A}_{ij})}.E_j = (ad_q F_i)^{(1-\mathrm{A}_{ij})}.F_j = 0$ with $i \neq j$.

The braid group $\mathscr{B}_W = \langle T_i \mid i \in \mathbf{I} \rangle$ associated with the Weyl group W acts on U as a group of $\mathbf{Q}(q)$-algebra automorphisms of U via

$$T_i(E_i) = -F_i K_i, \qquad T_i(E_j) = (ad_q E_i)^{(-\mathrm{A}_{ij})}.E_j \qquad (i \neq j), \tag{4.1}$$

$$T_i(F_i) = -K_i^{-1} E_i, \qquad T_i(F_j) = \Omega(T_i(E_j)) \qquad (i \neq j), \tag{4.2}$$

$$T_i(K_\lambda) = K_{s_i(\lambda)} = K_\lambda K_i^{-\langle \alpha_i^\vee, \lambda \rangle}, \tag{4.3}$$

where $i, j \in \mathbf{I}$, $\lambda \in P$ (cf. [14]). For each $x \in W$, we set $T_x := T_{i_1} T_{i_2} \dots T_{i_n}$, where $x = s_{i_1} s_{i_2} \dots s_{i_n}$ with $n = \ell(x)$ and $i_1, i_2, \dots, i_n \in \mathbf{I}$ is a reduced expression of x. The automorphism T_x does not depend on the reduced expressions.

Let $\mathscr{A}_1$ be the localization of the polynomial ring $\mathbf{Q}[q]$ at the maximal ideal $(q-1)$, that is, the $\mathbf{Q}$-subalgebra of $\mathbf{Q}(q)$ consisting of elements of $\mathbf{Q}(q)$ which have no pole at $q = 1$. For each $\mathscr{A}_1$-module M, we can define a vector space ${}_1M$ over $\mathbf{Q}$ by setting ${}_1M := \mathbf{Q} \otimes_{\mathscr{A}_1} M$, where $\mathbf{Q}$ is regarded as an

$\mathscr{A}_1$-algebra via $q \mapsto 1$, and call the canonical mapping $M \to {}_1M$ *the specialization at* $q = 1$. We note that ${}_1M \simeq M/\{(q-1)M\}$, and denote by $\overline{m}$ the image of $m \in M$ under the specialization at $q = 1$.

Let ${}_{\mathscr{A}_1}U'$ be the $\mathscr{A}_1$-subalgebra of U' generated by $\{E_i, F_i, K_i^{\pm 1} \mid i \in \mathbf{I}\}$, and ${}_{\mathscr{A}_1}U^+$ the $\mathscr{A}_1$-subalgebra of U' generated by $\{E_i \mid i \in \mathbf{I}\}$. Note that ${}_{\mathscr{A}_1}U'$ is stable under the action of $\mathscr{B}_W$ on U. Set ${}_{\mathscr{A}_1}U^+_\mu := {}_{\mathscr{A}_1}U^+ \cap U^+_\mu$ for each $\mu \in Q_+$. Then ${}_{\mathscr{A}_1}U^+ = \bigoplus_{\mu \in Q_+} {}_{\mathscr{A}_1}U^+_\mu$. We denote simply by ${}_1U^+$ and ${}_1U^+_\mu$ the image of ${}_{\mathscr{A}_1}U^+$ and ${}_{\mathscr{A}_1}U^+_\mu$ under the specialization at $q = 1$, respectively. Since ${}_{\mathscr{A}_1}U^+_\mu$ is a finitely generated $\mathscr{A}_1$-module without torsion and $\mathscr{A}_1$ is a principal ideal domain, we see that ${}_{\mathscr{A}_1}U^+_\mu$ is a free $\mathscr{A}_1$-module of finite rank.

Define sets $\tilde{\Delta}^{im}_+$ and $\tilde{\Delta}_+$ by setting $\tilde{\Delta}^{im}_+ := \{(m\delta, i) \mid m \in \mathbf{N}, i = 1, \dots, r\}$ and $\tilde{\Delta}_+ := \Delta^{re}_+ \amalg \tilde{\Delta}^{im}_+$. Set $\mathbf{Z}_+ := \mathbf{Z}_{\geq 0}$ and define $\kappa : Q_+ \to \mathbf{N}$ by setting

$$\kappa(\mu) := \#\left\{\mathbf{c} : \tilde{\Delta}_+ \to \mathbf{Z}_+ \mid \sum\nolimits_{\alpha \in \Delta^{re}_+} \mathbf{c}(\alpha)\alpha + \sum\nolimits_{m=1}^{\infty} \sum\nolimits_{i=1}^{r} \mathbf{c}((m\delta, i))m\delta = \mu\right\}.$$

Proposition 4.1 ([5], [14]). *The* $\mathbf{Q}$*-algebra* ${}_1U^+$ *is characterized as the associative* $\mathbf{Q}$*-algebra with the unit* $\overline{1}$ *defined by the generators* $\{\overline{E}_i \mid i \in \mathbf{I}\}$ *and the following relations:* $\sum_{k=0}^{1-\mathrm{A}_{ij}} (-1)^k \overline{E}_i^{(1-\mathrm{A}_{ij}-k)} \overline{E}_j \overline{E}_i^{(k)} = 0$ *with* $i \neq j$*, where* $\overline{E}_i^{(k)} = \overline{E}_i^k / k!$*. Moreover, for each* $\mu \in Q_+$*, the following equalities hold:* $\dim_{\mathbf{Q}} {}_1U^+_\mu = \dim_{\mathbf{Q}(q)} U^+_\mu = \mathrm{rank}_{\mathscr{A}_1}({}_{\mathscr{A}_1}U^+_\mu) = \kappa(\mu)$.

Lemma 4.2. *Let* V *be a vector space over* $\mathbf{Q}(q)$*,* W *a submodule of* V *over* $\mathscr{A}_1$*, and* $X = \{x_\lambda \mid \lambda \in \Lambda\}$ *a subset of* W *with* Λ *an index set. Suppose that the elements of* $\{\overline{x_\lambda} \mid \lambda \in \Lambda\}$ *are linearly independent over* $\mathbf{Q}$*. Then the elements of* X *are linearly independent over* $\mathbf{Q}(q)$*. Here,* $\overline{x_\lambda}$ *is the image of* x_λ *under the specialization at* $q = 1$*. Moreover, if, in addition, the subset* X *is a basis of* V*, then* X *is a basis of* W *over* $\mathscr{A}_1$.

Proof. Suppose that $\sum_{\lambda \in \mathrm{L}} k_\lambda x_\lambda = 0$ for some finite subset $\mathrm{L} \subset \Lambda$ with $k_\lambda \in \mathbf{Q}(q)^\times$. Multiplying by a power of $(q-1)$, we may further assume that $k_\lambda \in \mathscr{A}_1$ for all $\lambda \in \mathrm{L}$. Set $n := \max\{m \geq 0 \mid k_\lambda/(q-1)^m \in \mathscr{A}_1 \text{ for all } \lambda \in \mathrm{L}\}$. Then there exists an element $\lambda_* \in \mathrm{L}$ such that $k_{\lambda_*}/(q-1)^n \in \mathscr{A}_1 \backslash (q-1)\mathscr{A}_1$. Hence the equality $\sum_{\lambda \in \mathrm{L}} \overline{k_\lambda/(q-1)^n}\,\overline{x_\lambda} = 0$ holds in ${}_1W$ with $\overline{k_{\lambda_*}/(q-1)^n} \neq 0$. This contradicts the assumption.

Let us prove the second assertion. Let w be an arbitrary non-zero element of W. Then $w = \sum_{\lambda \in \mathrm{M}} c_\lambda x_\lambda$ for some finite subset $\mathrm{M} \subset \Lambda$ with $c_\lambda \in \mathbf{Q}(q)^\times$. Now we set $p := \min\{m \geq 0 \mid c_\lambda(q-1)^m \in \mathscr{A}_1 \text{ for all } \lambda \in \mathrm{M}\}$. We now assume that $p > 0$. Then there exists an element $\lambda_\# \in \mathrm{M}$ such that $c_{\lambda_\#}(q-1)^p \in \mathscr{A}_1 \backslash (q-1)\mathscr{A}_1$. Hence the equality $0 = \sum_{\lambda \in \mathrm{L}} \overline{c_\lambda(q-1)^p}\,\overline{x_\lambda}$ holds in ${}_1W$ with $\overline{c_{\lambda_\#}(q-1)^p} \neq 0$. This contradicts the assumption. Thus we get $p = 0$. Therefore, all c_λ with $\lambda \in \mathrm{M}$ are non-zero elements of $\mathscr{A}_1$. $\square$

Definition 4.3. For each $\boldsymbol{s} \in \mathscr{W}^*$ and $p \in \mathbf{N}_{\ell(\boldsymbol{s})}$, we define a weight vector $E_{\boldsymbol{s},p}$ of U^+ with weight $\phi_{\boldsymbol{s}}(p)$ by setting $E_{\boldsymbol{s},p} := T_{\boldsymbol{s}(1)} T_{\boldsymbol{s}(2)} \cdots T_{\boldsymbol{s}(p-1)}(E_{\boldsymbol{s}(p)})$. If $\phi_{\boldsymbol{s}}(p) = \beta$, we denote $E_{\boldsymbol{s},p}$ by $E_{\boldsymbol{s},\beta}$.

Lemma 4.4. (1) *Let β be an element of Δ_+^{re}, and $\boldsymbol{s}$ an element of $\mathscr{W}^*$ such that $\beta = \phi_{\boldsymbol{s}}(p)$ for some $p \in \mathbf{N}_{\ell(\boldsymbol{s})}$. Then $E_{\boldsymbol{s},\beta}$ belongs to ${}_{\mathscr{A}_1}U_\beta^+ \backslash (q-1)_{\mathscr{A}_1}U^+$. In particular, the image $\overline{E_{\boldsymbol{s},\beta}}$ of $E_{\boldsymbol{s},\beta}$ by the specialization at $q=1$ is a non-zero element of ${}_1U_\beta^+$.*

(2) *Let β be an element of Δ_+^{re}, and x an element of W such that $\beta \in \Phi(x)$. We assume that $E_{\boldsymbol{s}_1,\beta} \in \mathbf{Q}(q)^\times E_{\boldsymbol{s}_2,\beta}$ for all $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathscr{W}$ satisfying $[\boldsymbol{s}_1] = [\boldsymbol{s}_2] = x$. Then $E_{\boldsymbol{s}_1,\beta} = E_{\boldsymbol{s}_2,\beta}$ for all $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathscr{W}$ satisfying $[\boldsymbol{s}_1] = [\boldsymbol{s}_2] = x$.*

(3) *Let β be an element of Δ_+^{re}, and x an element of W such that $\beta \in \Phi(x)$. We assume that if $\beta = \sum_{\gamma \in \Phi(x)} \mathbf{c}(\gamma)\gamma$ with $\mathbf{c}(\gamma) \in \mathbf{Z}_+$ for all $\gamma \in \Phi(x)$ then $\mathbf{c}(\beta) = 1$ and $\mathbf{c}(\gamma) = 0$ for all $\gamma \neq \beta$. Then $E_{\boldsymbol{s}_1,\beta} = E_{\boldsymbol{s}_2,\beta}$ for all $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathscr{W}$ satisfying $[\boldsymbol{s}_1] = [\boldsymbol{s}_2] = x$.*

Proof. (1) By (4.1)–(4.3) and the equality $T_i^{-1} = \Psi T_i \Psi$, it is easy to see that $T_i({}_{\mathscr{A}_1}U') = {}_{\mathscr{A}_1}U'$, and hence $T_i((q-1)_{\mathscr{A}_1}U') = (q-1)_{\mathscr{A}_1}U'$ for all $i \in \mathbf{I}$. Thus we see that $T_i({}_{\mathscr{A}_1}U' \backslash (q-1)_{\mathscr{A}_1}U') = {}_{\mathscr{A}_1}U' \backslash (q-1)_{\mathscr{A}_1}U'$ for all $i \in \mathbf{I}$. Then, by Definition 4.3, we see that $E_{\boldsymbol{s},\beta} = E_{\boldsymbol{s},p} \in {}_{\mathscr{A}_1}U_\beta^+ \backslash (q-1)_{\mathscr{A}_1}U^+$, since $E_{\boldsymbol{s}(p)} \in {}_{\mathscr{A}_1}U^+ \backslash (q-1)_{\mathscr{A}_1}U^+$ and $E_{\boldsymbol{s},\beta} \in U_\beta^+$.

(2) Put $l = \ell(x)$. Let (p_1, p_2) be the unique pair of elements of $\mathbf{N}_l$ such that $\phi_{\boldsymbol{s}_i}(p_i) = \beta$ for $i = 1, 2$. To prove the assertion, it suffices to show the equality $E_{\boldsymbol{s}_1,p_1} = E_{\boldsymbol{s}_2,p_2}$. Since $\boldsymbol{s}_1$ can be transformed to $\boldsymbol{s}_2$ by a finite sequence of braid relations, we may assume that $\boldsymbol{s}_1$ can be transformed to $\boldsymbol{s}_2$ by one of the following (i)(ii)(iii)(iv).

(i): replacing two consecutive entries (s_i, s_j) in $\boldsymbol{s}_1$ by (s_j, s_i) when $\mathrm{A}_{ij}\mathrm{A}_{ji} = 0$;

(ii): replacing three consecutive entries (s_i, s_j, s_i) in $\boldsymbol{s}_1$ by (s_j, s_i, s_j) when $\mathrm{A}_{ij}\mathrm{A}_{ji} = 1$;

(iii): replacing four consecutive entries (s_i, s_j, s_i, s_j) in $\boldsymbol{s}_1$ by (s_j, s_i, s_j, s_i) when $\mathrm{A}_{ij}\mathrm{A}_{ji} = 2$;

(iv): replacing six consecutive entries $(s_i, s_j, s_i, s_j, s_i, s_j)$ in $\boldsymbol{s}_1$ by $(s_j, s_i, s_j, s_i, s_j, s_i)$ when $\mathrm{A}_{ij}\mathrm{A}_{ji} = 3$.

In the case (i), there exists a unique $m_0 \in \mathbf{N}$ such that $\boldsymbol{s}_1(m_0) = s_i$, $\boldsymbol{s}_1(m_0+1) = s_j$, $\boldsymbol{s}_2(m_0) = s_j$, $\boldsymbol{s}_2(m_0+1) = s_i$, and $\boldsymbol{s}_1(m) = \boldsymbol{s}_2(m)$ for all $m \neq m_0$, m_0+1. Suppose that $p_1 < p_2$. Then $p_1 = m_0$ and $p_2 = m_0 + 1$ since $\phi_{\boldsymbol{s}_1}(p_1) = \phi_{\boldsymbol{s}_2}(p_2)$. Thus we get $E_{\boldsymbol{s}_1,p_1} = E_{\boldsymbol{s}_2,p_2}$ since $E_i = T_j(E_i)$. Suppose that $p_1 = p_2$. Then $p_1 = p_2 < m_0$ or $m_0 + 1 < p_1 = p_2$ since $\phi_{\boldsymbol{s}_1}(p_1) = \phi_{\boldsymbol{s}_2}(p_2)$, hence the equality is valid since $T_i T_j = T_j T_i$ and $\boldsymbol{s}_1(m) = \boldsymbol{s}_2(m)$ for all $m \neq m_0$, $m_0 + 1$.

In the case (ii), there exists a unique $m_0 \in \mathbf{N}$ such that $\boldsymbol{s}_1(m_0) = s_i$, $\boldsymbol{s}_1(m_0+1) = s_j$, $\boldsymbol{s}_1(m_0+2) = s_i$, $\boldsymbol{s}_2(m_0) = s_j$, $\boldsymbol{s}_2(m_0+1) = s_i$, $\boldsymbol{s}_2(m_0+2) = s_j$ and $\boldsymbol{s}_1(m) = \boldsymbol{s}_2(m)$ for all $m \neq m_0$, m_0+1, m_0+2. Suppose that $p_1 < p_2$. Then $p_1 = m_0$ and $p_2 = m_0 + 2$, since $\phi_{\boldsymbol{s}_1}(p_1) = \phi_{\boldsymbol{s}_2}(p_2)$. Thus we get $E_{\boldsymbol{s}_1,p_1} = E_{\boldsymbol{s}_2,p_2}$, since $E_i = T_jT_i(E_j)$. Suppose that $p_1 = p_2$. Then there exist three cases (a)–(c) to be considered: (a) $p_1 = p_2 < m_0$, (b) $m_0 + 2 < p_1 = p_2$, (c) $p_1 = p_2 = m_0 + 1$, since $\phi_{\boldsymbol{s}_1}(p_1) = \phi_{\boldsymbol{s}_2}(p_2)$. In the case (a) or (b), the equality is valid since $T_iT_jT_i = T_jT_iT_j$ and $\boldsymbol{s}_1(m) = \boldsymbol{s}_2(m)$ for all $m \neq m_0$, m_0+1, m_0+2. In the case (c), $E_{\boldsymbol{s}_1,p_1}$ and $E_{\boldsymbol{s}_2,p_2}$ are not proportional since $T_i(E_j)$ and $T_j(E_i)$ are not proportional, which contradicts the assumption of (2). Therefore the assertion is valid in the case (ii). The arguments for the cases (iii) and (iv) are similar to that for the case (ii).

(3) Put $l = \ell(x)$. Let (p_1, p_2) be the unique pair of elements of $\mathbf{N}_l$ such that $\phi_{\boldsymbol{s}_i}(p_i) = \beta$ for $i = 1,2$. Then $E_{\boldsymbol{s}_i,\beta} = E_{\boldsymbol{s}_i,p_i}$ for $i = 1,2$. By Proposition 40.2.1 in [14], we see that

$$E_{\boldsymbol{s}_1,p_1} = \sum\nolimits_{(c_1,c_2,\ldots,c_l) \in (\mathbf{Z}_+)^l} k_{(c_1,c_2,\ldots,c_l)} E_{\boldsymbol{s}_2,1}^{c_1} E_{\boldsymbol{s}_2,2}^{c_2} \cdots E_{\boldsymbol{s}_2,l}^{c_l},$$

where $k_{(c_1,c_2,\ldots,c_l)} \in \mathbf{Q}(q)$. Now suppose that $(c_1, c_2, \ldots, c_l)$ is a sequence such that $k_{(c_1,c_2,\ldots,c_l)} \neq 0$. Then $\sum_{p=1}^{l} c_p \phi_{\boldsymbol{s}_2}(p) = \beta$. By the assumption, we see that $c_{p_2} = 1$ and $c_p = 0$ for all $p \neq p_2$. Thus $E_{\boldsymbol{s}_1,p_1} = kE_{\boldsymbol{s}_2,p_2}$ for some $k \in \mathbf{Q}(q)^{\times}$. By the part (2), we get $E_{\boldsymbol{s}_1,p_1} = E_{\boldsymbol{s}_2,p_2}$, i.e., $E_{\boldsymbol{s}_1,\beta} = E_{\boldsymbol{s}_2,\beta}$. □

5. The subalgebra $U_{\mathbf{J}}$ associated with $\Delta_{\mathbf{J}}$ and the braid group action

Lemma 5.1. *Let ε be an element of $\mathring{\Delta}_+$. If $(\boldsymbol{s}_1, \boldsymbol{s}_2)$ is a pair of elements of $\mathscr{W}$ such that $\delta - \varepsilon \in \Phi([\boldsymbol{s}_i]) \subset \Delta(1,-)$ for $i = 1,2$, then $E_{\boldsymbol{s}_1,\delta-\varepsilon} = E_{\boldsymbol{s}_2,\delta-\varepsilon}$.*

Proof. We may assume that $[\boldsymbol{s}_1] = [\boldsymbol{s}_2]$, and put $x = [\boldsymbol{s}_1] = [\boldsymbol{s}_2]$. Since $\gamma \in \Delta(1,-)$ for each $\gamma \in \Phi(x)$, there exists $\mathbf{d}(\gamma) \in \mathbf{N}$ such that $\gamma = \mathbf{d}(\gamma)\delta + \bar{\gamma}$ with $\bar{\gamma} \in \mathring{\Delta}_-$. Now suppose that $\delta - \varepsilon = \sum_{\gamma \in \Phi(x)} \mathbf{c}(\gamma)\gamma$ with $\mathbf{c}(\gamma) \in \mathbf{Z}_+$ for all $\gamma \in \Phi(x)$. Then $\delta - \varepsilon = (\sum_{\gamma \in \Phi(x)} \mathbf{c}(\gamma)\mathbf{d}(\gamma))\delta + \sum_{\gamma \in \Phi(x)} \mathbf{c}(\gamma)\bar{\gamma}$, which implies that $\mathbf{c}(\delta - \varepsilon) = \mathbf{d}(\delta - \varepsilon) = 1$ and $\mathbf{c}(\gamma) = 0$ for all $\gamma \neq \delta - \varepsilon$. Thus the assertion follows immediately from Lemma 4.4(3). □

Definition 5.2. For each $\varepsilon \in \mathring{\Delta}_+$, we define a weight vector $E_{\delta-\varepsilon}$ of U^+ with weight $\delta - \varepsilon$ by setting $E_{\delta-\varepsilon} := E_{\boldsymbol{s},\delta-\varepsilon}$, where $\boldsymbol{s}$ is an element of $\mathscr{W}$ such that $\delta - \varepsilon \in \Phi([\boldsymbol{s}]) \subset \Delta(1,-)$. By Lemma 5.1, we see that the vector $E_{\delta-\varepsilon}$ is independent of the choice of $\boldsymbol{s}$.

Definition 5.3. For each non-empty subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, we define subalgebras of U over $\mathbf{Q}(q)$ by setting

$$U_{\mathbf{J}} := \langle E_\alpha, K_\alpha^{\pm 1}, F_\alpha \,|\, \alpha \in \Pi_{\mathbf{J}} \rangle_{\mathbf{Q}(q)\text{-alg}}, \qquad U_{\mathbf{J}}^0 := \langle K_\alpha^{\pm 1} \,|\, \alpha \in \Pi_{\mathbf{J}} \rangle_{\mathbf{Q}(q)\text{-alg}},$$

$$U_{\mathbf{J}}^+ := \langle E_\alpha \,|\, \alpha \in \Pi_{\mathbf{J}} \rangle_{\mathbf{Q}(q)\text{-alg}}, \qquad U_{\mathbf{J}}^{\geq 0} := \langle E_\alpha, K_\alpha^{\pm 1} \,|\, \alpha \in \Pi_{\mathbf{J}} \rangle_{\mathbf{Q}(q)\text{-alg}},$$

$$U_{\mathbf{J}}^- := \langle F_\alpha \,|\, \alpha \in \Pi_{\mathbf{J}} \rangle_{\mathbf{Q}(q)\text{-alg}}, \qquad U_{\mathbf{J}}^{\leq 0} := \langle K_\alpha^{\pm 1}, F_\alpha \,|\, \alpha \in \Pi_{\mathbf{J}} \rangle_{\mathbf{Q}(q)\text{-alg}},$$

where $F_\alpha := \Omega(E_\alpha)$. Note that if $\mathbf{J} = \mathring{\mathbf{I}}$ then $U_{\mathbf{J}} = U'$ and $U_{\mathbf{J}}^{\pm} = U^{\pm}$. Define $\tilde{\Delta}_{\mathbf{J}+}^{im} \subset \tilde{\Delta}_{+}^{im}$ and $\tilde{\Delta}_{\mathbf{J}+} \subset \tilde{\Delta}_{+}$ by setting $\tilde{\Delta}_{\mathbf{J}+}^{im} := \{(m\delta, j) \,|\, m \in \mathbf{N}, j \in \mathbf{J}\}$ and $\tilde{\Delta}_{\mathbf{J}+} := \Delta_{\mathbf{J}+}^{re} \amalg \tilde{\Delta}_{\mathbf{J}+}^{im}$, and define $\kappa_{\mathbf{J}} : Q_{\mathbf{J}+} \to \mathbf{N}$ by setting for each $\mu \in Q_{\mathbf{J}+}$,

$$\kappa_{\mathbf{J}}(\mu) := \#\Big\{\mathbf{c} : \tilde{\Delta}_{\mathbf{J}+} \to \mathbf{Z}_+ \,|\, \textstyle\sum_{\alpha \in \Delta_{\mathbf{J}+}^{re}} \mathbf{c}(\alpha)\alpha + \sum_{m=1}^{\infty} \sum_{j \in \mathbf{J}} \mathbf{c}((m\delta, j)) m\delta = \mu\Big\},$$

where $Q_{\mathbf{J}+} = Q_+ \cap \mathrm{span}_{\mathbf{Z}} \Pi_{\mathbf{J}}$.

Lemma 5.4. *Let $\mathbf{J}$ and $\mathbf{J}'$ be connected subsets of $\mathring{\mathbf{I}}$ which are disjoint from each other, and j an arbitrary element of $\mathbf{J}_*$.*

(1) *Let j^- be the unique element of $\mathbf{J}_*$ such that $\rho_{\mathbf{J}j}(\alpha_{j^-}) = \delta - \theta_{\mathbf{J}}$. Then*

$$X_{\delta-\theta_{\mathbf{J}}} = T_{(\varepsilon_j)^{\mathbf{J}}}(X_{j^-}) \tag{5.1}$$

for each $X = E, K, F$. Here, t_{ε_j} is simply denoted by ε_j. Let $w_\circ$ and $w_{\circ j}$ be the longest element of $\mathring{W}_{\mathbf{J}}$ and $\mathring{W}_{\mathbf{J}\setminus\{j\}}$, respectively, and set $w_{\mathbf{J}j} := w_\circ w_{\circ j}$. Then

$$T_{(\varepsilon_j)^{\mathbf{J}}} = T_{\varepsilon_j} T_{w_{\mathbf{J}j}}^{-1}. \tag{5.2}$$

In particular, $X_{\delta-\alpha_j} = T_{\varepsilon_j} T_j^{-1}(X_j)$ for each $X = E, K, F$.

(2) *For each $i \in \mathring{\mathbf{I}}\setminus\mathbf{J}$, $z \in \mathring{W}_{\mathbf{J}'}$, and $j' \in \mathbf{J}'_*$, we have*

$$\text{(i)} \quad [T_{(\varepsilon_j)^{\mathbf{J}}}, T_{\varepsilon_i}] = 0, \qquad \text{(ii)} \quad [T_{(\varepsilon_j)^{\mathbf{J}}}, T_z] = 0, \qquad \text{(iii)} \quad [T_{(\varepsilon_j)^{\mathbf{J}}}, T_{(\varepsilon_{j'})^{\mathbf{J}'}}] = 0. \tag{5.3}$$

(3) *For each $i \in \mathring{\mathbf{I}}\setminus\mathbf{J}$, $z \in \mathring{W}_{\mathbf{J}'}$, and $(X, Y) \in U_{\mathbf{J}} \times U_{\mathbf{J}'}$, we have*

$$\text{(i)} \quad T_{\varepsilon_i}(X) = X, \qquad \text{(ii)} \quad T_z(X) = X, \qquad \text{(iii)} \quad T_{(\varepsilon_j)^{\mathbf{J}}}(Y) = Y. \tag{5.4}$$

(4) *For each $(X, Y) \in U_{\mathbf{J}} \times U_{\mathbf{J}'}$, we have $[X, Y] = 0$.*

Proof. (1) By Lemma 2.3(1)(i),(5)(i), and Definition 5.2, we get (5.1). The equality (5.2) follows from the following equalities: $\ell((\varepsilon_j)^{\mathbf{J}}) + \ell((\varepsilon_j)_{\mathbf{J}}) = \ell(\varepsilon_j)$ and $(\varepsilon_j)_{\mathbf{J}} = w_{\mathbf{J}j}$. In the case where $\mathbf{J} = \{j\}$, we see that $j^- = j$ and $w_{\mathbf{J}j} = s_j$, and hence $X_{\delta-\alpha_j} = T_{\varepsilon_j} T_j^{-1}(X_j)$ for each $X = E, K, F$ by (5.1)(5.2).

(2) This part follows from Lemma 2.3(2).

(3) Since $t_{\varepsilon_i}(\alpha) = \alpha$, we see that $T_{\varepsilon_i}(X_\alpha) = X_\alpha$ for each $X = E, K, F$ and $\alpha \in \mathring{\Pi}_{\mathbf{J}}$. By (5.1) and (2)(i), we have $T_{\varepsilon_i}(X_{\delta-\theta_{\mathbf{J}}}) = X_{\delta-\theta_{\mathbf{J}}}$ for each $X = E, K, F$, and hence we get (i). Since $z(\alpha) = \alpha$, we see that $T_z(X_\alpha) = X_\alpha$ for each $X = E, K, F$ and $\alpha \in \mathring{\Pi}_{\mathbf{J}}$. By (5.1) and (2)(ii), we have $T_z(X_{\delta-\theta_{\mathbf{J}}}) = X_{\delta-\theta_{\mathbf{J}}}$ for

each $X = E, K, F$, hence we get (ii). The assertion (iii) follows from (i)(ii) and (5.2).

(4) Since $(\alpha|\alpha') = 0$, it is clear that $[E_\alpha, K_{\alpha'}] = [F_\alpha, K_{\alpha'}] = 0$. Let us prove that $[E_\alpha, E_{\alpha'}] = 0$. Suppose that $j \in \mathbf{J}_*$ and $j' \in \mathbf{J}'_*$. In the case where $(\alpha, \alpha') \in \mathring{\Pi}_\mathbf{J} \times \mathring{\Pi}_{\mathbf{J}'}$, it is clear that $[E_\alpha, E_{\alpha'}] = 0$. In the case where $\alpha = \delta - \theta_\mathbf{J}$ and $\alpha' \in \mathring{\Pi}_{\mathbf{J}'}$, we have $[E_\alpha, E_{\alpha'}] = T_{(\varepsilon_j)^\mathbf{J}}([E_{j_*}, E_{\alpha'}]) = 0$ by (5.1) and (3), where j^- is the unique element of $\mathbf{J}$ such that $\rho_{\mathbf{J}j}(\alpha_{j^-}) = \alpha$. In the case where $\alpha = \delta - \theta_\mathbf{J}$ and $\alpha' = \delta - \theta_{\mathbf{J}'}$, we have $[E_\alpha, E_{\alpha'}] = T_{(\varepsilon_j)^\mathbf{J}} T_{(\varepsilon_{j'})^{\mathbf{J}'}}([E_{j^-}, E_{j'^-}]) = 0$ by (5.1) and (3), where j'^- is the unique element of $\mathbf{J}'_*$ such that $\rho_{\mathbf{J}'j'}(\alpha_{j'^-}) = \alpha'$. Similarly, we can prove that $[E_\alpha, F_{\alpha'}] = [F_\alpha, F_{\alpha'}] = 0$. □

Proposition 5.5. *Let* $\mathbf{J}$ *be a non-empty subset of* $\mathring{\mathbf{I}}$, *and* $\mathbf{J}_1, \ldots, \mathbf{J}_{\mathrm{C}(\mathbf{J})}$ *the connected components of* $\mathbf{J}$ *with* $\mathrm{C}(\mathbf{J})$ *the number of the connected components. If* $\mathbf{J}_c$ *and* $\mathbf{J}_{c'}$ *are different connected components of* $\mathbf{J}$, *then* $[X, X'] = 0$ *for all* $(X, X') \in U_{\mathbf{J}_c} \times U_{\mathbf{J}_{c'}}$. *Moreover, the following equality holds:*

$$U_\mathbf{J} = \mathrm{span}_{\mathbf{Q}(q)}\left\{\prod\nolimits_{c=1}^{\mathrm{C}(\mathbf{J})} X_c \mid X_c \in U_{\mathbf{J}_c}\right\}. \tag{5.5}$$

Proof. The first assertion follows from Lemma 5.4(4), and the second assertion follows from the first assertion and Definition 5.3. □

Proposition 5.6. (1) *Let us use the notation introduced in Definition* 3.4. *If* $\rho \in \Omega_\mathbf{J}$ *and* $\alpha \in \Pi_\mathbf{J}$, *then* $T_{\tilde{\rho}}(X_\alpha) = X_{\rho(\alpha)}$ *for each* $X = E, K, F$. *In particular, the restriction* $T_{\tilde{\rho}}|_{U_\mathbf{J}}$ *is an automorphism of* $U_\mathbf{J}$.

(2) *Let* $\mathbf{J}$ *and* $\mathbf{J}'$ *be connected subsets of* $\mathring{\mathbf{I}}$ *which are disjoint from each other. Then* $[T_{\tilde{\tau}}, T_{\tilde{\sigma}}] = 0$ *for all* $(\tau, \sigma) \in \Omega_\mathbf{J} \times \Omega_{\mathbf{J}'}$. *Moreover,* $T_{\tilde{\tau}}(X) = X$ *for all* $\tau \in \Omega_\mathbf{J}$ *and* $X \in U_{\mathbf{J}'}$.

Proof. (1) By Proposition 2.1, we may assume that $\rho = \rho_{\mathbf{J}j}$ with $j \in \mathbf{J}_*$. Then we have $T_{\tilde{\rho}} = T_{(\varepsilon_j)^\mathbf{J}}$ by Definition 3.5. By Lemma 2.3(4)(i)(iii), we have $\ell(\{(\varepsilon_j)^\mathbf{J}\}^2) = 2\ell((\varepsilon_j)^\mathbf{J})$, and hence $T_{\{(\varepsilon_j)^\mathbf{J}\}^2} = (T_{(\varepsilon_j)^\mathbf{J}})^2$. In the case where $\alpha = \delta - \theta_\mathbf{J}$, by (5.1), we see that $T_{\tilde{\rho}}(E_\alpha) = (T_{(\varepsilon_j)^\mathbf{J}})^2(E_{j^-}) = T_{\{(\varepsilon_j)^\mathbf{J}\}^2}(E_{j^-}) = E_{\rho(\alpha)}$ since $\rho(\alpha) = \rho(\delta - \theta_\mathbf{J}) = \{(\varepsilon_j)^\mathbf{J}\}^2(\alpha_{j^-}) = \alpha_j \in \mathring{\Pi}_\mathbf{J}$. In the case where $\alpha = \alpha_{j^-}$, the required equalities are nothing but the equalities in (5.1). In the case where $\alpha \in \mathring{\Pi}_\mathbf{J} \backslash \{\alpha_{j^-}\}$, since $\rho(\alpha) \in \mathring{\Pi}_\mathbf{J}$, the required equalities are clear.

(2) The first assertion follows from Lemma 5.4(2)(iii), and the second assertion follows from Lemma 5.4(3)(iii). □

Proposition 5.7. (1) *Let* $\mathbf{J}$ *be an arbitrary connected subset of* $\mathring{\mathbf{I}}$. *Then the* $\mathbf{Q}(q)$*-subalgebra* $U_\mathbf{J}$ *of* U *is characterized as the associative* $\mathbf{Q}(q)$*-algebra with the unit* 1 *defined by the generators* $\{E_\alpha, K_\alpha^{\pm 1}, F_\alpha \mid \alpha \in \Pi_\mathbf{J}\}$ *and the following relations:*

$$[K_\alpha, K_\beta] = 0, \qquad K_\alpha K_\alpha^{-1} = K_\alpha^{-1} K_\alpha = 1, \tag{5.6}$$

$$K_\alpha E_\beta K_\alpha^{-1} = q^{(\beta|\alpha)} E_\beta, \qquad K_\alpha F_\beta K_\alpha^{-1} = q^{-(\beta|\alpha)} F_\beta, \tag{5.7}$$

$$[E_\alpha, F_\beta] = \delta_{\alpha\beta}(K_\alpha - K_\alpha^{-1})/(q_\alpha - q_\alpha^{-1}), \tag{5.8}$$

$$(ad_q E_\alpha)^{(1-\mathrm{A}_{\alpha\beta})}.E_\beta = (ad_q F_\alpha)^{(1-\mathrm{A}_{\alpha\beta})}.F_\beta = 0 \quad (\alpha \neq \beta), \tag{5.9}$$

where $\alpha, \beta \in \Pi_{\mathbf{J}}$. *Moreover, the following equalities hold:*

$$U_{\mathbf{J}}^+ = \bigoplus\nolimits_{\mu \in Q_{\mathbf{J}+}} U_{\mathbf{J}\mu}^+, \qquad \dim_{\mathbf{Q}(q)} U_{\mathbf{J}\mu}^+ = \kappa_{\mathbf{J}}(\mu). \tag{5.10}$$

(2) *For each non-empty subset* $\mathbf{J} \subset \mathring{\mathbf{I}}$, *the multiplication defines the following isomorphism* m *of* $\mathbf{Q}(q)$*-vector spaces:* $m : U_{\mathbf{J}}^+ \otimes U_{\mathbf{J}}^0 \otimes U_{\mathbf{J}}^- \xrightarrow{\sim} U_{\mathbf{J}}$.

Proof. It is clear that all of the claims in (1) and (2) are valid in the case where $\mathbf{J} = \mathring{\mathbf{I}}$. Hence we may assume that $\mathbf{J}$ is a non-empty proper subset of $\mathring{\mathbf{I}}$. Then we see that the irreducible root system $\mathring{\Delta}_{\mathbf{J}}$ is not of type E_8 or F_4 or G_2, and hence $\#\mathbf{J}_* \geq 1$.

Let $\check{U}_{\mathbf{J}}$ be the associative $\mathbf{Q}(q)$-algebra with the unit 1 defined by the generators $\{\check{E}_\alpha, \check{F}_\alpha, \check{K}_\alpha^{\pm 1} \mid \alpha \in \Pi_{\mathbf{J}}\}$ and the relations (5.6)–(5.9) with X_α replaced by $\check{X}_\alpha$ for $X = E, K^{\pm 1}, F$ with $\alpha \in \Pi_{\mathbf{J}}$. To prove the part (1), it suffices to prove the claim that the assignment $\check{X}_\alpha \mapsto X_\alpha$ for $X = E, K^{\pm 1}, F$ with $\alpha \in \Pi_{\mathbf{J}}$ defines a $\mathbf{Q}(q)$-algebra isomorphism $h_{\mathbf{J}} : \check{U}_{\mathbf{J}} \to U_{\mathbf{J}}$. In the case where $\#\mathbf{J} = 1$, the claim is nothing but that of Proposition 3.8 in [1]. Hence we may assume that $\#\mathbf{J} \geq 2$. To prove the well-definedness of $h_{\mathbf{J}}$, we show that the generators $\{E_\alpha, K_\alpha^{\pm 1}, F_\alpha \mid \alpha \in \Pi_{\mathbf{J}}\}$ of $U_{\mathbf{J}}$ satisfies the relations (5.6)–(5.9). The relations (5.6), (5.7), and (5.8) for $\alpha = \beta$ are clear. Thus it suffices to prove the relations (5.11) for $\alpha \neq \beta$ and (5.9). In the case where $\{\alpha, \beta\} \subset \mathring{\Pi}_{\mathbf{J}}$, the relations (5.8) for $\alpha \neq \beta$ and (5.12) are clear.

Suppose that $\{\alpha, \beta\} = \{\alpha_j, \delta - \theta_{\mathbf{J}}\}$ with $j \in \mathbf{J}_*$ satisfying $\mathrm{ord}(\rho_{\mathbf{J}j}) \geq 3$. Then, there exists an element τ of the cyclic group generated by $\rho_{\mathbf{J}j}$ such that $\tau(\alpha)$ and $\tau(\beta)$ are distinct elements of $\mathring{\Pi}_{\mathbf{J}}$. Since $\mathrm{A}_{\alpha\beta} = \mathrm{A}_{\tau(\alpha)\tau(\beta)}$, it follows from Proposition 5.6 that

$$T_{\tilde{\tau}}([E_\alpha, F_\beta]) = [E_{\tau(\alpha)}, F_{\tau(\beta)}] = 0, \tag{5.11}$$

$$T_{\tilde{\tau}}((ad_q E_\alpha)^{(1-\mathrm{A}_{\alpha\beta})}.E_\beta) = (ad_q E_{\tau(\alpha)})^{(1-\mathrm{A}_{\tau(\alpha)\tau(\beta)})}.E_{\tau(\beta)} = 0, \tag{5.12}$$

$$T_{\tilde{\tau}}((ad_q F_\alpha)^{(1-\mathrm{A}_{\alpha\beta})}.F_\beta) = (ad_q F_{\tau(\alpha)})^{(1-\mathrm{A}_{\tau(\alpha)\tau(\beta)})}.F_{\tau(\beta)} = 0. \tag{5.13}$$

Since $T_{\tilde{\tau}}$ is an automorphism of $U_{\mathbf{J}}$, the equalities (5.11)(5.12)(5.13) imply that the relations (5.8) for $\alpha \neq \beta$ and (5.9) are valid in this case.

Suppose that $\{\alpha,\beta\} = \{\alpha_j, \delta - \theta_{\mathbf{J}}\}$ with $j \in \mathbf{J}_*$ satisfying $(\rho_{\mathbf{J}j})^2 = 1$. By Lemma 2.4 and Definition 5.2, we see that $E_\alpha = T_z(E_i)$, $E_\beta = T_z(E_{i'})$, and $F_\beta = T_z(F_{i'})$ for some $z \in W(\Delta(1,-))$ and distinct elements $i, i' \in \mathbf{I}$. Since $\mathrm{A}_{\alpha\beta} = \mathrm{A}_{ii'}$, it follows that $[E_\alpha, F_\beta] = T_z([E_i, F_{i'}]) = 0$,

$$(ad_q E_\alpha)^{(1-\mathrm{A}_{\alpha\beta})}.E_\beta = T_z((ad_q E_i)^{(1-\mathrm{A}_{ii'})}.E_{i'}) = 0,$$

$$(ad_q F_\alpha)^{(1-\mathrm{A}_{\alpha\beta})}.F_\beta = T_z((ad_q F_i)^{(1-\mathrm{A}_{ii'})}.F_{i'}) = 0.$$

Suppose that $\{\alpha,\beta\} = \{\alpha_{j'}, \delta - \theta_{\mathbf{J}}\}$ with $j' \in \mathbf{J}\backslash\mathbf{J}_*$. By Proposition 2.1 and the first assertion of Lemma 2.3(5), we see that $\rho_{\mathbf{J}j}(\alpha)$ and $\rho_{\mathbf{J}j}(\beta)$ are distinct elements of $\mathring{\Pi}_{\mathbf{J}}$ for each $j \in \mathbf{J}_*$. Put $\rho = \rho_{\mathbf{J}j}$. Since $\mathrm{A}_{\alpha\beta} = \mathrm{A}_{\rho(\alpha)\rho(\beta)}$, it follows from Proposition 5.6 that the equalities (5.11)(5.12)(5.13) hold with τ replaced by ρ. Hence the relations (5.8) for $\alpha \neq \beta$ and (5.9) are valid in this case.

We next prove (2). It is clear that $U_{\mathbf{J}}^+ \subset U^+$, $U_{\mathbf{J}}^0 \subset U^0$, and $U_{\mathbf{J}}^- \subset U^-$, and hence the multiplication mapping m is an injective $\mathbf{Q}(q)$-linear mapping. In the case where $\mathbf{J}$ is connected, by (5.6)–(5.8), we see that m is surjective. In the general case, the surjectivity of m follows from Lemma 5.4(4).

The surjectivity of $h_{\mathbf{J}}$ is clear. We prove the injectivity of $h_{\mathbf{J}}$. Let $\check{U}_{\mathbf{J}}^+$ be the subalgebra of $\check{U}_{\mathbf{J}}$ generated by $\{\check{E}_\alpha \mid \alpha \in \Pi_{\mathbf{J}}\}$, $\check{U}_{\mathbf{J}}^0$ the subalgebra of $\check{U}_{\mathbf{J}}$ generated by $\{\check{K}_\alpha^{\pm 1} \mid \alpha \in \Pi_{\mathbf{J}}\}$, and $\check{U}_{\mathbf{J}}^-$ the subalgebra of $\check{U}_{\mathbf{J}}$ generated by $\{\check{F}_\alpha \mid \alpha \in \Pi_{\mathbf{J}}\}$. Then $h_{\mathbf{J}}(\check{U}_{\mathbf{J}}^+) = (U_{\mathbf{J}}^+)$, $h_{\mathbf{J}}(\check{U}_{\mathbf{J}}^0) = (U_{\mathbf{J}}^0)$, and $h_{\mathbf{J}}(\check{U}_{\mathbf{J}}^-) = (U_{\mathbf{J}}^-)$. Set $h_{\mathbf{J}}^{\pm} := h_{\mathbf{J}}|_{U_{\mathbf{J}}^{\pm}}$ and $h_{\mathbf{J}}^0 := h_{\mathbf{J}}|_{U_{\mathbf{J}}^0}$. Then we see that $h_{\mathbf{J}} \circ \check{m} = m \circ (h_{\mathbf{J}}^+ \otimes h_{\mathbf{J}}^0 \otimes h_{\mathbf{J}}^-)$, where $\check{m}$ is the multiplication mapping $\check{U}_{\mathbf{J}}^+ \otimes \check{U}_{\mathbf{J}}^0 \otimes \check{U}_{\mathbf{J}}^- \to \check{U}_{\mathbf{J}}$. Since both m and $\check{m}$ are isomorphisms of $\mathbf{Q}(q)$-vector spaces, it suffices to show that $\check{U}_{\mathbf{J}}^{\pm} \cap \mathrm{Ker}\, h_{\mathbf{J}} = \{0\}$ and $\check{U}_{\mathbf{J}}^0 \cap \mathrm{Ker}\, h_{\mathbf{J}} = \{0\}$. It is clear that $\check{U}_{\mathbf{J}}^0 \cap \mathrm{Ker}\, h_{\mathbf{J}} = \{0\}$. Now suppose that $u \in \check{U}_{\mathbf{J}}^- \cap \mathrm{Ker}\, h_{\mathbf{J}}$. Let λ be an element of $\mathfrak{h}^*$ such that $2(\alpha|\lambda)/(\alpha|\alpha) = 1$ for all $\alpha \in \Pi_{\mathbf{J}}$. For each $n \in \mathbf{N}$, let $\rho_n : U \to \mathrm{End}(M(n\lambda))$ be the representation of U on the Verma module $M(n\lambda)$ with highest weight $n\lambda$, and v_n a highest weight vector of $M(n\lambda)$. Set $M_n := \rho_n(U_{\mathbf{J}})v_n$. Since $\rho_n(U_{\mathbf{J}}^+)v_n = \{0\}$, we see that $M_n = \rho_n(U_{\mathbf{J}}^-)v_n$ and $U^0 M_n = M_n$. It follows that $M_n = \bigoplus_{\alpha \in Q_{\mathbf{J}+}} (M_n \cap M(n\lambda)_{n\lambda-\alpha})$ and $\dim_{\mathbf{Q}(q)}(M_n \cap M(n\lambda)_{n\lambda}) = 1$, where $M(n\lambda)_{n\lambda-\alpha}$ is the weight space of $M(n\lambda)$ with weight $n\lambda - \alpha$. Therefore we may regard the composition $\rho_n \circ h_{\mathbf{J}}$ as a highest weight representation of $\check{U}_{\mathbf{J}}$ on M_n with highest weight $n\lambda$. Hence there exists a unique irreducible quotient L_n of M_n as $\check{U}_{\mathbf{J}}$-module. Since $u \in \check{U}_{\mathbf{J}}^- \cap \mathrm{Ker}\, h_{\mathbf{J}}$, we see that $uL_n = \{0\}$ for all $n \in \mathbf{N}$. By the assumptions on λ, we see that L_n is an integrable highest weight $\check{U}_{\mathbf{J}}$-module for each $n \in \mathbf{N}$. Thus we get $u \in \bigcap_{n>0}(\sum_{\alpha \in \Pi_{\mathbf{J}}} \check{U}_{\mathbf{J}}^- \check{F}_\alpha^{n+1})$, and hence $u = 0$. Similarly, we have $\check{U}_{\mathbf{J}}^+ \cap \mathrm{Ker}\, h_{\mathbf{J}} = \{0\}$ by considering lowest weight modules.

The equalities (5.10) follow from the characterization of $U_{\mathbf{J}}$ and Proposition 4.1. □

Remark 5.8. (1) *In the case where* $\#\mathbf{J} = 1$, *the characterization of* $U_{\mathbf{J}}$ *described in the part* (1) *of Proposition* 5.7 *is given by J. Beck in* [1].

(2) *We will show that the part* (1) *of Proposition* 5.7 *is still valid in the case where* $\mathbf{J}$ *is an arbitrary non-empty subset of* $\mathring{\mathbf{I}}$ *(see Proposition* 7.1).

Lemma 5.9. *Let* $\mathbf{J}$ *be an arbitrary connected subset of* $\mathring{\mathbf{I}}$. *Then, for each* $j \in \mathbf{J}$, *the following equality holds:*

$$T_j|_{U_{\mathbf{J}}} = h_{\mathbf{J}} \circ \check{T}_j \circ h_{\mathbf{J}}^{-1}, \tag{5.14}$$

where $h_{\mathbf{J}} : \check{U}_{\mathbf{J}} \to U_{\mathbf{J}}$ *is the* $\mathbf{Q}(q)$*-algebra isomorphism introduced in the proof of Proposition* 5.7 *and* $\check{T}_j$ *is Lusztig's automorphism of* $\check{U}_{\mathbf{J}}$.

Proof. We note that the proof is similar to that of Corollary (a) of Proposition 3.8 in [1]. Let M be an arbitrary integrable $U_q(\mathfrak{g})$-module. Then M can be regarded as an integrable $\check{U}_{\mathbf{J}}$-module via $h_{\mathbf{J}}$. Let us denote by T_{jM} the $\mathbf{Q}(q)$-linear isomorphism $T''_{j,1} : M \to M$ introduced in 5.2.1 of [14]. It follows from Proposition 37.1.2 of [14] that

$$\begin{aligned}
&T_j(E_{\delta-\theta_{\mathbf{J}}}).T_{jM}(m)\\
&\quad = T_{jM}(E_{\delta-\theta_{\mathbf{J}}}.m)\\
&\quad = \sum\nolimits_{a,b,c\geq 0;\, -a+b-c=n+\langle \alpha_j^{\vee}, \delta-\theta_{\mathbf{J}}\rangle} (-1)^b q_j^{b-ac} E_j^{(a)} F_j^{(b)} E_j^{(c)} E_{\delta-\theta_{\mathbf{J}}}.m\\
&\quad = h_{\mathbf{J}}\Big(\sum\nolimits_{a,b,c\geq 0;\, -a+b-c=n+\langle \alpha_j^{\vee}, \delta-\theta_{\mathbf{J}}\rangle} (-1)^b q_j^{b-ac} \check{E}_j^{(a)} \check{F}_j^{(b)} \check{E}_j^{(c)} \check{E}_{\delta-\theta_{\mathbf{J}}}\Big).m\\
&\quad = h_{\mathbf{J}}(\check{T}_j(\check{E}_{\delta-\theta_{\mathbf{J}}}) \sum\nolimits_{a,b,c\geq 0;\, -a+b-c=n} (-1)^b q_j^{b-ac} \check{E}_j^{(a)} \check{F}_j^{(b)} \check{E}_j^{(c)}).m\\
&\quad = h_{\mathbf{J}}(\check{T}_j(\check{E}_{\delta-\theta_{\mathbf{J}}})).T_{jM}(m) = h_{\mathbf{J}} \circ \check{T}_j \circ h_{\mathbf{J}}^{-1}(E_{\delta-\theta_{\mathbf{J}}}).T_{jM}(m)
\end{aligned}$$

for all $n \in \mathbf{Z}$ and $m \in M_j^n = \{m \in M \mid K_j.m = q_j^n m\}$. Thus we get $T_j(E_{\delta-\theta_{\mathbf{J}}}) = h_{\mathbf{J}} \circ \check{T}_j \circ h_{\mathbf{J}}^{-1}(E_{\delta-\theta_{\mathbf{J}}})$ by Proposition 3.5.4 of [14]. Similarly, we see that $T_j(u) = h_{\mathbf{J}} \circ \check{T}_j \circ h_{\mathbf{J}}^{-1}(u)$ for $u = E_\alpha, F_\alpha, K_\alpha$ with $\alpha \in \Pi_{\mathbf{J}}$. Hence (5.14) is valid since the both sides are automorphisms of $U_{\mathbf{J}}$. □

Proposition 5.10. *Let us use the notations introduced in Definition* 3.5. *Let* $\mathbf{J}$ *be an arbitrary connected subset of* $\mathring{\mathbf{I}}$. *Then the following equalities hold:*

$$T_{\widetilde{s_\alpha}}(E_\alpha) = -F_\alpha K_\alpha, \qquad T_{\widetilde{s_\alpha}}(E_\beta) = (ad_q E_\alpha)^{(-\mathrm{A}_{\alpha\beta})}.E_\beta \qquad (\alpha \neq \beta), \tag{5.15}$$

$$T_{\widetilde{s_\alpha}}(F_\alpha) = -K_\alpha^{-1} E_\alpha, \qquad T_{\widetilde{s_\alpha}}(F_\beta) = \Omega(T_{\widetilde{s_\alpha}}(E_\beta)) \qquad (\alpha \neq \beta), \tag{5.16}$$

$$T_{\widetilde{s_\alpha}}(K_\beta) = K_{s_\alpha(\beta)} = K_\beta K_\alpha^{-\mathrm{A}_{\alpha\beta}}, \tag{5.17}$$

where $\alpha, \beta \in \Pi_{\mathbf{J}}$. *In particular, the restriction of* $T_{\widetilde{s_\alpha}}|_{U_{\mathbf{J}}}$ *is an automorphism of* $U_{\mathbf{J}}$. *If, in addition,* $\mathbf{J}'$ *is a connected subset of* $\mathring{\mathbf{I}}$ *which is disjoint from* $\mathbf{J}$, *then the following equalities hold:*

$$\text{(i)} \quad T_{\widetilde{s_\alpha}}(X) = X, \qquad \text{(ii)} \quad [T_{\tilde{\tau}}, T_{\widetilde{s_{\alpha'}}}] = 0, \qquad \text{(iii)} \quad [T_{\widetilde{s_\alpha}}, T_{\widetilde{s_{\alpha'}}}] = 0 \tag{5.18}$$

for all $X \in U_{\mathbf{J}'}$, $(\alpha, \alpha') \in \Pi_{\mathbf{J}} \times \Pi_{\mathbf{J}'}$, *and* $\tau \in \Omega_{\mathbf{J}}$.

Proof. By Lemma 2.3(5) and Definition 3.4, we have

$$T_{\widetilde{s_{\delta-\theta_{\mathbf{J}}}}} = T_{(\varepsilon_{j_0})^{\mathbf{J}}} T_{j_0^-} T_{(\varepsilon_{j_0^-})^{\mathbf{J}}}, \tag{5.19}$$

where j_0 and j_0^- are the fixed elements of $\mathbf{J}_*$ such that $\delta - \theta_{\mathbf{J}} = \rho_{\mathbf{J}j_0}(\alpha_{j_0^-})$ and $(\rho_{\mathbf{J}j_0})^{-1} = \rho_{\mathbf{J}j_0^-}$. Let us prove (5.17). In the case where $\alpha = \alpha_j$ with $j \in \mathbf{J}$, the equality is clear since $T_{\widetilde{s_\alpha}} = T_j$. In the case where $\alpha = \delta - \theta_{\mathbf{J}}$, Lemma 2.3(5)(iii) implies (5.17). Let us prove the left equalities of (5.15) and (5.16). In the case where $\alpha = \alpha_j$ with $j \in \mathbf{J}$, the equalities are clear since $T_{\widetilde{s_\alpha}} = T_j$. In the case where $\alpha = \delta - \theta_{\mathbf{J}}$, it follows from (5.1), (5.19), and Proposition 5.6(1) that

$$T_{\widetilde{s_\alpha}}(E_\alpha) = T_{(\varepsilon_{j_0})^{\mathbf{J}}} T_{j_0^-}(E_{j_0^-}) = T_{(\varepsilon_{j_0})^{\mathbf{J}}}(-F_{j_0^-} K_{j_0^-}) = -F_\alpha K_\alpha.$$

Since $\Omega T_{\widetilde{s_\alpha}} = T_{\widetilde{s_\alpha}} \Omega$, we have $T_{\widetilde{s_\alpha}}(F_\alpha) = -K_\alpha^{-1} E_\alpha$.

Let us prove the right equalities of (5.15) and (5.16). Since $F_\beta = \Omega(E_\beta)$, the right equality of (5.16) follows from the right equality of (5.15) and the equality $T_i^{-1} = \Psi T_i \Psi$. Hence it suffices to prove the right equality of (5.15). In the case where $\alpha = \alpha_j$ with $j \in \mathbf{J}$, since $T_{\widetilde{s_\alpha}} = T_j$, it follows from Lemma 5.9 that

$$\begin{aligned} T_{\widetilde{s_\alpha}}(E_\beta) &= T_j(E_\beta) = h_{\mathbf{J}} \circ \check{T}_j \circ h_{\mathbf{J}}^{-1}(E_\beta) = h_{\mathbf{J}} \circ \check{T}_j(\check{E}_\beta) \\ &= h_{\mathbf{J}}((ad_q \check{E}_j)^{(-\mathrm{A}_{\alpha_j\beta})}.\check{E}_\beta) = (ad_q E_j)^{(-\mathrm{A}_{\alpha_j\beta})}.E_\beta = (ad_q E_\alpha)^{(-\mathrm{A}_{\alpha\beta})}.E_\beta. \end{aligned}$$

In the case where $\alpha = \delta - \theta_{\mathbf{J}}$ and $\beta = \alpha_j$ with $j \in \mathbf{J}$, set $\gamma := \rho_{\mathbf{J}j_0}^{-1}(\beta) = \rho_{\mathbf{J}j_0}^{-1}(\alpha_j)$, then we see that $T_{(\varepsilon_{j_0^-})^{\mathbf{J}}}(E_\beta) = E_\gamma$ and $T_{(\varepsilon_{j_0})^{\mathbf{J}}}(E_\gamma) = E_\beta$, and hence

$$\begin{aligned} T_{\widetilde{s_\alpha}}(E_\beta) &= T_{(\varepsilon_{j_0})^{\mathbf{J}}} T_{j_0^-} T_{(\varepsilon_{j_0^-})^{\mathbf{J}}}(E_\beta) = T_{(\varepsilon_{j_0})^{\mathbf{J}}} T_{j_0^-}(E_\gamma) \\ &= T_{(\varepsilon_{j_0})^{\mathbf{J}}}((ad_q E_{j_0^-})^{(-\mathrm{A}_{j_0^-\gamma})}.E_\gamma) = (ad_q E_\alpha)^{(-\mathrm{A}_{\alpha\beta})}.E_\beta, \end{aligned}$$

where $\mathrm{A}_{j_0^-\gamma} = 2(\alpha_{j_0^-}|\gamma)/(\alpha_{j_0^-}|\alpha_{j_0^-})$.

The equality (i) of (5.18) follows from Lemma 5.4(3)(ii)(iii), and the equalities (ii)(iii) of (5.18) follow from Lemma 5.4(2)(ii)(iii). □

Definition 5.11. For each non-empty subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, we define a group $\mathscr{B}_{\hat{W}_{\mathbf{J}}}$, called the braid group associated with $\hat{W}_{\mathbf{J}}$, by the generators $\{{}_{\mathbf{J}}T_{s_\alpha}, {}_{\mathbf{J}}T_\tau \mid \alpha \in \Pi_{\mathbf{J}}, \tau \in \Omega_{\mathbf{J}}\}$ and the following relations:

(i) ${}_{\mathbf{J}}T_{s_\alpha} \cdot {}_{\mathbf{J}}T_{s_\beta} = {}_{\mathbf{J}}T_{s_\beta} \cdot {}_{\mathbf{J}}T_{s_\alpha}$ if $\mathrm{ord}(s_\alpha s_\beta) = 2$,

(ii) ${}_{\mathbf{J}}T_{s_\alpha} \cdot {}_{\mathbf{J}}T_{s_\beta} \cdot {}_{\mathbf{J}}T_{s_\alpha} = {}_{\mathbf{J}}T_{s_\beta} \cdot {}_{\mathbf{J}}T_{s_\alpha} \cdot {}_{\mathbf{J}}T_{s_\beta}$ if $\mathrm{ord}(s_\alpha s_\beta) = 3$,

(iii) $({}_{\mathbf{J}}T_{s_\alpha} \cdot {}_{\mathbf{J}}T_{s_\beta})^2 = ({}_{\mathbf{J}}T_{s_\beta} \cdot {}_{\mathbf{J}}T_{s_\alpha})^2$ if $\mathrm{ord}(s_\alpha s_\beta) = 4$,

(iv) $({}_{\mathbf{J}}T_{s_\alpha} \cdot {}_{\mathbf{J}}T_{s_\beta})^3 = ({}_{\mathbf{J}}T_{s_\beta} \cdot {}_{\mathbf{J}}T_{s_\alpha})^3$ if $\mathrm{ord}(s_\alpha s_\beta) = 6$,

(v) ${}_{\mathbf{J}}T_\tau \cdot {}_{\mathbf{J}}T_{s_\alpha} = {}_{\mathbf{J}}T_{s_{\tau(\alpha)}} \cdot {}_{\mathbf{J}}T_\tau$, (vi) ${}_{\mathbf{J}}T_\tau \cdot {}_{\mathbf{J}}T_{\tau'} = {}_{\mathbf{J}}T_{\tau\tau'}$, (vii) ${}_{\mathbf{J}}T_1 = 1$,

where $\mathrm{ord}(x)$ is the order of x. The braid group $\mathscr{B}_{\hat{W}_{\mathbf{J}}}$ is also defined by the generators $\{{}_{\mathbf{J}}T_x \mid x \in \hat{W}_{\mathbf{J}}\}$ and the following relations:

$${}_{\mathbf{J}}T_x \cdot {}_{\mathbf{J}}T_y = {}_{\mathbf{J}}T_{xy} \qquad \text{if } \ell_{\mathbf{J}}(x) + \ell_{\mathbf{J}}(y) = \ell_{\mathbf{J}}(xy).$$

In the case where $\mathbf{J} = \mathring{\mathbf{I}}$, we can denote ${}_{\mathbf{J}}T_x$ simply by T_x.

Theorem 5.12. *Let us use the notations introduced in Definition* 3.4. *For each non-empty subset* $\mathbf{J} \subset \mathring{\mathbf{I}}$, *the braid group* $\mathscr{B}_{\hat{W}_{\mathbf{J}}}$ *acts on* $U_{\mathbf{J}}$ *as a group of* $\mathbf{Q}(q)$-*algebra automorphisms of* $U_{\mathbf{J}}$ *via*

$${}_{\mathbf{J}}T_s \mapsto T_{\tilde{s}}|_{U_{\mathbf{J}}}, \tag{5.20}$$

where $s \in \hat{S}_{\mathbf{J}}$. *Moreover, the action of* ${}_{\mathbf{J}}T_x$ *on* $U_{\mathbf{J}}$ *is given by*

$${}_{\mathbf{J}}T_x \mapsto T_{[\tilde{\boldsymbol{s}}]}|_{U_{\mathbf{J}}} \tag{5.21}$$

for each $x \in \hat{W}_{\mathbf{J}}$, *where* $\boldsymbol{s}$ *is an element of* $\hat{\mathscr{W}}_{\mathbf{J}}$ *such that* $[\boldsymbol{s}] = x$.

Proof. By direct calculations as in the section 39.2 of Lusztig's book [14] using Proposition 5.5–5.7 and 5.10, we see that the automorphisms $T_{\tilde{s}}|_{U_{\mathbf{J}}}$ satisfy the relations (i)–(vii) of $\mathscr{B}_{\hat{W}_{\mathbf{J}}}$ with ${}_{\mathbf{J}}T_s$ replaced by $T_{\tilde{s}}|_{U_{\mathbf{J}}}$, and hence the assignment (5.20) defines a group homomorphism from $\mathscr{B}_{\hat{W}_{\mathbf{J}}}$ to the automorphism group $\mathrm{Aut}(U_{\mathbf{J}})$.

We next prove (5.21). Denote the sequence $\boldsymbol{s}$ by $\boldsymbol{s} = (\boldsymbol{s}(p))_{p \in \mathbf{N}_n}$ with $n \in \mathbf{N}$. Then

$$x = [\boldsymbol{s}] = \boldsymbol{s}(1)\boldsymbol{s}(2)\cdots\boldsymbol{s}(n), \qquad \ell_{\mathbf{J}}(x) = \ell_{\mathbf{J}}(\boldsymbol{s}(1)) + \ell_{\mathbf{J}}(\boldsymbol{s}(2)) + \cdots + \ell_{\mathbf{J}}(\boldsymbol{s}(n)),$$

hence the following equality in $\mathscr{B}_{\hat{W}_{\mathbf{J}}}$ holds: ${}_{\mathbf{J}}T_x = {}_{\mathbf{J}}T_{\boldsymbol{s}(1)} \cdot {}_{\mathbf{J}}T_{\boldsymbol{s}(2)} \cdots {}_{\mathbf{J}}T_{\boldsymbol{s}(n)}$. By (3.5) and Lemma 3.5(1)(v), we see that

$$[\tilde{\boldsymbol{s}}] = \widetilde{\boldsymbol{s}(1)}\widetilde{\boldsymbol{s}(2)}\cdots\widetilde{\boldsymbol{s}(n)}, \qquad \ell([\tilde{\boldsymbol{s}}]) = \ell(\widetilde{\boldsymbol{s}(1)}) + \ell(\widetilde{\boldsymbol{s}(2)}) + \cdots + \ell(\widetilde{\boldsymbol{s}(n)}),$$

which implies the following equality in $\mathscr{B}_{\hat{W}}$: $T_{[\tilde{\boldsymbol{s}}]} = T_{\widetilde{\boldsymbol{s}(1)}} \cdot T_{\widetilde{\boldsymbol{s}(2)}} \cdots T_{\widetilde{\boldsymbol{s}(n)}}$. Thus we see that $T_{[\tilde{\boldsymbol{s}}]}|_{U_{\mathbf{J}}} = T_{\widetilde{\boldsymbol{s}(1)}}|_{U_{\mathbf{J}}} \cdot T_{\widetilde{\boldsymbol{s}(2)}}|_{U_{\mathbf{J}}} \cdots T_{\widetilde{\boldsymbol{s}(n)}}|_{U_{\mathbf{J}}}$. Therefore the action of ${}_{\mathbf{J}}T_x$ on $U_{\mathbf{J}}$ is given by (5.21). □

Remark 5.13. *Note that* ${}_{\mathbf{J}}T_w(u) = T_w(u)$ *for each* $w \in \mathring{W}_{\mathbf{J}}$ *and* $u \in U_{\mathbf{J}}$ *and that if* $\mathbf{J} = \mathring{\mathbf{I}}$ *then* ${}_{\mathbf{J}}T_x(u) = T_x(u)$ *for each* $x \in \hat{W}_{\mathbf{J}}$ *and* $u \in U_{\mathbf{J}}$. *In Proposition* 5.20, *we will prove that the action of* $\mathscr{B}_{\hat{W}_{\mathbf{J}}}$ *on* $U_{\mathbf{J}}$ *is faithful.*

Lemma 5.14. *Let* $\mathbf{J}$ *be an arbitrary non-empty subset of* $\mathring{\mathbf{I}}$, *and* $\mathbf{K}$ *an arbitrary connected subset of* $\mathbf{J}$.

(1) *The equality* $[{}_{\mathbf{J}}T_x, T_{\varepsilon_i}] = 0$ *in* $\mathrm{Aut}(U_{\mathbf{J}})$ *holds for all* $x \in \hat{W}_{\mathbf{J}}$ *and* $i \in \mathring{\mathbf{I}} \backslash \mathbf{J}$.

(2) *For each* $k \in \mathbf{K}_*$, *we have*

$$E_{\delta-\theta_{\mathbf{K}}} = {}_{\mathbf{J}}T_{\varepsilon_k} T^{-1}_{(\varepsilon_k)_{\mathbf{K}}}(E_{k^-}), \tag{5.22}$$

where k^- *is the unique element of* $\mathbf{K}_*$ *such that* $\rho_{\mathbf{K}k}(\alpha_{k^-}) = \delta - \theta_{\mathbf{K}}$. *In particular,* $E_{\delta-\theta_{\mathbf{K}}} \in U^+_{\mathbf{J}}$. *Moreover,* $U_{\mathbf{K}} \subset U_{\mathbf{J}}$.

Proof. (1) This follows from Lemma 5.4(2)(i) and the equality $[T_j, T_{\varepsilon_i}] = 0$ in $\mathrm{Aut}(U_{\mathbf{J}})$ for all $j \in \mathbf{J}$.

(2) Let $\boldsymbol{s}$ be an element of $\hat{\mathscr{W}}_{\mathbf{J}}$ such that $[\boldsymbol{s}] = t_{\varepsilon_k}$. Then, by Theorem 5.12, we have the following equality in $\mathrm{Aut}(U_{\mathbf{J}})$:

$${}_{\mathbf{J}}T_{\varepsilon_k} = T_{[\tilde{\boldsymbol{s}}]}|_{U_{\mathbf{J}}}. \tag{5.23}$$

By (1)(ii),(5)(i) of Lemma 2.3, and (i) of Lemma 3.8, we have

$$[\tilde{\boldsymbol{s}}]^{\mathbf{K}}(\alpha_{k^-}) = \delta - \theta_{\mathbf{K}} = (\varepsilon_k)^{\mathbf{K}}(\alpha_{k^-}).$$

Since $\Phi(\varepsilon_k) \subset \Delta(1,-)$, we have $\Phi((\varepsilon_k)^{\mathbf{K}}) \subset \Delta(1,-)$. Moreover, by Lemma 3.5(1)(vi)(vii), we have $\Phi([\tilde{\boldsymbol{s}}]) \subset \Delta(1,-)$, hence $\Phi([\tilde{\boldsymbol{s}}]^{\mathbf{K}}) \subset \Delta(1,-)$. Therefore, by Lemma 5.1 and Definition 5.2, we have $T_{[\tilde{\boldsymbol{s}}]^{\mathbf{K}}}(E_{k^-}) = E_{\delta-\theta_{\mathbf{K}}} = T_{(\varepsilon_k)^{\mathbf{K}}}(E_{k^-})$. By Lemma 3.8(ii), we have $T_{[\tilde{\boldsymbol{s}}]} = T_{[\tilde{\boldsymbol{s}}]^{\mathbf{K}}} T_{(\varepsilon_k)_{\mathbf{K}}}$. Since $T^{-1}_{(\varepsilon_k)_{\mathbf{K}}}(E_{k^-}) \in U_{\mathbf{J}}$, by (5.23), we get (5.22) as follows: $E_{\delta-\theta_{\mathbf{K}}} = T_{[\tilde{\boldsymbol{s}}]}|_{U_{\mathbf{J}}} T^{-1}_{(\varepsilon_k)_{\mathbf{K}}}(E_{k^-}) = {}_{\mathbf{J}}T_{\varepsilon_k} T^{-1}_{(\varepsilon_k)_{\mathbf{K}}}(E_{k^-})$.

Proposition 5.15. *For each non-empty subset* $\mathbf{J} \subset \mathring{\mathbf{I}}$ *and* $j \in \mathbf{J}$, *we have*

$${}_{\mathbf{J}}T_{\varepsilon_j} = T_{\varepsilon_j}|_{U_{\mathbf{J}}}. \tag{5.24}$$

Proof. Put $\mathbf{K} = \{j\}$. Then $\theta_{\mathbf{K}} = \alpha_j$, $(\varepsilon_j)_{\mathbf{K}} = s_j$, and $j^- = j$. By Lemma 5.14(2), we have

$$E_{\delta-\alpha_j} = {}_{\mathbf{J}}T_{\varepsilon_j} T_j^{-1}(E_j). \tag{5.25}$$

Suppose that $j \in \mathbf{J}_*$. Then $\varepsilon_j = \rho_{\mathbf{J}j}(\varepsilon_j)_{\mathbf{J}}$, $\ell_{\mathbf{J}}(\varepsilon_j) = \ell_{\mathbf{J}}(\rho_{\mathbf{J}j}) + \ell_{\mathbf{J}}((\varepsilon_j)_{\mathbf{J}})$, and $\widetilde{\rho_{\mathbf{J}j}} = (\varepsilon_j)^{\mathbf{J}}$. Since ${}_{\mathbf{J}}T_{\varepsilon_j} = {}_{\mathbf{J}}T_{\rho_{\mathbf{J}j}} \cdot {}_{\mathbf{J}}T_{(\varepsilon_j)_{\mathbf{J}}}$, we get (5.24) as follows: ${}_{\mathbf{J}}T_{\varepsilon_j} = T_{(\varepsilon_j)^{\mathbf{J}}}|_{U_{\mathbf{J}}} \cdot T_{(\varepsilon_j)_{\mathbf{J}}}|_{U_{\mathbf{J}}} = T_{\varepsilon_j}|_{U_{\mathbf{J}}}$. Suppose that $j \in \mathbf{J} \backslash \mathbf{J}_*$. It suffices to show that

$${}_{\mathbf{J}}T_{\varepsilon_j}(X) = T_{\varepsilon_j}(X) \tag{5.26}$$

for $X = E_\alpha, F_\alpha, K_\alpha^{\pm 1}$ with $\alpha \in \Pi_{\mathbf{J}}$. In the case where $X = K_\alpha^{\pm 1}$, the equality (5.26) is clear. By (5.25), we have ${}_{\mathbf{J}}T_{\varepsilon_j}(F_j) = -K^{-1}_{\delta-\alpha_j} E_{\delta-\alpha_j} = T_{\varepsilon_j}(F_j)$ for each

$j \in \mathbf{J}$. In the case where $j' \in \mathbf{J}\backslash\{j\}$, we have $t_{\varepsilon_j}(\alpha_{j'}) = \alpha_{j'}$, and hence ${}_{\mathbf{J}}T_{\varepsilon_j}(F_{j'}) = F_{j'} = T_{\varepsilon_j}(F_{j'})$. Thus (5.26) holds in the case where $X = F_\alpha$ with $\alpha \in \mathring{\Pi}_{\mathbf{J}}$. Since $[\Omega, {}_{\mathbf{J}}T_{\varepsilon_j}] = 0$, the equality (5.26) holds in the case where $X = E_\alpha$ with $\alpha \in \mathring{\Pi}_{\mathbf{J}}$. Therefore we have ${}_{\mathbf{J}}T_{\varepsilon_j}|_{\mathring{U}_{\mathbf{J}}} = T_{\varepsilon_j}|_{\mathring{U}_{\mathbf{J}}}$, where $\mathring{U}_{\mathbf{J}}$ is the $\mathbf{Q}(q)$-subalgebra of $U_{\mathbf{J}}$ generated by $\{E_\alpha, F_\alpha, K_\alpha^{\pm 1} \mid \alpha \in \mathring{\Pi}_{\mathbf{J}}\}$. We next prove (5.26) in the case where $X = E_{\delta-\theta_{\mathbf{J}}}$. By Lemma 5.14(2), we have $E_{\delta-\theta_{\mathbf{J}}} = {}_{\mathbf{J}}T_{\varepsilon_k}T^{-1}_{(\varepsilon_k)_{\mathbf{J}}}(E_{k^-})$, where $k, k^- \in \mathbf{J}_*$ such that $\rho_{\mathbf{J}k}(\alpha_{k^-}) = \delta - \theta_{\mathbf{J}}$. Since $k \in \mathbf{J}_*$, we have ${}_{\mathbf{J}}T_{\varepsilon_k} = T_{\varepsilon_k}|_{U_{\mathbf{J}}}$. Since $T^{-1}_{(\varepsilon_k)_{\mathbf{J}}}(E_{k^-}) \in \mathring{U}_{\mathbf{J}}$, we have ${}_{\mathbf{J}}T_{\varepsilon_j}T^{-1}_{(\varepsilon_k)_{\mathbf{J}}}(E_{k^-}) = T_{\varepsilon_j}T^{-1}_{(\varepsilon_k)_{\mathbf{J}}}(E_{k^-})$. In addition, it is clear that $[{}_{\mathbf{J}}T_{\varepsilon_j}, {}_{\mathbf{J}}T_{\varepsilon_k}] = 0$. Therefore we see that

$$
\begin{aligned}
{}_{\mathbf{J}}T_{\varepsilon_j}(E_{\delta-\theta_{\mathbf{J}}}) &= {}_{\mathbf{J}}T_{\varepsilon_k} \cdot {}_{\mathbf{J}}T_{\varepsilon_j}T^{-1}_{(\varepsilon_k)_{\mathbf{J}}}(E_{k^-}) = {}_{\mathbf{J}}T_{\varepsilon_k}T_{\varepsilon_j}T^{-1}_{(\varepsilon_k)_{\mathbf{J}}}(E_{k^-}) \\
&= T_{\varepsilon_k}T_{\varepsilon_j}T^{-1}_{(\varepsilon_k)_{\mathbf{J}}}(E_{k^-}) = T_{\varepsilon_j}T_{\varepsilon_k}T^{-1}_{(\varepsilon_k)_{\mathbf{J}}}(E_{k^-}) = T_{\varepsilon_j}(E_{\delta-\theta_{\mathbf{J}}}).
\end{aligned}
$$

The equality (5.26) for $X = F_{\delta-\theta_{\mathbf{J}}}$ also holds, since $[\Omega, {}_{\mathbf{J}}T_{\varepsilon_j}] = 0$. □

Definition 5.16. Let $\mathbf{J}$ be an arbitrary non-empty subset of $\mathring{\mathbf{I}}$. For each $y \in \hat{W}_{\mathbf{J}}$, we define $\mathbf{Q}(q)$-subalgebras $A_{\mathbf{J}}(y)$ and $A_{\mathbf{J}}(y)^c$ of $U^+_{\mathbf{J}}$ by setting

$$
A_{\mathbf{J}}(y) := \{u \in U^+_{\mathbf{J}} \mid {}_{\mathbf{J}}T_y^{-1}(u) \in U^{\leq 0}_{\mathbf{J}}\}, \qquad A_{\mathbf{J}}(y)^c := \{u \in U^+_{\mathbf{J}} \mid {}_{\mathbf{J}}T_y^{-1}(u) \in U^+_{\mathbf{J}}\}.
$$

Note that $A_{\mathbf{J}}(y) = A_{\mathbf{J}}(|y|)$ and $A_{\mathbf{J}}(y)^c = A_{\mathbf{J}}(|y|)^c$, where $y = |y|\tau_y$, $|y| \in W_{\mathbf{J}}$, and $\tau_y \in \Omega_{\mathbf{J}}$. For each $B \in \mathfrak{B}^*_{\mathbf{J}}$, we set

$$
A_{\mathbf{J}}(B) := \bigcup_{y \in W_{\mathbf{J}}(B)} A_{\mathbf{J}}(y), \qquad A_{\mathbf{J}}(B)^c := \bigcap_{y \in W_{\mathbf{J}}(B)} A_{\mathbf{J}}(y)^c,
$$

where $W_{\mathbf{J}}(B) = \{y \in W_{\mathbf{J}} \mid \Phi_{\mathbf{J}}(y) \subset B\}$. Here note that $A_{\mathbf{J}}(B)^c$ is a $\mathbf{Q}(q)$-subalgebra of $U^+_{\mathbf{J}}$. In Proposition 7.2(2), we will show that $A_{\mathbf{J}}(B)$ is also a $\mathbf{Q}(q)$-subalgebra of $U^+_{\mathbf{J}}$. For each $w \in \mathring{W}_{\mathbf{J}}$, we set

$$
A_{\mathbf{J}}(w,-) := A_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,-)), \qquad A_{\mathbf{J}}(w,-)^c := A_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,-))^c, \tag{5.27}
$$

$$
A_{\mathbf{J}}(w,+) := \Psi A_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,+)), \qquad A_{\mathbf{J}}(w,+)^c := \Psi A_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,+))^c. \tag{5.28}
$$

In addition, we define a $\mathbf{Q}(q)$-subalgebra $A_{\mathbf{J}}(w,0)$ of $U^+_{\mathbf{J}}$ by setting

$$
A_{\mathbf{J}}(w,0) := A_{\mathbf{J}}(w,-)^c \cap A_{\mathbf{J}}(w,+)^c. \tag{5.29}
$$

In the case where $\mathbf{J} = \mathring{\mathbf{I}}$, we will denote the symbols above more simply by removing $\mathbf{J}$ from them.

Lemma 5.17. (1) *For each $y \in W_{\mathbf{J}}$, the multiplication defines the following injective $\mathbf{Q}(q)$-linear mapping: $A_{\mathbf{J}}(y) \otimes A_{\mathbf{J}}(y)^c \hookrightarrow U^+_{\mathbf{J}}$.*

(2) *Let $w_\circ$ be the longest element of $\mathring{W}_{\mathbf{J}}$. Then the following equalities hold:*

$$
A_{\mathbf{J}}(w,+) = \Psi A_{\mathbf{J}}(ww_\circ,-), \qquad A_{\mathbf{J}}(w,+)^c = \Psi A_{\mathbf{J}}(ww_\circ,-)^c, \qquad A_{\mathbf{J}}(w,0) = \Psi A_{\mathbf{J}}(ww_\circ,0).
$$

Proof. (1) Since ${}_{\mathbf{J}}T_y$ is an automorphism of the $\mathbf{Q}(q)$-algebra $U_{\mathbf{J}}$, the assignment $a\otimes b \mapsto {}_{\mathbf{J}}T_y^{-1}(a)\otimes {}_{\mathbf{J}}T_y^{-1}(b)$ defines an automorphism $({}_{\mathbf{J}}T_y^{-1})^{\otimes 2}$ of the $\mathbf{Q}(q)$-algebra $U_{\mathbf{J}}^{\otimes 2}$. Let m be the multiplication mapping $A_{\mathbf{J}}(y)\otimes A_{\mathbf{J}}(y)^c \to U_{\mathbf{J}}^+$, and m' the multiplication mapping $U_{\mathbf{J}}^{\le 0}\otimes U_{\mathbf{J}}^+ \simeq U_{\mathbf{J}}$. Then we see that

$$ {}_{\mathbf{J}}T_y^{-1}\circ m = m'\circ ({}_{\mathbf{J}}T_y^{-1})^{\otimes 2}|_{A_{\mathbf{J}}(y)\otimes A_{\mathbf{J}}(y)^c}, $$

where $({}_{\mathbf{J}}T_y^{-1})^{\otimes 2}|_{A_{\mathbf{J}}(y)\otimes A_{\mathbf{J}}(y)^c}$ is the restriction of $({}_{\mathbf{J}}T_y^{-1})^{\otimes 2}$ to $A_{\mathbf{J}}(y)\otimes A_{\mathbf{J}}(y)^c$. Since the right hand side is injective, we see that m is injective.

(2) Since $\Delta_{\mathbf{J}}(w,+)=\Delta_{\mathbf{J}}(ww_\circ,-)$ we get the left and the middle equalities. The middle equality implies that $\Psi A_{\mathbf{J}}(ww_\circ,+)^c = \Psi\Psi A_{\mathbf{J}}(ww_\circ w_\circ,-)^c = A_{\mathbf{J}}(w,-)^c$. Hence, by (5.29), we see that

$$ \Psi A_{\mathbf{J}}(ww_\circ,0) = \Psi A_{\mathbf{J}}(ww_\circ,-)^c \cap \Psi A_{\mathbf{J}}(ww_\circ,+)^c = A_{\mathbf{J}}(w,0). \quad \square $$

Definition 5.18. Let A be an associative algebra with the unit 1 over a commutative ring R, and $\{X_\lambda \mid \lambda\in\Lambda\}$ a subset of A indexed by a totally ordered set Λ with $\preceq$ the total order on Λ. For each function $f: X\to\mathbf{Z}_+$, we set $\mathrm{supp}(f) := \{x\in X \mid f(x)>0\}$, and call $\mathrm{supp}(f)$ the *support* of f. If $\#\mathrm{supp}(f)<\infty$, we call f a *finitely supported function*. For each finitely supported function $\mathbf{c}:\Lambda\to\mathbf{Z}_+$, we set

$$ X_{\prec}^{\mathbf{c}} := X_{\lambda_1}^{\mathbf{c}(\lambda_1)}\cdot X_{\lambda_2}^{\mathbf{c}(\lambda_2)}\cdots X_{\lambda_m}^{\mathbf{c}(\lambda_m)}, \qquad X_{\succ}^{\mathbf{c}} := X_{\lambda_m}^{\mathbf{c}(\lambda_m)}\cdots X_{\lambda_2}^{\mathbf{c}(\lambda_2)}\cdot X_{\lambda_1}^{\mathbf{c}(\lambda_1)}, \quad (5.30) $$

where $\mathrm{supp}(\mathbf{c})=\{\lambda_1,\lambda_2,\dots,\lambda_m\}$ with $\lambda_1\prec\lambda_2\prec\cdots\prec\lambda_m$, and call the element $X_{\prec}^{\mathbf{c}}$ (resp. $X_{\succ}^{\mathbf{c}}$) a *normally ordered* (resp. *opposite ordered*) *monomial* of $\{X_\lambda\mid\lambda\in\Lambda\}$. Here we set $X_{\prec}^{\mathbf{c}} = X_{\succ}^{\mathbf{c}} := 1$ if $\mathrm{supp}(\mathbf{c})=\varnothing$. We denote by $X_{\prec}^{*}$ (resp. $X_{\succ}^{*}$) the set of all $X_{\prec}^{\mathbf{c}}$ (resp. $X_{\succ}^{\mathbf{c}}$). In addition, for each $\Sigma\subset\Lambda$, we set

$$ X_{\prec}^{*}(\Sigma) := \{X_{\prec}^{\mathbf{c}} \mid \mathrm{supp}(\mathbf{c})\subset\Sigma\}, \qquad X_{\succ}^{*}(\Sigma) := \{X_{\succ}^{\mathbf{c}} \mid \mathrm{supp}(\mathbf{c})\subset\Sigma\}. \quad (5.31) $$

Note that $X_{\prec}^{*} = X_{\prec}^{*}(\Lambda)$ and $X_{\succ}^{*} = X_{\succ}^{*}(\Lambda)$.

For each $s\in S_{\mathbf{J}}$, we set

$$ E_s := \begin{cases} E_{\delta-\theta_{\mathbf{J}_c}} & \text{if } s = s_{\delta-\theta_{\mathbf{J}_c}} \text{ with } c=1,\dots,\mathrm{C}(\mathbf{J}), \\ E_j & \text{if } s = s_j \text{ with } j\in\mathbf{J}, \end{cases} \quad (5.32) $$

where $E_{\delta-\theta_{\mathbf{J}_c}}$ is introduced in Definition 5.2. For each $\boldsymbol{s}\in\mathscr{W}_{\mathbf{J}}^*$ and $p\in\mathbf{N}_{\ell(\boldsymbol{s})}$, we define a weight vector $E_{\boldsymbol{s},p}$ of $U_{\mathbf{J}}^+$ with weight $\phi_{\boldsymbol{s}}(p)$ by setting

$$ E_{\boldsymbol{s},p} := {}_{\mathbf{J}}T_{\boldsymbol{s}(1)}\cdot\cdots\cdot{}_{\mathbf{J}}T_{\boldsymbol{s}(p-1)}(E_{\boldsymbol{s}(p)}). \quad (5.33) $$

If $\phi_{\boldsymbol{s}}(p)=\beta$, we denote $E_{\boldsymbol{s},p}$ by $E_{\boldsymbol{s},\beta}$.

Proposition 5.19. (1) *Let B be a real biconvex set in $\Delta_{\mathbf{J}+}$, $\boldsymbol{s}$ an element of $\mathscr{W}_{\mathbf{J}}^*$ such that $B=\Phi_{\mathbf{J}}^*([\boldsymbol{s}])$, and $\le$ the usual total order on $\mathbf{N}_{\ell(\boldsymbol{s})}$. Then the set*

$E^*_{s,<}$ *(resp.* $E^*_{s,>}$*) (see Definition* 5.18*) forms a basis of a subspace* $U_{\mathbf{J},<}(B)$ *(resp.* $U_{\mathbf{J},>}(B)$*) of* $U^+_{\mathbf{J}}$ *which does not depend on the choice of* $\boldsymbol{s}$*. Moreover, the multiplication defines the following injective* $\mathbf{Q}(q)$*-linear mappings:*

$$U_{\mathbf{J},<}(B) \otimes A_{\mathbf{J}}(B)^c \hookrightarrow U^+_{\mathbf{J}}, \qquad U_{\mathbf{J},>}(B) \otimes A_{\mathbf{J}}(B)^c \hookrightarrow U^+_{\mathbf{J}}. \tag{5.34}$$

(2) *Let* $\mathbf{J}_1$ *and* $\mathbf{J}_2$ *be non-empty subsets of* $\mathring{\mathbf{I}}$ *which are disjoint from each other, and* (B_1, B_2) *an element of* $\mathfrak{B}^*_{\mathbf{J}_1} \times \mathfrak{B}^*_{\mathbf{J}_2}$*. Then the multiplication defines the following injective* $\mathbf{Q}(q)$*-linear mappings:*

$$U_{\mathbf{J}_1,<}(B_1) \otimes U_{\mathbf{J}_2,<}(B_2) \hookrightarrow U^+, \qquad U_{\mathbf{J}_1,>}(B_1) \otimes U_{\mathbf{J}_2,>}(B_2) \hookrightarrow U^+. \tag{5.35}$$

Proof. (1) We first consider the linear independence over $\mathbf{Q}(q)$ of the sets $E_{s,<}$. Since the proof of the linear independence is similar to that of Lemma 8.21 in [10], we omit the detailed proof, but we give a key point. Since E^k_j is a non-zero element of $A_{\mathbf{J}}(s_j)$ with weight $k\alpha_j$ for each $j \in \mathbf{J}$ and $k = 0, 1, \ldots, m$, the elements E^k_j $(k = 0, 1, \ldots, m)$ of $A_{\mathbf{J}}(s_j)$ are linearly independent over $\mathbf{Q}(q)$. Thus it follows from Lemma 5.17(1) that the equalities $\sum_{k=0}^m E^k_j u_k = 0$ with $u_k \in A_{\mathbf{J}}(s_j)^c$ $(k = 0, 1, \ldots, m)$ imply that $u_k = 0$ for all k.

We next prove the independence of $U_{\mathbf{J},<}(B)$ from the choice of $\boldsymbol{s}$. For convenience, we denote by $U_{\mathbf{J},<}(\boldsymbol{s})$ the $\mathbf{Q}(q)$-subspace of $U^+_{\mathbf{J}}$ spanned by $E^*_{s,<}$. Then it suffices to show that $U_{\mathbf{J},<}(\boldsymbol{s}) = U_{\mathbf{J},<}(\boldsymbol{s}')$ for another element $\boldsymbol{s}' \in \mathscr{W}^\infty$ such that $\Phi^\infty([\boldsymbol{s}']) = B$. In the case where B is a finite biconvex set in $\Delta_{\mathbf{J}+}$, since $\boldsymbol{s}$ is a finite reduced word, the proof of the assertion is similar to that of Proposition 8.22 in [10], so we omit that. We will prove the case where B is an infinite real biconvex set in $\Delta_{\mathbf{J}+}$. Since $\Phi^\infty_{\mathbf{J}}([\boldsymbol{s}]) = \Phi^\infty_{\mathbf{J}}([\boldsymbol{s}'])$, for each $(m, n) \in \mathbf{N}^2$, there exists $(m', n') \in \mathbf{Z}_{>m} \times \mathbf{Z}_{>n}$ such that $\Phi_{\mathbf{J}}([\boldsymbol{s}|_m]) \subset \Phi_{\mathbf{J}}([\boldsymbol{s}'|_{m'}])$ and $\Phi_{\mathbf{J}}([\boldsymbol{s}'|_n]) \subset \Phi_{\mathbf{J}}([\boldsymbol{s}|_{n'}])$, which implies $U_{\mathbf{J},<}([\boldsymbol{s}|_m]) \subset U_{\mathbf{J},<}([\boldsymbol{s}'|_{m'}])$ and $U_{\mathbf{J},<}([\boldsymbol{s}|_n]) \subset U_{\mathbf{J},<}([\boldsymbol{s}'|_{n'}])$. Since $U_{\mathbf{J},<}(\boldsymbol{s}) = \bigcup_{p \in \mathbf{N}} U_{\mathbf{J},<}([\boldsymbol{s}|_p])$ and $U_{\mathbf{J},<}(\boldsymbol{s}') = \bigcup_{p \in \mathbf{N}} U_{\mathbf{J},<}([\boldsymbol{s}'|_p])$, we get $U_{\mathbf{J},<}(\boldsymbol{s}) = U_{\mathbf{J},<}(\boldsymbol{s}')$. The proof of the assertion for $E^*_{s,>}$ is quite similar.

We next prove (5.34). By Lemma 5.17(1), we see that the multiplication $A_{\mathbf{J}}([\boldsymbol{s}|_p]) \otimes A_{\mathbf{J}}([\boldsymbol{s}|_p])^c \to U^+_{\mathbf{J}}$ is injective for each $p \in \mathbf{N}_{\ell(\boldsymbol{s})}$. It is clear that $U_{\mathbf{J},<}(\boldsymbol{s}|_p) \subset A_{\mathbf{J}}([\boldsymbol{s}|_p])^c$ and $A_{\mathbf{J}}(B)^c \subset A_{\mathbf{J}}([\boldsymbol{s}|_p])^c$. It follows that the multiplication $m_p : U_{\mathbf{J},<}(\boldsymbol{s}|_p) \otimes A_{\mathbf{J}}(B)^c \to U^+_{\mathbf{J}}$ is injective. Suppose that two elements (a_1, b_1) and (a_2, b_2) of $U_{\mathbf{J},<}(\boldsymbol{s}) \times A_{\mathbf{J}}(B)^c$ satisfy $a_1 b_1 = a_2 b_2$. We may assume that both a_1 and a_2 belong to $U_{\mathbf{J},<}(\boldsymbol{s}|_p)$ for some $p \in \mathbf{N}_{\ell(\boldsymbol{s})}$. Then the injectivity of m_p implies that $a_1 = a_2$ and $b_1 = b_2$. Therefore the multiplication mapping $U_{\mathbf{J},<}(B) \otimes A_{\mathbf{J}}(B)^c \to U^+_{\mathbf{J}}$ is injective. The proof of the remains are quite similar.

(2) Set $\mathbf{J} = \mathbf{J}_1 \amalg \mathbf{J}_2$. Then we see that B_1 is a real biconvex subset in $\Delta_{\mathbf{J}+}$. From (5.18)(i) in Proposition 5.10, it follows that both $U_{\mathbf{J}_2,<}(B_2)$ and $U_{\mathbf{J}_2,>}(B_2)$ are subspaces of $A_{\mathbf{J}}(B)^c$. Thus (5.35) follows from (5.34). □

Proposition 5.20. (1) *Suppose that B is a real biconvex set in $\Delta_{\mathbf{J}+}$ and set $B_c := B \cap \Delta_{\mathbf{J}_c}$ for each $c = 1, \dots, \mathrm{C}(\mathbf{J})$. Then the multiplication defines the following isomorphisms of $\mathbf{Q}(q)$-vector spaces:*

$$\bigotimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c,<}(B_c) \xrightarrow{\sim} U_{\mathbf{J},<}(B), \qquad \bigotimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c,>}(B_c) \xrightarrow{\sim} U_{\mathbf{J},>}(B). \quad (5.36)$$

(2) *Let C be a real biconvex set in $\Delta_{\mathbf{J}+}$, and y an element of $W_{\mathbf{J}}(C)$. Set $D := y^{-1}\{C \backslash \Phi_{\mathbf{J}}(y)\}$. Then the multiplication defines the following isomorphisms of $\mathbf{Q}(q)$-vector spaces:*

$$U_{\mathbf{J},<}(y) \otimes {}_{\mathbf{J}}T_y U_{\mathbf{J},<}(D) \xrightarrow{\sim} U_{\mathbf{J},<}(C), \qquad {}_{\mathbf{J}}T_y U_{\mathbf{J},>}(D) \otimes U_{\mathbf{J},>}(y) \xrightarrow{\sim} U_{\mathbf{J},>}(C), \quad (5.37)$$

where $U_{\mathbf{J},<}(y) := U_{\mathbf{J},<}(\Phi_{\mathbf{J}}(y))$ and $U_{\mathbf{J},>}(y) := U_{\mathbf{J},>}(\Phi_{\mathbf{J}}(y))$. In particular, we have $U_{\mathbf{J},<}(y) \subset U_{\mathbf{J},<}(C)$ and $U_{\mathbf{J},>}(y) \subset U_{\mathbf{J},>}(C)$. Moreover, we have:

$$U_{\mathbf{J},<}(y) = U_{\mathbf{J},<}(C) \cap A_{\mathbf{J}}(y), \qquad {}_{\mathbf{J}}T_y U_{\mathbf{J},<}(D) = U_{\mathbf{J},<}(C) \cap A_{\mathbf{J}}(y)^c, \quad (5.38)$$

$$U_{\mathbf{J},>}(y) = U_{\mathbf{J},>}(C) \cap A_{\mathbf{J}}(y), \qquad {}_{\mathbf{J}}T_y U_{\mathbf{J},>}(D) = U_{\mathbf{J},>}(C) \cap A_{\mathbf{J}}(y)^c. \quad (5.39)$$

(3) *For each $B \in \mathfrak{B}_{\mathbf{J}}^*$, we have $U_{\mathbf{J},<}(B) \cup U_{\mathbf{J},>}(B) \subset A_{\mathbf{J}}(B)$.*

(4) *The action of $\mathscr{B}_{\hat{W}_{\mathbf{J}}}$ on $U_{\mathbf{J}}$ is faithful.*

Proof. (1) For each $c = 1, \dots, \mathrm{C}(\mathbf{J})$, we set $\boldsymbol{s}^{-1}(S_{\mathbf{J}_c}) := \{p \in \mathbf{N}_{\ell(\boldsymbol{s})} \mid \boldsymbol{s}(p) \in S_{\mathbf{J}_c}\}$ and $n_c := \#\boldsymbol{s}^{-1}(S_{\mathbf{J}_c})$, and denote by ι_c the unique strictly increasing function from $\mathbf{N}_{n_c}$ to $\mathbf{N}_{\ell(\boldsymbol{s})}$ such that $\operatorname{Im} \iota_c = \boldsymbol{s}^{-1}(S_{\mathbf{J}_c})$. Then, for each $c = 1, \dots, \mathrm{C}(\mathbf{J})$, we define a sequence $\boldsymbol{s}_c = (\boldsymbol{s}_c(p))_{p \in \mathbf{N}_{n_c}} \in S_{\mathbf{J}_c}^{\mathbf{N}_{n_c}}$ by setting $\boldsymbol{s}_c(p) := \boldsymbol{s}(\iota_c(p))$ for each $p \in \mathbf{N}_{n_c}$. We see that $\boldsymbol{s}_c$ is an element of $\mathscr{W}_{\mathbf{J}_c}^{n_c}$ such that $\Phi_{\mathbf{J}_c}^*([\boldsymbol{s}_c]) = B_c$. By Proposition 5.19(2), we see that the multiplication defines the following injective $\mathbf{Q}(q)$-linear mappings: $\bigotimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c,<}(B_c) \hookrightarrow U^+$ and $\bigotimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c,>}(B_c) \hookrightarrow U^+$. By Lemma 5.4(4) and Proposition 5.10(2), we see that $\prod_{c=1}^{\mathrm{C}(\mathbf{J})} E_{\boldsymbol{s}_c,<}^* = E_{\boldsymbol{s},<}^*$ and $\prod_{c=1}^{\mathrm{C}(\mathbf{J})} E_{\boldsymbol{s}_c,>}^* = E_{\boldsymbol{s},>}^*$, which implies (5.36).

(2) By definition, we see that for each $\boldsymbol{s} \in \mathscr{W}_{\mathbf{J}}^*$ and $m \in \mathbf{N}_{\ell(\boldsymbol{s})}$ the multiplication defines the following isomorphisms of $\mathbf{Q}(q)$-vector spaces:

$$U_{\mathbf{J},<}(\boldsymbol{s}|_m) \otimes T_{[\boldsymbol{s}|_m]} U_{\mathbf{J},<}(\boldsymbol{s}|^m) \xrightarrow{\sim} U_{\mathbf{J},<}(\boldsymbol{s}), \qquad a \otimes b \mapsto ab, \quad (5.40)$$

$$T_{[\boldsymbol{s}|_m]} U_{\mathbf{J},>}(\boldsymbol{s}|^m) \otimes U_{\mathbf{J},>}(\boldsymbol{s}|_m) \xrightarrow{\sim} U_{\mathbf{J},>}(\boldsymbol{s}), \qquad a \otimes b \mapsto ab, \quad (5.41)$$

where $\boldsymbol{s}|_m$ is the initial m-section of $\boldsymbol{s}$ and $\boldsymbol{s}|^m$ is the m-shift of $\boldsymbol{s}$ (see [7]).

Let us prove (5.37) in the case where $B = \Phi_{\mathbf{J}}(z) \in \mathfrak{B}_{\mathbf{J}}$ with $z \in W_{\mathbf{J}}$. Since $\Phi_{\mathbf{J}}(y) \subset \Phi_{\mathbf{J}}(z)$, we have $\Phi_{\mathbf{J}}(z) = \Phi_{\mathbf{J}}(y) \amalg y\Phi_{\mathbf{J}}(y^{-1}z)$, and hence $D = \Phi_{\mathbf{J}}(y^{-1}z)$. Thus we see that there exists an element $\boldsymbol{s} \in \mathscr{W}_{\mathbf{J}}^n$ such that $[\boldsymbol{s}] = z$, $[\boldsymbol{s}|_m] = y$, and $[\boldsymbol{s}|^m] = y^{-1}z$ with $n = \ell(z)$ and $m = \ell(y)$. Hence, (5.40)(5.41) imply (5.37).

Let us prove (5.37) in the case where $B \in \mathfrak{B}^{\infty}$. By Lemma 2.5 in [8], we see that $D \in \mathfrak{B}_{\mathbf{J}}^{\infty}$ and $B = \Phi_{\mathbf{J}}(y) \amalg yD$. Thus there exists an element $\boldsymbol{s} \in \mathscr{W}_{\mathbf{J}}^{\infty}$ such that $\Phi_{\mathbf{J}}^{*}([\boldsymbol{s}]) = B$, $[\boldsymbol{s}|_m] = y$, and $\Phi_{\mathbf{J}}^{\infty}([\boldsymbol{s}|^m]) = D$ with $m = \ell(y)$. Hence, (5.40)(5.41) imply (5.37).

It is easy to see that both $U_{\mathbf{J},<}(y)$ and $U_{\mathbf{J},>}(y)$ are subsets of $A_{\mathbf{J}}(y)$ and that both ${}_{\mathbf{J}}T_y U_{\mathbf{J},<}(D)$ and ${}_{\mathbf{J}}T_y U_{\mathbf{J},>}(D)$ are subsets of $A_{\mathbf{J}}(y)^c$, and hence (5.38)(5.39) follow from (5.37).

(3) We see that $E_{\boldsymbol{s},<}^{*} = \bigcup_{p \in \mathbf{N}} E_{\boldsymbol{s}|_p,<}^{*}$ and $E_{\boldsymbol{s},>}^{*} = \bigcup_{p \in \mathbf{N}} E_{\boldsymbol{s}|_p,>}^{*}$ with $E_{\boldsymbol{s}|_p,<}^{*} \subset E_{\boldsymbol{s}|_{p+1},<}^{*}$ and $E_{\boldsymbol{s}|_p,>}^{*} \subset E_{\boldsymbol{s}|_{p+1},>}^{*}$, which implies that $U_{\mathbf{J},<}(B) = \bigcup_{p \in \mathbf{N}} U_{\mathbf{J},<}([\boldsymbol{s}|_p])$ and $U_{\mathbf{J},>}(B) = \bigcup_{p \in \mathbf{N}} U_{\mathbf{J},>}([\boldsymbol{s}|_p])$. Thus both $U_{\mathbf{J},<}(B)$ and $U_{\mathbf{J},>}(B)$ are subsets of $A_{\mathbf{J}}(B)$, since $U_{\mathbf{J},<}([\boldsymbol{s}|_p]) \cup U_{\mathbf{J},>}([\boldsymbol{s}|_p]) \subset A_{\mathbf{J}}([\boldsymbol{s}|_p])$ and $[\boldsymbol{s}|_p] \in W_{\mathbf{J}}(B)$ for all $p \in \mathbf{N}$.

(4) Suppose that ${}_{\mathbf{J}}T_y|_{U_{\mathbf{J}}} = id$ for $y \in W_{\mathbf{J}}$. Then $A_{\mathbf{J}}(y) = \mathbf{Q}(q)$. Since $U_{\mathbf{J},<}(y) \subset A_{\mathbf{J}}(y)$, it follows that $U_{\mathbf{J},<}(y) = \mathbf{Q}(q)$. Thus we get $y = 1$ by Proposition 5.19(1). □

Definition 5.21. For each $w \in \mathring{W}_{\mathbf{J}}$, we set

$$U_{\mathbf{J},<}(w,-) := U_{\mathbf{J},<}(\Delta_{\mathbf{J}}(w,-)), \qquad U_{\mathbf{J},>}(w,-) := U_{\mathbf{J},>}(\Delta_{\mathbf{J}}(w,-)), \tag{5.42}$$

$$U_{\mathbf{J},>}(w,+) := \Psi U_{\mathbf{J},<}(\Delta_{\mathbf{J}}(w,+)), \qquad U_{\mathbf{J},<}(w,+) := \Psi U_{\mathbf{J},>}(\Delta_{\mathbf{J}}(w,+)). \tag{5.43}$$

Note that

$$U_{\mathbf{J},>}(w,+) = \Psi U_{\mathbf{J},<}(ww_\circ,-), \qquad U_{\mathbf{J},<}(w,+) = \Psi U_{\mathbf{J},>}(ww_\circ,-) \tag{5.44}$$

with $w_\circ$ the longest element of $\mathring{W}_{\mathbf{J}}$. In the case where $\mathbf{J} = \mathring{\mathbf{I}}$, we will denote the symbols above more simply by removing $\mathbf{J}$ from them.

Proposition 5.22. *For each $w \in \mathring{W}_{\mathbf{J}}$ and $y \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,-))$, the multiplication defines the following isomorphisms of $\mathbf{Q}(q)$-vector spaces:*

$$U_{\mathbf{J},<}(y) \otimes {}_{\mathbf{J}}T_y U_{\mathbf{J},<}(\bar{y}^{-1}w,-) \xrightarrow{\sim} U_{\mathbf{J},<}(w,-), \tag{5.45}$$

$${}_{\mathbf{J}}T_y U_{\mathbf{J},>}(\bar{y}^{-1}w,-) \otimes U_{\mathbf{J},>}(y) \xrightarrow{\sim} U_{\mathbf{J},>}(w,-). \tag{5.46}$$

Proof. Since $\Phi_{\mathbf{J}}(y) \subset \Delta_{\mathbf{J}}(w,-)$, we have

$$\Phi_{\mathbf{J}}(y) \amalg y\Delta_{\mathbf{J}}(\bar{y}^{-1}w,-) = \Delta_{\mathbf{J}}(w,-). \tag{5.47}$$

Thus the assertions follow from Proposition 5.20(2). □

Lemma 5.23. *Let w be an arbitrary element of $\mathring{W}_{\mathbf{J}}$.*

(1) *For each $y \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,-))$, we have*

$${}_{\mathbf{J}}T_y^{-1} U_{\mathbf{J},>}(w,+) \subset U_{\mathbf{J},>}(\bar{y}^{-1}w,+), \qquad {}_{\mathbf{J}}T_y^{-1} U_{\mathbf{J},<}(w,+) \subset U_{\mathbf{J},<}(\bar{y}^{-1}w,+).$$

In particular, we have $U_{\mathbf{J},>}(w,+) \cup U_{\mathbf{J},<}(w,+) \subset A_{\mathbf{J}}(w,-)^c$.

(2) *For each* $y \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,+))$, *we have*

$$ {}_{\mathbf{J}}T_{y^{-1}} U_{\mathbf{J},<}(w,-) \subset U_{\mathbf{J},<}(\bar{y}^{-1}w,-), \qquad {}_{\mathbf{J}}T_{y^{-1}} U_{\mathbf{J},>}(w,-) \subset U_{\mathbf{J},>}(\bar{y}^{-1}w,-). $$

In particular, we have $U_{\mathbf{J},<}(w,-) \cup U_{\mathbf{J},>}(w,-) \subset A_{\mathbf{J}}(w,+)^{c}$.

(3) *The following* (i) *and* (ii) *hold:*

$$ \text{(i)} \quad u \in \Psi T_{w_\circ} U_{\mathbf{J},<}(1,-) \Rightarrow T_w(u) \in U_{\mathbf{J},>}(w,+), $$

$$ \text{(ii)} \quad u \in \Psi T_{w_\circ} U_{\mathbf{J},>}(1,-) \Rightarrow T_w(u) \in U_{\mathbf{J},<}(w,+). $$

(4) *The following inclusions hold:*

$$ U_{\mathbf{J},<}(w,-) \cup U_{\mathbf{J},>}(w,-) \subset A_{\mathbf{J}}(w,-), \qquad U_{\mathbf{J},<}(w,+) \cup U_{\mathbf{J},>}(w,+) \subset A_{\mathbf{J}}(w,+). $$

Proof. (1) By (5.47), we have $\Phi_{\mathbf{J}}(y^{-1}) \subset \Delta_{\mathbf{J}}(\bar{y}^{-1}ww_\circ,-)$. By Proposition 5.22, we have ${}_{\mathbf{J}}T_{y^{-1}} U_{\mathbf{J},<}(ww_\circ,-) \subset U_{\mathbf{J},<}(\bar{y}^{-1}ww_\circ,-)$. Thus, by (5.44), we get

$$ {}_{\mathbf{J}}T_y^{-1} U_{\mathbf{J},>}(w,+) = \Psi {}_{\mathbf{J}}T_{y^{-1}} U_{\mathbf{J},<}(ww_\circ,-) \subset U_{\mathbf{J},>}(\bar{y}^{-1}w,+). $$

(2) Since $\Phi_{\mathbf{J}}(y) \subset \Delta_{\mathbf{J}}(ww_\circ,-)$ we have $\Phi_{\mathbf{J}}(y^{-1}) \subset \Delta_{\mathbf{J}}(\bar{y}^{-1}w,-)$. Thus, by Proposition 5.22, we get ${}_{\mathbf{J}}T_{y^{-1}} U_{\mathbf{J},<}(w,-) \subset U_{\mathbf{J},<}(\bar{y}^{-1}w,-)$.

(3) By (5.44) and (5.45), we see that the multiplication mapping

$$ \Psi U_{\mathbf{J},<}(ww_\circ) \otimes \Psi T_{ww_\circ} U_{\mathbf{J},<}(1,-) \to U_{\mathbf{J},>}(1,+) $$

is an isomorphism of $\mathbf{Q}(q)$-vector spaces. On the other hand, we see that $T_{w_\circ} = T_{w^{-1}} T_{ww_\circ}$ and $T_w \Psi = \Psi T_{w^{-1}}^{-1}$, and hence

$$ T_w(u) \in T_w \Psi T_{w_\circ} U_{\mathbf{J},<}(1,-) = \Psi T_{ww_\circ} U_{\mathbf{J},<}(1,-) \subset U_{\mathbf{J},>}(w,+). $$

The proof of (ii) is similar.

(4) The left inclusion follows from Proposition 5.20(3) and (5.42) and the left part of (5.27). The right inclusion follows from Proposition 5.20(3) and (5.43) and the left part of (5.28). □

6. Imaginary root vectors of $U_{\mathbf{J}}^{+}$

In this section, we introduce imaginary root vectors of $U_{\mathbf{J}}^{+}$, where $\mathbf{J}$ is an arbitrary non-empty subset of $\mathring{\mathbf{I}}$.

Definition 6.1. For each $(i,m) \in \mathring{\mathbf{I}} \times \mathbf{Z}$, we set

$$ x_{i,m}^{-} := T_{\varepsilon_i}^{m} T_i^{-1}(E_i), \qquad x_{i,m}^{+} := T_{\varepsilon_i}^{-m}(E_i). \tag{6.1} $$

Lemma 6.2. (1) *Suppose that* $n \in \mathbf{N}$ *and* $m \in \mathbf{Z}_{+}$. *Then*

$$ T_w(x_{i,n}^{-}) \in U_{<}(w,-)_{n\delta - w(\alpha_i)}, \qquad T_w(x_{i,m}^{+}) \in U_{<}(w,+)_{m\delta + w(\alpha_i)} \tag{6.2} $$

for each $i \in \mathring{\mathbf{I}}$ *and* $w \in \mathring{W}$. *Moreover,*

$$T_w(x_{i,n}^-) \in {}_{\mathscr{A}_1}U^+ \backslash (q-1)_{\mathscr{A}_1}U^+, \qquad T_w(x_{i,m}^+) \in {}_{\mathscr{A}_1}U^+ \backslash (q-1)_{\mathscr{A}_1}U^+. \tag{6.3}$$

(2) *For each* $(j,m) \in \mathbf{J} \times \mathbf{Z}$, *we have*

$$x_{j,m}^- = ({}_{\mathbf{J}}T_{\varepsilon_j})^m T_j^{-1}(E_j), \qquad x_{j,m}^+ = ({}_{\mathbf{J}}T_{\varepsilon_j})^{-m}(E_j). \tag{6.4}$$

Therefore, both $x_{j,m}^-$ *and* $x_{j,m}^+$ *are elements of* $U_{\mathbf{J}}$. *Moreover, we have* $x_{j,m}^- \in A_{\mathbf{J}}(1,-)$ *if* $m > 0$, *and* $x_{j,m}^+ \in A_{\mathbf{J}}(1,+)$ *if* $m \geq 0$.

(3) *Let* (j,m) *be an arbitrary element of* $\mathbf{J} \times \mathbf{Z}_+$, *and* $\boldsymbol{s}$ *an arbitrary element of* $\mathscr{W}_{\mathbf{J}}$ *such that* $m\delta + \alpha_j \in \Phi_{\mathbf{J}}([\boldsymbol{s}]) \subset \Delta_{\mathbf{J}}(1,+)$. *Then* $\Psi E_{\boldsymbol{s},m\delta+\alpha_j} = x_{j,m}^+$.

Proof. (1) By Definition 5.2, Definition 5.21(5.42), and Definition 6.1, we see that $x_{i,1}^- = E_{\delta-\alpha_i} \in U_<(1,-)_{\delta-\alpha_i}$, and hence $T_w(x_{i,n}^-) = T_w T_{\varepsilon_i}^{n-1}(E_{\delta-\alpha_i}) \in U_<(w,-)_{n\delta-w(\alpha_i)}$ by (5.45). Let $w_\circ$ be the longest element of $\mathring{W}$. Then, by (5.44)(5.46), the multiplication mapping $\Psi U_>(w_\circ) \otimes \Psi T_{w_\circ} U_>(1,-) \to U_<(1,+)$ is an isomorphism of $\mathbf{Q}(q)$-vector spaces, and hence

$$U_<(1,+)_{m\delta+\alpha_i} \subset \Psi T_{w_\circ} U_>(1,-). \tag{6.5}$$

On the other hand, by Lemma 5.23(1), we see that $x_{i,m}^+ = T_{\varepsilon_i}^{-m}(E_i)$ is an element of $U_<(1,+)_{m\delta+\alpha_i}$, since $E_i \in U_<(1,+)$. Combining with (6.5), we see that $x_{i,m}^+ \in U_<(1,+)_{m\delta+\alpha_i} \subset \Psi T_{w_\circ} U_>(1,-)$. Thus, by Lemma 5.23(3)(ii), we see that $T_w(x_{i,m}^+)$ is an element of $U_<(w,+)_{m\delta+w(\alpha_i)}$.

It is easy to see that the set ${}_{\mathscr{A}_1}U' \backslash (q-1)_{\mathscr{A}_1}U'$ is stable under the action of $\mathscr{B}_{\hat{W}}$ on U, which implies that both $T_w(x_{i,n}^-)$ and $T_w(x_{i,m}^+)$ are elements of ${}_{\mathscr{A}_1}U' \backslash (q-1)_{\mathscr{A}_1}U'$. Moreover, by (6.2) we see that both $T_w(x_{i,n}^-)$ and $T_w(x_{i,m}^+)$ are elements of U^+. Thus we get (6.3).

(2) Since both E_j and $T_j^{-1}(E_j)$ are elements of $U_{\mathbf{J}}$, the equalities (6.4) follow immediately from (6.1) and Proposition 5.15. Since ${}_{\mathbf{J}}T_{\varepsilon_j}$ is an automorphism of $U_{\mathbf{J}}$, by (6.4) we see that $x_{j,m}^{\pm} \in U_{\mathbf{J}}$. In the case where $m > 0$, by (1) we have $x_{j,m}^- \in U_{\mathbf{J}} \cap U^+ = U_{\mathbf{J}}^+$. In addition, by the left equality in (6.4), we see that $({}_{\mathbf{J}}T_{\varepsilon_j})^{-m}(x_{j,m}^-) = -K_j^{-1}F_j \in U_{\mathbf{J}}^{\leq 0}$, and hence $x_{j,m}^- \in A_{\mathbf{J}}(1,-)$. The proof of remains is similar.

(3) Firstly, we prove the fact that if $(\boldsymbol{s}_1, \boldsymbol{s}_2)$ is a pair of elements of $\mathscr{W}_{\mathbf{J}}$ such that $m\delta + \alpha_j \in \Phi_{\mathbf{J}}([\boldsymbol{s}_i]) \subset \Delta_{\mathbf{J}}(1,+)$ for $i = 1,2$ then $E_{\boldsymbol{s}_1,m\delta+\alpha_j} = E_{\boldsymbol{s}_2,m\delta+\alpha_j}$. We may assume that $[\boldsymbol{s}_1] = [\boldsymbol{s}_2]$, and put $x = [\boldsymbol{s}_1] = [\boldsymbol{s}_2]$. Since $\gamma \in \Delta_{\mathbf{J}}(1,+)$ for each $\gamma \in \Phi_{\mathbf{J}}(x)$, there exists $\mathbf{d}(\gamma) \in \mathbf{Z}_+$ such that $\gamma = \mathbf{d}(\gamma)\delta + \bar{\gamma}$ with $\bar{\gamma} \in \mathring{\Delta}_{\mathbf{J}+}$. Now suppose that $m\delta + \alpha_j = \sum_{\gamma \in \Phi_{\mathbf{J}}(x)} \mathbf{c}(\gamma)\gamma$ with $\mathbf{c}(\gamma) \in \mathbf{Z}_+$ for all $\gamma \in \Phi_{\mathbf{J}}(x)$. Then $m\delta + \alpha_j = (\sum_{\gamma \in \Phi_{\mathbf{J}}(x)} \mathbf{c}(\gamma)\mathbf{d}(\gamma))\delta + \sum_{\gamma \in \Phi_{\mathbf{J}}(x)} \mathbf{c}(\gamma)\bar{\gamma}$, which implies that $\mathbf{c}(m\delta + \alpha_j) = 1$ and $\mathbf{c}(\gamma) = 0$ for all $\gamma \neq m\delta + \alpha_j$. Thanks to Theorem 5.12 and (5.33), by applying Lemma 4.4(3), we get $E_{\boldsymbol{s}_1,m\delta+\alpha_j} = E_{\boldsymbol{s}_2,m\delta+\alpha_j}$.

Thanks to the fact above, to prove the required equality, it suffices to show that $\Psi E_{s', m\delta+\alpha_j} = x^+_{j,m}$ for some $s' \in \mathscr{W}_{\mathbf{J}}$ such that $m\delta + \alpha_j \in \Phi_{\mathbf{J}}([s']) \subset \Delta_{\mathbf{J}}(1,+)$. Put $l = \ell_{\mathbf{J}}(t_{-m\varepsilon_j})$ and write $t_{-m\varepsilon_j}$ as $s_1 s_2 \dots s_l \rho$ with $s_1, s_2, \dots, s_l \in S_{\mathbf{J}}$ and $\rho \in \Omega_{\mathbf{J}}$. Let $E_{s_{l+1}} = {}_{\mathbf{J}}T_\rho(E_j)$ with $s_{l+1} \in S_{\mathbf{J}}$, and define $s' = (s'(p))_{p \in \mathbf{N}_{l+1}} \in S_{\mathbf{J}}^{l+1}$ by setting $s'(p) := s_p$ for each $p \in \mathbf{N}_{l+1}$. Then we see that the sequence s' is an element of $\mathscr{W}_{\mathbf{J}}$ satisfying $\Phi_{\mathbf{J}}([s']) \subset \Delta_{\mathbf{J}}(1,+)$ and $\phi_{s'}(l+1) = m\delta + \alpha_j$. Thus it follows from the right equality in (6.4) that

$$x^+_{j,m} = \Psi_{\mathbf{J}} T_{-m\varepsilon_j}(E_j) = \Psi_{\mathbf{J}} T_{s_1} {}_{\mathbf{J}}T_{s_2} \dots {}_{\mathbf{J}}T_{s_l}(E_{s_{l+1}}) = \Psi E_{s', m\delta+\alpha_j}. \quad \square$$

Definition 6.3. For each $(i,n) \in \mathring{\mathbf{I}} \times \mathbf{N}$, we set

$$\varphi_{i,n} := [x^-_{i,n}, E_i]_q = x^-_{i,n} E_i - q_i^{-2} E_i x^-_{i,n}, \tag{6.6}$$

and also define $\varphi_i(z) \in U^+[[z]]$ by setting $\varphi_i(z) := (q_i - q_i^{-1}) \sum_{n \geq 1} \varphi_{i,n} z^n$. In addition, we define $I_{i,n} \in U^+_{n\delta}$ by the following equality in $U^+[[z]]$:

$$I_i(z) = \log(1 + \varphi_i(z)) \equiv \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \varphi_i(z)^n, \tag{6.7}$$

where $I_i(z) := (q_i - q_i^{-1}) \sum_{n \geq 1} I_{i,n} z^n$.

Lemma 6.4. (1) *For each $w \in \mathring{W}$ and $(i,n) \in \mathring{\mathbf{I}} \times \mathbf{N}$, both $T_w(\varphi_{i,n})$ and $T_w(I_{i,n})$ are elements of ${}_{\mathscr{A}_1}U^+_{n\delta} \backslash (q-1)_{\mathscr{A}_1}U^+$. Moreover, both $\overline{T_w(\varphi_{i,n})}$ and $\overline{T_w(I_{i,n})}$ are non-zero elements of ${}_1U^+_{n\delta}$.*

(2) *For each $w \in \mathring{W}$, the elements of $\{\overline{T_w(I_{i,n})} \mid (i,n) \in \mathring{\mathbf{I}} \times \mathbf{N}\}$ are linearly independent over $\mathbf{Q}$.*

(3) *Suppose that $j \in \mathbf{J} \subset \mathring{\mathbf{I}}$. Then both $T_w(\varphi_{j,n})$ and $T_w(I_{j,n})$ are elements of $U^+_{\mathbf{J}}$ for each $w \in \mathring{W}_{\mathbf{J}}$ and $n \in \mathbf{N}$.*

Proof. (1) Suppose that $w(\alpha_i) > 0$. Then we have $T_w(E_i) \in {}_{\mathscr{A}_1}U^+_{w(\alpha_i)}$. Hence, by Lemma 6.2(1), we have $T_w(x^-_{i,n}) \in {}_{\mathscr{A}_1}U^+_{n\delta - w(\alpha_i)}$, and hence $T_w(\varphi_{i,n}) = [T_w(x^-_{i,n}), T_w(E_i)]_q \in {}_{\mathscr{A}_1}U^+$. Suppose that $w(\alpha_i) < 0$. Then we see that $w = w' s_i$ and $\ell(w) = \ell(w') + 1$ for some $w' \in \mathring{W}$, and hence $T_w = T_{w'} T_i$ and $T_{w'}(E_i) \in {}_{\mathscr{A}_1}U^+$. Thus $T_w(x^-_{i,0}) = T_{w'} T_i(T_i^{-1}(E_i)) = T_{w'}(E_i) \in {}_{\mathscr{A}_1}U^+$. Combining with Lemma 6.2(1), we get

$$T_w(\varphi_{i,1}) = T_w([x^-_{i,0}, x^+_{i,1}]_q) = [T_{w'}(E_i), T_w(x^+_{i,1})]_q \in {}_{\mathscr{A}_1}U^+.$$

In the case where $n \geq 2$, by Lemma 6.2(1)(6.3), we have

$$T_w(\varphi_{i,n}) = T_w([x^-_{i,n-1}, x^+_{i,1}]_q) = [T_w(x^-_{i,n-1}), T_w(x^+_{i,1})]_q \in {}_{\mathscr{A}_1}U^+.$$

Therefore we see that $T_w(\varphi_{i,n}) \in {}_{\mathscr{A}_1}U^+$ for each $w \in \mathring{W}$ and $(i,n) \in \mathring{\mathbf{I}} \times \mathbf{N}$. By the definition, we see that

$$T_w(I_{i,n}) = T_w(\varphi_{i,n}) + \sum\nolimits_{\sum_{k=1}^{n-1} kp_k=n} \frac{(\sum_k p_k - 1)!}{p_1!p_2!\dots p_{n-1}!}$$
$$\times (q_i^{-1} - q_i)^{\sum_k p_k - 1} T_w(\varphi_{i,1}^{p_1}\varphi_{i,2}^{p_2}\dots\varphi_{i,n-1}^{p_{n-1}}), \tag{6.8}$$

and hence $T_w(I_{i,n}) \in {}_{\mathscr{A}_1}U^+$. By Theorem 4.7 of [1], we have

$$[x_{j,m}^-, I_{i,n}] = (\mathrm{sgn}(A_{ij}))^n [nA_{ij}]_{q_i} x_{j,m+n}^-/n. \tag{6.9}$$

Combining (6.9) with Lemma 6.2(1), we see that $T_w(I_{i,n}) \in {}_{\mathscr{A}_1}U^+ \backslash (q-1)_{\mathscr{A}_1}U^+$, and hence $T_w(\varphi_{i,n}) \in {}_{\mathscr{A}_1}U^+ \backslash (q-1)_{\mathscr{A}_1}U^+$ by (6.8).

(2) We may assume that $w = 1$. Hence, it suffices to show the linear independence over $\mathbf{Q}$ of the elements of $\{\overline{I_{i,n}} \mid i \in \mathring{\mathbf{I}}\}$ for each $n \in \mathbf{N}$. Now we suppose that $\sum_{i=1}^l v_i \overline{I_{i,n}} = 0$ with $v_i \in \mathbf{Q}$. By (6.9), we see that $[\overline{E_{\delta-\alpha_j}}, \overline{I_{i,n}}] = (\mathrm{sgn}(A_{ij}))^m A_{ij} \overline{x_{j,n+1}^-}/n$ for all $j \in \mathring{\mathbf{I}}$, which implies that $\sum_{i=1}^l v_i (\mathrm{sgn}(A_{ij}))^n A_{ij} = 0$ for all $j \in \mathring{\mathbf{I}}$. Thus $[v_1, \dots, v_l][(\mathrm{sgn}(A_{ij}))^n A_{ij}]_{i,j \in \mathring{\mathbf{I}}} = [0, \dots, 0]$. Since the matrix $[(\mathrm{sgn}(A_{ij}))^n A_{ij}]_{i,j \in \mathring{\mathbf{I}}}$ is invertible, we get $[v_1, \dots, v_l] = [0, \dots, 0]$. Therefore the assertion is valid.

(3) By Lemma 6.2(2), we have $x_{j,n}^- \in U_{\mathbf{J}}^+$, and hence $\varphi_{j,n} = [x_{j,n}^-, E_j]_q \in U_{\mathbf{J}}^+$. By (6.8), we have $I_{j,n} \in U_{\mathbf{J}}^+$. Since $U_{\mathbf{J}}$ is stable under the action of T_w, both $T_w(\varphi_{j,n})$ and $T_w(I_{j,n})$ are elements of $U_{\mathbf{J}}$. Thus we get the assertion in (3) by combining with the first assertion in (1). □

Definition 6.5. For each $w \in \mathring{W}_{\mathbf{J}}$, we define a $\mathbf{Q}(q)$-subalgebra $U_{\mathbf{J}}(w,0)$ of $U_{\mathbf{J}}^+$ by setting $U_{\mathbf{J}}(w,0) := \langle T_w(I_{j,n}) \mid (j,n) \in \mathbf{J} \times \mathbf{N} \rangle_{\mathbf{Q}(q)\text{-alg}}$. Note that Lemma 6.6(3) implies $U_{\mathbf{J}}(w,0) \subset U_{\mathbf{J}}^+$.

Proposition 6.6. *Let w be an arbitrary element of $\mathring{W}_{\mathbf{J}}$, and $\preceq$ an arbitrary total order on $\mathbf{J} \times \mathbf{N}$. Then $U_{\mathbf{J}}(w,0)$ is a commutative $\mathbf{Q}(q)$-subalgebra of $U_{\mathbf{J}}^+$, and the set $T_w(I_{\preceq}^*)$ (see Definition 5.18) is a basis of $U_{\mathbf{J}}(w,0)$.*

Proof. We may assume that $w = 1$. By Theorem 4.7 of [1], we see that $U_{\mathbf{J}}(1,0)$ is a commutative $\mathbf{Q}(q)$-subalgebra of $U_{\mathbf{J}}^+$, and hence the set $I_{\preceq}^*$ spans $U_{\mathbf{J}}(1,0)$. Thus it suffices to show the linear independence over $\mathbf{Q}(q)$ of the set $I_{\preceq}^*$. Let us denote by $\overline{I_{\preceq}^*}$ the image of $I_{\preceq}^*$ by the specialization at $q = 1$. By Proposition 4.1, Lemma 6.4(2), and the PBW Theorem of Lie algebras over $\mathbf{Q}$, we see that $\overline{I_{\preceq}^*}$ is linearly independent over $\mathbf{Q}$. Combining with Lemma 4.2, we see that $I_{\preceq}^*$ is linearly independent over $\mathbf{Q}(q)$. □

Lemma 6.7. (1) *For each $t \in T_{\mathbf{J}}$, there exists an element $t' \in T_{\mathbf{J}} \cap W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$ such that $tt' \in T_{\mathbf{J}} \cap W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$.*

(2) *Let w be an element of $\mathring{W}_{\mathbf{J}}$, and y an element of $W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,-))$. Set $w' = \bar{y}^{-1}w$. Then there exist elements $t, t' \in T_{\mathbf{J}} \cap W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$ such that $wt = yw't'$ and $\ell_{\mathbf{J}}(wt) = \ell(w) + \ell_{\mathbf{J}}(t) = \ell_{\mathbf{J}}(y) + \ell(w') + \ell_{\mathbf{J}}(t')$.*

Proof. (1) Clear. Let us prove (2). Since $\Phi_{\mathbf{J}}(y) \subset \Delta_{\mathbf{J}}(w,-)$ we have

$$\Phi(w) \amalg w\Delta_{\mathbf{J}}(1,-) = \Delta_{\mathbf{J}}(w,-) = \Phi_{\mathbf{J}}(y) \amalg y\{\Phi(w') \amalg w'\Delta_{\mathbf{J}}(1,-)\}.$$

Hence we see that $\ell_{\mathbf{J}}(wz) = \ell(w) + \ell_{\mathbf{J}}(z)$ and $\ell_{\mathbf{J}}(yw'z') = \ell_{\mathbf{J}}(y) + \ell(w') + \ell_{\mathbf{J}}(z')$ for all $z, z' \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$. Since $\overline{yw'} = \bar{y}w' = w$, we have $yw' = wt_\mu$ with $\mu \in \mathring{Q}_{\mathbf{J}}^\vee$. By (1), there exists an element $t_\nu \in T_{\mathbf{J}} \cap W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$ such that $t_\mu t_\nu \in T_{\mathbf{J}} \cap W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$. Set $t = t_\mu t_\nu$ and $t' = t_\nu$. Then $wt = yw't'$. Since $t, t' \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$ we have

$$\ell(w) + \ell_{\mathbf{J}}(t) = \ell_{\mathbf{J}}(wt) = \ell_{\mathbf{J}}(yw't') = \ell_{\mathbf{J}}(y) + \ell(w') + \ell_{\mathbf{J}}(t'). \quad \square$$

Proposition 6.8. (1) *If* $t \in T_{\mathbf{J}}$ *and* $u \in U_{\mathbf{J}}(1,0)$, *then* ${}_{\mathbf{J}}T_t(u) = u$.

(2) *If* $w \in \mathring{W}_{\mathbf{J}}$ *and* $y \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,-))$, *then* ${}_{\mathbf{J}}T_y^{-1} U_{\mathbf{J}}(w,0) = U_{\mathbf{J}}(\bar{y}^{-1}w,0)$. *In particular,* $U_{\mathbf{J}}(w,0) \subset A_{\mathbf{J}}(w,-)^c$.

(3) *If* $w \in \mathring{W}_{\mathbf{J}}$ *and* $y \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,+))$, *then* ${}_{\mathbf{J}}T_{y^{-1}} U_{\mathbf{J}}(w,0) = U_{\mathbf{J}}(\bar{y}^{-1}w,0)$. *In particular,* $U_{\mathbf{J}}(w,0) \subset A_{\mathbf{J}}(w,+)^c$.

(4) *If* $w \in \mathring{W}_{\mathbf{J}}$, *then* $U_{\mathbf{J}}(w,0) \subset A_{\mathbf{J}}(w,0)$.

Proof. (1) This follows from Proposition 5.15, Proposition 6.6, and Proposition 3.12 of [1].

(2) Set $w' = \bar{y}^{-1}w$. By Lemma 6.7(2), we see that $T_w \cdot {}_{\mathbf{J}}T_t = {}_{\mathbf{J}}T_y \cdot T_{w'} \cdot {}_{\mathbf{J}}T_{t'}$ for some $t, t' \in T_{\mathbf{J}} \cap W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$. By (1), for each $u \in U_{\mathbf{J}}(1,0)$, we have ${}_{\mathbf{J}}T_t(u) = {}_{\mathbf{J}}T_{t'}(u) = u$, and hence ${}_{\mathbf{J}}T_y^{-1} T_w(u) = T_{w'}(u) \in U_{\mathbf{J}}(\bar{y}^{-1}w,0)$. Thus the assertion is valid.

(3) Since $\Phi_{\mathbf{J}}(y) \subset \Delta_{\mathbf{J}}(ww_\circ,-)$, we have $\Phi_{\mathbf{J}}(y) \amalg y\Delta_{\mathbf{J}}(\bar{y}^{-1}ww_\circ,-) = \Delta_{\mathbf{J}}(ww_\circ,-)$, and hence $\Phi_{\mathbf{J}}(y^{-1}) \subset \Delta_{\mathbf{J}}(\bar{y}^{-1}w,-)$. By (2), we get $U_{\mathbf{J}}(\bar{y}^{-1}w,0) = {}_{\mathbf{J}}T_{y^{-1}} U_{\mathbf{J}}(w,0)$.

(4) This follows from (5.29), (2), and (3). $\square$

7. Decompositions of $U_{\mathbf{J}}^+$ into tensor products of subalgebras

In this section, we give several decompositions of $U_{\mathbf{J}}^+$ into tensor products of subalgebras, where $\mathbf{J}$ is an arbitrary non-empty subset of $\mathring{\mathbf{I}}$.

Proposition 7.1. (1) *For each* $w \in \mathring{W}_{\mathbf{J}}$, *the multiplication defines the following isomorphism of* $\mathbf{Q}(q)$*-vector spaces:*

$$U_{\mathbf{J},<}(w,-) \otimes U_{\mathbf{J}}(w,0) \otimes U_{\mathbf{J},>}(w,+) \xrightarrow{\sim} U_{\mathbf{J}}^+. \tag{7.1}$$

Moreover, the following equality holds:

$$U_{\mathbf{J}}(w,0) = A_{\mathbf{J}}(w,0). \tag{7.2}$$

(2) *The multiplication defines the following isomorphism of* $\mathbf{Q}(q)$*-algebras:*

$$\bigotimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c} \xrightarrow{\sim} U_{\mathbf{J}}. \tag{7.3}$$

(3) *The part* (1) *of Proposition* 5.7 *is still valid in the case where* $\mathbf{J}$ *is an arbitrary non-empty subset of* $\mathring{\mathbf{I}}$.

PROOF. Let us prove (1) and (2). Let s and s' be elements of $\mathscr{W}_{\mathbf{J}}^{\infty}$ such that $\Phi_{\mathbf{J}}^{\infty}([s]) = \Delta_{\mathbf{J}}(w,-)$ and $\Phi_{\mathbf{J}}^{\infty}([s']) = \Delta_{\mathbf{J}}(w,+)$. By Proposition 5.19(1), we see that $E_{s,<}^{*}$ and $\Psi E_{s',<}^{*}$ are bases of $U_{\mathbf{J},<}(w,-)$ and $U_{\mathbf{J},>}(w,+)$ respectively. Let $\preceq$ be a total order on $\mathbf{J}\times\mathbf{N}$. By Proposition 6.6, we see that $T_w(I_{\preceq}^{*})$ is a basis of $U(w,0)$. By Proposition 5.19(1), (5.29), the right part of (5.38), and the left part of (5.43), we see that the multiplication defines the following injective $\mathbf{Q}(q)$-linear mapping:

$$U_{\mathbf{J},<}(w,-)\otimes A_{\mathbf{J}}(w,0)\otimes\{A_{\mathbf{J}}(w,-)^{c}\cap U_{\mathbf{J},>}(w,+)\}\hookrightarrow U_{\mathbf{J}}^{+}. \tag{7.4}$$

By Proposition 6.8(3), we have $U_{\mathbf{J}}(w,0)\subset A_{\mathbf{J}}(w,0)$. By Lemma 5.23(1), we have $U_{\mathbf{J},>}(w,+)\subset A_{\mathbf{J}}(w,-)^{c}$. Thus we see that the multiplication defines the following injective $\mathbf{Q}(q)$-linear mapping:

$$m_1 : U_{\mathbf{J},<}(w,-)\otimes U_{\mathbf{J}}(w,0)\otimes U_{\mathbf{J},>}(w,+)\hookrightarrow U_{\mathbf{J}}^{+}. \tag{7.5}$$

Hence the elements of the subset $E_{s,<}^{*}T_w(I_{\preceq}^{*})\Psi(E_{s',<}^{*})$ of $U_{\mathbf{J}}^{+}$ are linearly independent. In the case where $\mathbf{J}$ is connected, by (5.14), we see that the set $E_{s,<}^{*}T_w(I_{\preceq}^{*})\Psi(E_{s',<}^{*})$ is a basis of $U_{\mathbf{J}}^{+}$, and hence the mapping (7.5) is bijective. To consider the general case, we write w uniquely as $w=\prod_{c=1}^{\mathrm{C}(\mathbf{J})} w_c$ with $w_c\in \mathring{W}_{\mathbf{J}_c}$. Then the multiplication defines the following isomorphism of $\mathbf{Q}(q)$-vector spaces:

$$U_{\mathbf{J}_c,<}(w_c,-)\otimes U_{\mathbf{J}_c}(w_c,0)\otimes U_{\mathbf{J}_c,>}(w_c,+)\xrightarrow{\sim} U_{\mathbf{J}_c}^{+} \tag{7.6}$$

for each $c=1,\dots,\mathrm{C}(\mathbf{J})$. By Proposition 5.20(1) and Proposition 6.6, we see that the multiplication defines the following isomorphisms of $\mathbf{Q}(q)$-vector spaces:

$$\bigotimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c,<}(w_c,-)\xrightarrow{\sim} U_{\mathbf{J},<}(w,-),\qquad \bigotimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c,>}(w_c,+)\xrightarrow{\sim} U_{\mathbf{J},>}(w,+), \tag{7.7}$$

$$\bigotimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c}(w_c,0)\xrightarrow{\sim} U_{\mathbf{J}}(w,0). \tag{7.8}$$

Therefore we have the following diagram:

$$\begin{array}{ccc} U_{\mathbf{J},<}(w,-)\otimes U_{\mathbf{J}}(w,0)\otimes U_{\mathbf{J},>}(w,+) & \xhookrightarrow{m_1} & U_{\mathbf{J}}^{+} \\ \varphi\Big\uparrow & & \Big\uparrow m_2^{+} \\ \bigotimes_{c=1}^{\mathrm{C}(\mathbf{J})}\big(U_{\mathbf{J}_c,<}(w_c,-)\otimes U_{\mathbf{J}_c}(w_c,0)\otimes U_{\mathbf{J}_c,>}(w_c,+)\big) & \xrightarrow{\sim} & \bigotimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c}^{+}. \end{array}$$

Here, m_2^{+} is defined by the multiplication and φ is defined by setting

$$\varphi(\bigotimes_{c=1}^{\mathrm{C}(\mathbf{J})} u_c(-) \otimes u_c(0) \otimes u_c(+))$$

$$:= \left(\prod_{c=1}^{\mathrm{C}(\mathbf{J})} u_c(-)\right) \otimes \left(\prod_{c=1}^{\mathrm{C}(\mathbf{J})} u_c(0)\right) \otimes \left(\prod_{c=1}^{\mathrm{C}(\mathbf{J})} u_c(+)\right),$$

where $u_c(-) \in U_{\mathbf{J}_c,<}(w_c,-)$, $u_c(0) \in U_{\mathbf{J}_c}(w_c,0)$, and $u_c(+) \in U_{\mathbf{J}_c,>}(w_c,+)$. By Lemma 5.4(4), we see that the diagram above is commutative. By (7.6)–(7.8), we see that φ is an isomorphism of $\mathbf{Q}(q)$-vector spaces, which implies the injectivity of m_2^+. By Proposition 5.5, we have the surjectivity of m_2^+. Thus both of m_2^+ and m_1 are isomorphisms of $\mathbf{Q}(q)$-vector spaces. Moreover it is easy to see that the multiplication defines the following isomorphisms of $\mathbf{Q}(q)$-vector spaces:

$$m_2^- : \bigotimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c}^- \xrightarrow{\sim} U_{\mathbf{J}}^-, \qquad m_2^0 : \bigotimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c}^0 \xrightarrow{\sim} U_{\mathbf{J}}^0. \tag{7.9}$$

Thus, by Proposition 5.7(2), we see that the multiplication defines the following isomorphism of $\mathbf{Q}(q)$-vector spaces:

$$m_2 : \bigotimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c} \xrightarrow{\sim} U_{\mathbf{J}}. \tag{7.10}$$

It is easy to see that m_2 is compatible with the standard $\mathbf{Q}(q)$-algebra structure of the tensor product $\bigotimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c}$. The equality (7.2) follows from Lemma 6.8(4), (7.1), (5.29), and Lemma 5.23(4).

Let us prove (3). The characterization of $U_{\mathbf{J}}$ in terms of the generators and the defining relations follows from the part (2) and the first assertion of Proposition 5.7(1). The equalities in (5.14) follow from the part (1) and Proposition 6.6. □

Proposition 7.2. (1) *For each $y \in W_{\mathbf{J}}$, the multiplication defines the following isomorphism of $\mathbf{Q}(q)$-vector spaces:*

$$A_{\mathbf{J}}(y) \otimes A_{\mathbf{J}}(y)^c \xrightarrow{\sim} U_{\mathbf{J}}^+. \tag{7.11}$$

(2) *Let B be an arbitrary real biconvex set in $\Delta_{\mathbf{J}+}$. Then the following equality holds:*

$$U_{\mathbf{J},<}(B) = A_{\mathbf{J}}(B) = U_{\mathbf{J},>}(B). \tag{7.12}$$

Moreover, $A_{\mathbf{J}}(B)$ is a $\mathbf{Q}(q)$-subalgebra of $U_{\mathbf{J}}^+$.

Proof. Let us prove (7.11). We may assume that $\Phi_{\mathbf{J}}(y) \subset \Delta_{\mathbf{J}}(w,-)$ for some $w \in \mathring{W}_{\mathbf{J}}$. Then there exists an element $s \in \mathscr{W}_{\mathbf{J}}^{\infty}$ such that $\Phi_{\mathbf{J}}^{\infty}([s]) = \Delta_{\mathbf{J}}(w,-)$ and $[s|_p] = y$ with $p = \ell(y)$. Let s' be an element of $\mathscr{W}_{\mathbf{J}}^*$ such that $\Phi_{\mathbf{J}}([s']) = \Delta_{\mathbf{J}}(w,+)$, and $\preceq$ a total order on $\mathbf{J} \times \mathbf{N}$. Then the product set $E_{s,<}^* T_w(I_{\preceq}^*) \Psi(E_{s',<}^*)$ is a basis of $U_{\mathbf{J}}^+$. From Lemma 5.17(1), it follows that the multiplication $A_{\mathbf{J}}(y) \otimes A_{\mathbf{J}}(y)^c \to U_{\mathbf{J}}^+$ is injective. Moreover, we see that

$$E^*_{s|_p,<} \subset A_{\mathbf{J}}(y), \qquad {}_{\mathbf{J}}T_y(E^*_{s|^p,<})T_w(I^*_{\prec})\Psi E^*_{s',<} \subset A_{\mathbf{J}}(y)^c,$$

and $E^*_{s|_p,<}\,{}_{\mathbf{J}}T_y(E^*_{s|^p,<}) = E^*_{s,<}$, where $s|_p$ is the initial p-section of s and $s|^p$ is the p-shift of s (see [7]). Therefore we see that the multiplication $A_{\mathbf{J}}(y) \otimes A_{\mathbf{J}}(y)^c \to U^+_{\mathbf{J}}$ is bijective and that the sets $E^*_{s|_p,<}$ and ${}_{\mathbf{J}}T_y(E^*_{s|^p,<})T_w(I^*_{\prec})\Psi(E^*_{s',<})$ are bases of $A_{\mathbf{J}}(y)$ and $A_{\mathbf{J}}(y)^c$, respectively. Since $E^*_{s|_p,<}$ is also a basis of $U_{\mathbf{J},<}(y)$, we get $U_{\mathbf{J},<}(y) = A_{\mathbf{J}}(y)$. Similarly, we can prove the equality $U_{\mathbf{J},>}(y) = A_{\mathbf{J}}(y)$. Hence (7.12) is proved in the case where $B = \Phi_{\mathbf{J}}(y)$ with $y \in W_{\mathbf{J}}$.

Let us prove (7.12) in the case where $B \in \mathfrak{B}^\infty_{\mathbf{J}}$. Suppose that $\Phi^\infty_{\mathbf{J}}([s]) = B$ with $s \in \mathscr{W}^\infty_{\mathbf{J}}$. Then $U_{\mathbf{J},<}([s|_p]) = A_{\mathbf{J}}([s|_p]) = U_{\mathbf{J},>}([s|_p])$ for all $p \in \mathbf{N}$. Since $U_{\mathbf{J},<}(B) = \bigcup_{p\in\mathbf{N}} U_{\mathbf{J},<}([s|_p])$ and $U_{\mathbf{J},>}(B) = \bigcup_{p\in\mathbf{N}} U_{\mathbf{J},>}([s|_p])$, we have $U_{\mathbf{J},<}(B) = U_{\mathbf{J},>}(B) = \bigcup_{p\in\mathbf{N}} A_{\mathbf{J}}([s|_p])$. It follows that $U_{\mathbf{J},<}(B) = U_{\mathbf{J},>}(B) \subset A_{\mathbf{J}}(B)$, since $A_{\mathbf{J}}([s|_p]) \subset A_{\mathbf{J}}(B)$ for all $p \in \mathbf{N}$. Let y be an arbitrary element of $W_{\mathbf{J}}(B)$. Then we have $A_{\mathbf{J}}(y) = U_{\mathbf{J},<}(y) \subset U_{\mathbf{J},<}(B)$. Thus we get $A_{\mathbf{J}}(B) \subset U_{\mathbf{J},<}(B) = U_{\mathbf{J},>}(B)$. Therefore (7.12) is valid.

Let us prove the second assertion of the part (2). Suppose that u_1 and u_2 are elements of $A_{\mathbf{J}}(B)$. By Definition 5.16, we may assume that $u_i \in A_{\mathbf{J}}(y_i)$ with $y_i \in W_{\mathbf{J}}(B)$ for $i = 1, 2$. By Lemma 3.6(1), there exists an elements $y_3 \in W_{\mathbf{J}}(B)$ such that $\Phi_{\mathbf{J}}(y_1) \cup \Phi_{\mathbf{J}}(y_2) \subset \Phi_{\mathbf{J}}(y_3)$. By Proposition 5.20(2) and the equality $U_{\mathbf{J},<}(y) = A(y)$ for $y \in W$, we see that $A_{\mathbf{J}}(y_3)$ is a $\mathbf{Q}(q)$-subalgebra of $U^+_{\mathbf{J}}$ such that $A_{\mathbf{J}}(y_1) \cup A_{\mathbf{J}}(y_2) \subset A_{\mathbf{J}}(y_3)$, and hence $u_1 + u_2$, $u_1u_2 \in A_{\mathbf{J}}(y_3) \subset A_{\mathbf{J}}(B)$. Therefore $A_{\mathbf{J}}(B)$ is a $\mathbf{Q}(q)$-subalgebra of $U^+_{\mathbf{J}}$. □

Proposition 7.3. *For each $w \in \mathring{W}_{\mathbf{J}}$, the multiplication defines the following isomorphisms of vector spaces:*

$$A_{\mathbf{J}}(w,-) \otimes A_{\mathbf{J}}(w,0) \otimes A_{\mathbf{J}}(w,+) \xrightarrow{\sim} U^+_{\mathbf{J}}, \tag{7.13}$$

$$A_{\mathbf{J}}(w,0) \otimes A_{\mathbf{J}}(w,+) \xrightarrow{\sim} A_{\mathbf{J}}(w,-)^c, \qquad A_{\mathbf{J}}(w,-) \otimes A_{\mathbf{J}}(w,0) \xrightarrow{\sim} A_{\mathbf{J}}(w,+)^c, \tag{7.14}$$

$$A_{\mathbf{J}}(w,-) \otimes A_{\mathbf{J}}(w,-)^c \xrightarrow{\sim} U^+_{\mathbf{J}}, \qquad A_{\mathbf{J}}(w,+)^c \otimes A_{\mathbf{J}}(w,+) \xrightarrow{\sim} U^+_{\mathbf{J}}. \tag{7.15}$$

Proof. The isomorphism (7.13) follows from (7.1), (7.2), and (7.12). The left isomorphism in (7.14) follows from (7.13), Lemma 5.23(1), and Proposition 6.8(2). The right isomorphism in (7.14) follows from (7.13), Lemma 5.23(2), and Proposition 6.8(3). The isomorphisms in (7.15) follow from (7.13)(7.14). □

Proposition 7.4. *Let B and B_1 be arbitrary real biconvex sets in $\Delta_{\mathbf{J}+}$ such that $B \subset B_1$. Then the multiplication defines the following isomorphisms of $\mathbf{Q}(q)$-vector spaces:*

$$A_{\mathbf{J}}(B) \otimes \{A_{\mathbf{J}}(B)^c \cap A_{\mathbf{J}}(B_1)\} \xrightarrow{\sim} A_{\mathbf{J}}(B_1), \tag{7.16}$$

$$\{A_{\mathbf{J}}(B)^c \cap A_{\mathbf{J}}(B_1)\} \otimes A_{\mathbf{J}}(B_1)^c \xrightarrow{\sim} A_{\mathbf{J}}(B)^c, \tag{7.17}$$

$$A_{\mathbf{J}}(B) \otimes A_{\mathbf{J}}(B)^c \xrightarrow{\sim} U_{\mathbf{J}}^+. \tag{7.18}$$

Proof. Let us prove (7.16). By Proposition 5.19(1) and (7.12), we see that the multiplication defines the following injective $\mathbf{Q}(q)$-linear mapping:

$$A_{\mathbf{J}}(B) \otimes A_{\mathbf{J}}(B)^c \hookrightarrow U_{\mathbf{J}}^+. \tag{7.19}$$

Since $A_{\mathbf{J}}(B) \subset A_{\mathbf{J}}(B_1)$, we see that the multiplication defines the following injective $\mathbf{Q}(q)$-linear mapping:

$$A_{\mathbf{J}}(B) \otimes \{A_{\mathbf{J}}(B)^c \cap A_{\mathbf{J}}(B_1)\} \hookrightarrow A_{\mathbf{J}}(B_1). \tag{7.20}$$

Hence it suffices to show the surjectivity of (7.20). In the case where $B \in \mathfrak{B}_{\mathbf{J}}$ and $B = \Phi_{\mathbf{J}}(y)$ with $y \in W_{\mathbf{J}}$, the surjectivity of (7.20) follows from Proposition 5.20(2) and (7.12). We next suppose that $B \in \mathfrak{B}_{\mathbf{J}}^{\infty}$ and $\Phi_{\mathbf{J}}^{\infty}([\boldsymbol{s}]) = B$ with $\boldsymbol{s} \in \mathscr{W}_{\mathbf{J}}^{\infty}$. Let u be an arbitrary weight vector of $A_{\mathbf{J}}(B_1)$ with weight β. We use the induction on $\mathrm{wt}(u) = \beta$. In the case where $u \in A_{\mathbf{J}}(B)^c \cap A_{\mathbf{J}}(B_1)$, there is nothing to prove. So we may assume that there exists $y \in W\mathbf{J}(B)\backslash\{1\}$ such that $u = \sum_{\lambda \in \Lambda} X_\lambda Y_\lambda$, where X_λ and Y_λ are weight vectors of $A_{\mathbf{J}}(y)$ and $A_{\mathbf{J}}(y)^c \cap A_{\mathbf{J}}(B_1)$ respectively with $\mathrm{wt}(X_\lambda) > 0$. Since all $\mathrm{wt}(Y_\lambda)$ are lower than β, by the induction, we see that all Y_λ belong to the image of the (7.20). Since $A_{\mathbf{J}}(y)$ is a subalgebra of $A_{\mathbf{J}}(B)$, we see that $u = \sum_{\lambda \in \Lambda} X_\lambda Y_\lambda$ belongs to the image of the mapping (7.20).

Let us prove (7.18), i.e., the surjectivity of (7.19). Let w be an element of $W_{\mathbf{J}}$ such that $B \subset \Delta_{\mathbf{J}}(w,-)$. By using (7.16) with $B_1 = \Delta_{\mathbf{J}}(w,-)$, we see that $A_{\mathbf{J}}(B) \otimes \{A_{\mathbf{J}}(B)^c \cap A_{\mathbf{J}}(w,-)\} \xrightarrow{\sim} A_{\mathbf{J}}(w,-)$. Moreover, since $A_{\mathbf{J}}(w,-)^c \subset A_{\mathbf{J}}(B)^c$, we see that $\{A_{\mathbf{J}}(B)^c \cap A_{\mathbf{J}}(w,-)\} \otimes A_{\mathbf{J}}(w,-)^c \hookrightarrow A_{\mathbf{J}}(B)^c$. Combining with (7.13), we get (7.18). The assertion (7.17) can be proved by using (7.16) and (7.18). □

Lemma 7.5. *Let y be an arbitrary element of $W_{\mathbf{J}}$, $\boldsymbol{s}$ an element of $\mathscr{W}_{\mathbf{J}}$ such that $[\boldsymbol{s}] = y$, and ε an element of $\mathring{P}^{\vee}$ such that $(\varepsilon|\alpha_i) > 0$ for all $i \in \mathring{\mathbf{I}}\backslash\mathbf{J}$ and $(\varepsilon|\alpha_j) = 0$ for all $j \in \mathbf{J}$. Then, for each $n \in \mathbf{Z}_{\geq 0}$, we have*

$$A([\tilde{\boldsymbol{s}}]t_\varepsilon^n) \cap U_{\mathbf{J}}^+ = A_{\mathbf{J}}(y), \qquad A([\tilde{\boldsymbol{s}}]t_\varepsilon^n)^c \cap U_{\mathbf{J}}^+ = A_{\mathbf{J}}(y)^c. \tag{7.21}$$

Proof. By the definitions of the action of ${}_{\mathbf{J}}T_y$ on $U_{\mathbf{J}}$ and the subalgebra $A([\tilde{\boldsymbol{s}}]t_\varepsilon^n)$ of U^+, we have

$$A([\tilde{\boldsymbol{s}}]t_\varepsilon^n) \cap U_{\mathbf{J}}^+ = \{u \in U_{\mathbf{J}}^+ \mid T_\varepsilon^{-n}{}_{\mathbf{J}}T_y^{-1}(u) \in U^{\leq 0}\} = A_{\mathbf{J}}(y),$$

where the second equality follows from Lemma 5.4(3)(i) and Lemma 5.14(1). The proof of the right equality in (7.21) is similar. □

Proposition 7.6. *Let B be a real biconvex set in $\Delta_{\mathbf{J}+}$, and set $\tilde{B} := B \amalg \Delta^{\mathbf{J}}(1,-)$. Then $\tilde{B}$ is a real biconvex set and the following equalities hold:*

$$A_{\mathbf{J}}(B) = A(\tilde{B}) \cap U_{\mathbf{J}}^{+}, \qquad A_{\mathbf{J}}(B)^{c} = A(\tilde{B})^{c} \cap U_{\mathbf{J}}^{+}. \tag{7.22}$$

Proof. We may assume that $\mathbf{J} \subsetneq \mathring{\mathbf{I}}$. Suppose that $B \in \mathfrak{B}_{\mathbf{J}}$. Then $B = \Phi_{\mathbf{J}}(y)$ for some $y \in W_{\mathbf{J}}$. Let s be an element of $\mathscr{W}_{\mathbf{J}}$ such that $[s] = y$, and ε an element of $\mathring{P}^{\vee}$ such that $(\varepsilon|\alpha_i) > 0$ for all $i \in \mathring{\mathbf{I}} \backslash \mathbf{J}$ and $(\varepsilon|\alpha_j) = 0$ for all $j \in \mathbf{J}$. By Proposition 3.7, we see that $\bigcup_{n \geq 0} \Phi([\tilde{s}]t_{\varepsilon}^{n}) = \tilde{B}$, hence $\tilde{B} \in \mathfrak{B}^{\infty}$ by Lemma 3.6(2). Thus, by Lemma 7.5, we have

$$A(\tilde{B}) \cap U_{\mathbf{J}}^{+} = \{\textstyle\bigcup_{n \geq 0} A([\tilde{s}]t_{\varepsilon}^{n})\} \cap U_{\mathbf{J}}^{+} = \bigcup_{n \geq 0}\{A([\tilde{s}]t_{\varepsilon}^{n}) \cap U_{\mathbf{J}}^{+}\} = A_{\mathbf{J}}(B),$$

$$A(\tilde{B})^{c} \cap U_{\mathbf{J}}^{+} = \{\textstyle\bigcap_{n \geq 0} A([\tilde{s}]t_{\varepsilon}^{n})^{c}\} \cap U_{\mathbf{J}}^{+} = \bigcap_{n \geq 0}\{A([\tilde{s}]t_{\varepsilon}^{n})^{c} \cap U_{\mathbf{J}}^{+}\} = A_{\mathbf{J}}(B)^{c}.$$

Suppose that $B \in \mathfrak{B}_{\mathbf{J}}^{\infty}$. Let s be an element of $\mathscr{W}_{\mathbf{J}}^{\infty}$ such that $\Phi_{\mathbf{J}}^{\infty}([s]) = B$. By the definitions of $A_{\mathbf{J}}([s|_p])$ and $A_{\mathbf{J}}([s|_p])^{c}$, for each $p \in \mathbf{N}$, we see that $A_{\mathbf{J}}([s|_p]) = A(\widetilde{[s|_p]}) \cap U_{\mathbf{J}}^{+}$ and $A_{\mathbf{J}}([s|_p])^{c} = A(\widetilde{[s|_p]})^{c} \cap U_{\mathbf{J}}^{+}$. By Lemma 3.5(2), we have $\bigcup_{p \in \mathbf{N}} \Phi(\widetilde{[s|_p]}) = \tilde{B}$, and hence $\tilde{B} \in \mathfrak{B}^{\infty}$ by Lemma 3.6(2). Thus we get

$$A_{\mathbf{J}}(B) = \textstyle\bigcup_{p \in \mathbf{N}} A_{\mathbf{J}}([s|_p]) = \{\bigcup_{p \in \mathbf{N}} A(\widetilde{[s|_p]})\} \cap U_{\mathbf{J}}^{+} = A(\tilde{B}) \cap U_{\mathbf{J}}^{+},$$

$$A_{\mathbf{J}}(B)^{c} = \textstyle\bigcap_{p \in \mathbf{N}} A_{\mathbf{J}}([s|_p])^{c} = \{\bigcap_{p \in \mathbf{N}} A(\widetilde{[s|_p]})^{c}\} \cap U_{\mathbf{J}}^{+} = A(\tilde{B})^{c} \cap U_{\mathbf{J}}^{+}. \quad \square$$

8. Convex bases of $U_{\mathbf{J}}^{+}$

The aim of this section is to construct convex bases of $U_{\mathbf{J}}^{+}$ associated with all convex orders on $\Delta_{\mathbf{J}+}$, where $\mathbf{J}$ is an arbitrary non-empty subset of $\mathring{\mathbf{I}}$.

Proposition 8.1. *Let C_1 and C_2 be real biconvex sets in $\Delta_{\mathbf{J}+}$ such that $C_1 \subset C_2$. Write C_1 and C_2 uniquely as $C_1 = \nabla_{\mathbf{J}}(\mathbf{K}, w, y)$ and $C_2 = C_1 \amalg wyB$ with $\mathbf{K} \subset \mathbf{J}$, $w \in \mathring{W}_{\mathbf{J}}^{\mathbf{K}}$, $y \in W_{\mathbf{K}}$, and $B \in \mathfrak{B}_{\mathbf{K}}^{*}$. Then the following equality holds:*

$$A_{\mathbf{J}}(C_1)^{c} \cap A_{\mathbf{J}}(C_2) = T_w \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(B) \tag{8.1}$$

and the multiplication defines the following isomorphism of $\mathbf{Q}(q)$-vector spaces:

$$A_{\mathbf{J}}(C_1) \otimes T_w \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(B) \xrightarrow{\sim} A_{\mathbf{J}}(C_2). \tag{8.2}$$

Proof. In the case where $C_1 \in \mathfrak{B}_{\mathbf{J}}$, we have $\mathbf{K} = \mathbf{J}$ and $C_1 = \Phi_{\mathbf{J}}(y)$, which implies that $w = 1$ and $B = y^{-1}\{C_1 \backslash \Phi_{\mathbf{J}}(y)\}$. Thus $A_{\mathbf{J}}(C_1)^{c} \cap A_{\mathbf{J}}(C_2) = {}_{\mathbf{J}}T_y A_{\mathbf{J}}(B)$ by Proposition 5.20(2) and Proposition 7.2(2). Therefore (8.1) is valid in this case.

We next suppose that $C_1 \in \mathfrak{B}_{\mathbf{J}}^{\infty}$. Then $\mathbf{K} \subsetneq \mathbf{J}$. Let ε be an element of $\overset{\circ}{P}{}^{\vee}$ such that $(\varepsilon|\alpha_i) > 0$ for all $j \in \overset{\circ}{\mathbf{I}} \backslash \mathbf{K}$ and $(\varepsilon|\alpha_k) = 0$ for all $k \in \mathbf{K}$, $\boldsymbol{s}$ an element of $\mathscr{W}_{\mathbf{K}}$ such that $[\boldsymbol{s}] = y$, and $\boldsymbol{s}_2$ an element of $\mathscr{W}_{\mathbf{K}}^*$ such that $\Phi_{\mathbf{K}}^*([\boldsymbol{s}_2]) = B$. Set $\widetilde{C_1} := C_1 \amalg \Delta^{\mathbf{J}}(1,-)$ and $\widetilde{C_2} := C_2 \amalg \Delta^{\mathbf{J}}(1,-)$. Then $\widetilde{C_1} = \nabla(\mathbf{K}, w, y)$ and $\widetilde{C_2} = \widetilde{C_1} \amalg wyB$. Thus, by Proposition 3.7, we have

$$\widetilde{C_1} = \bigcup_{n \geq 0} \Phi(w[\tilde{\boldsymbol{s}}]t_{\varepsilon}^n), \qquad \widetilde{C_2} = \bigcup_{n \geq 0} \bigcup_{p=1}^{\ell(\boldsymbol{s}_2)} \Phi(w[\tilde{\boldsymbol{s}}]t_{\varepsilon}^n[\widetilde{\boldsymbol{s}_2|_p}]). \tag{8.3}$$

By Proposition 7.6, it follows that

$$A_{\mathbf{J}}(C_1)^c = \bigcap_{n \geq 0} \{A(w[\tilde{\boldsymbol{s}}]t_{\varepsilon}^n)^c \cap U_{\mathbf{J}}^+\}, \tag{8.4}$$

$$A_{\mathbf{J}}(C_2) = \bigcup_{n \geq 0} \bigcup_{p=1}^{\ell(\boldsymbol{s}_2)} \{A(w[\tilde{\boldsymbol{s}}]t_{\varepsilon}^n[\widetilde{\boldsymbol{s}_2|_p}]) \cap U_{\mathbf{J}}^+\}, \tag{8.5}$$

where

$$A(w[\tilde{\boldsymbol{s}}]t_{\varepsilon}^n)^c \cap U_{\mathbf{J}}^+ = \{u \in U_{\mathbf{J}}^+ \mid T_{\varepsilon}^{-n} {}_{\mathbf{J}}T_y^{-1} T_w^{-1}(u) \in U^+\}, \tag{8.6}$$

$$A(w[\tilde{\boldsymbol{s}}]t_{\varepsilon}^n[\widetilde{\boldsymbol{s}_2|_p}]) \cap U_{\mathbf{J}}^+ = \{u \in U_{\mathbf{J}}^+ \mid T_{[\widetilde{\boldsymbol{s}_2|_p}]}^{-1} T_{\varepsilon}^{-n} {}_{\mathbf{J}}T_y^{-1} T_w^{-1}(u) \in U^{\leq 0}\}. \tag{8.7}$$

Here, by Lemma 5.4(3), we see the fact that $T_{\varepsilon}^{-n}(x) = x$ for all $x \in A_{\mathbf{K}}(B)$. Combining the fact with (8.4) and (8.6), we get $T_w \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(B) \subset A_{\mathbf{J}}(C_1)^c$. Combining the fact with (8.7) and the equality ${}_{\mathbf{K}}T_{[\boldsymbol{s}_2|_p]} = T_{[\widetilde{\boldsymbol{s}_2|_p}]}|_{U_{\mathbf{K}}}$, we see that $T_w \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}([\boldsymbol{s}_2|_p]) \subset A(w[\tilde{\boldsymbol{s}}]t_{\varepsilon}^n[\widetilde{\boldsymbol{s}_2|_p}]) \cap U_{\mathbf{J}}^+$ for all $1 \leq p \leq \ell(\boldsymbol{s}_2)$, and hence $T_w \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(B) \subset A_{\mathbf{J}}(C_2)$ by (8.5). Therefore we get $T_w \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(B) \subset A_{\mathbf{J}}(C_1)^c \cap A_{\mathbf{J}}(C_2)$. By (7.16) in Proposition 7.4, we see that the multiplication $m : A_{\mathbf{J}}(C_1) \otimes \{A_{\mathbf{J}}(C_1)^c \cap A_{\mathbf{J}}(C_2)\} \to A_{\mathbf{J}}(C_2)$ is an isomorphism of vector spaces, which induces the injective linear mapping:

$$\varphi : A_{\mathbf{J}}(C_1) \otimes T_w \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(B) \hookrightarrow A_{\mathbf{J}}(C_2). \tag{8.8}$$

Since $C_1 \amalg wyB = C_2$, by Proposition 5.19(1) and (7.12), we see that

$$\dim_{\mathbf{Q}(q)}(\mathrm{Im}\,\varphi)_{\mu} = \dim_{\mathbf{Q}(q)} A_{\mathbf{J}}(C_2)_{\mu} = \#\left\{\mathbf{c} : C_2 \to \mathbf{Z}_+ \mid \sum_{\beta \in C_2} \mathbf{c}(\beta)\beta = \mu\right\}$$

for each $\mu \in Q_{\mathbf{J}+}$. This implies that $\varphi = m$ with the equality (8.1).

The (8.2) follows immediately from (8.1) and (7.16). □

Corollary 8.2. *Suppose that B is a real biconvex set in $\Delta_{\mathbf{J}+}$ satisfying $B \subset \Delta_{\mathbf{J}}(w,-)$ for some $w \in \overset{\circ}{W}_{\mathbf{J}}$ and that*

$$B = \nabla_{\mathbf{J}}(\mathbf{K}, w^{\mathbf{K}}, y), \qquad B \amalg w^{\mathbf{K}} y \Delta_{\mathbf{K}}(\bar{y}^{-1} w_{\mathbf{K}}, -) = \Delta_{\mathbf{J}}(w,-)$$

for some $\mathbf{K} \subset \mathbf{J}$ and $y \in W_{\mathbf{K}}$. Then the multiplication defines the following isomorphisms of $\mathbf{Q}(q)$-vector spaces:

$$A_{\mathbf{J}}(B) \otimes T_{w^{\mathbf{K}}} \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(\bar{y}^{-1}w_{\mathbf{K}}, -) \xrightarrow{\sim} A_{\mathbf{J}}(w, -), \tag{8.9}$$

$$T_{w^{\mathbf{K}}} \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(\bar{y}^{-1}w_{\mathbf{K}}, -) \otimes A_{\mathbf{J}}(w, -)^c \xrightarrow{\sim} A_{\mathbf{J}}(B)^c. \tag{8.10}$$

Proof. By (8.1), we have

$$A_{\mathbf{J}}(B)^c \cap A_{\mathbf{J}}(w, -) = T_{w^{\mathbf{K}}} \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(\bar{y}^{-1}w_{\mathbf{K}}, -). \tag{8.11}$$

Hence, (8.9) and (8.10) follow from (7.16) and (7.17) respectively.

Definition 8.3. Let Λ be a totally ordered set with $\preceq$ the total order on Λ. Then we call a subset $I \subset \Lambda$ a *section* of Λ with respect to $\preceq$ if $[\lambda, \mu]_{\preceq} \subset I$ for all $\lambda, \mu \in I$ satisfying $\lambda \prec \mu$, where $[\lambda, \mu]_{\preceq} := \{\nu \in \Lambda \mid \lambda \preceq \nu \preceq \mu\}$. If, in addition, $I \prec (\Lambda \backslash I)$ then we call I an *initial section* of Λ with respect to $\preceq$. Moreover, for each $\lambda \in \Lambda$ we set $(*, \lambda]_{\preceq} := \{\mu \in \Lambda \mid \mu \preceq \lambda\}$, $(\lambda, *)_{\preceq} := \Lambda \backslash (*, \lambda]_{\preceq}$, $(*, \lambda)_{\preceq} := \{\mu \in \Lambda \mid \mu \prec \lambda\}$, and $[\lambda, *)_{\preceq} := \Lambda \backslash (*, \lambda)_{\preceq}$.

Let A be an associative algebra with the unit 1 over a commutative ring R, and $\{X_\lambda \mid \lambda \in \Lambda\}$ a subset of A indexed by the totally ordered set Λ. Then we call the set $X^*_{\prec} = X^*_{\prec}(\Lambda)$ (resp. $X^*_{\succ} = X^*_{\succ}(\Lambda)$) (see Definition 5.18) a *convex basis* of A if $X^*_{\prec}(I)$ (resp. $X^*_{\succ}(I)$) is a free R-basis of the R-subalgebra $\langle X_\lambda \mid \lambda \in I \rangle_{R\text{-}alg}$ of A for each section I of Λ with respect to $\preceq$.

Theorem 8.4. *Let $(n, \mathbf{K}_\bullet, y_\bullet, \mathbf{s}_\bullet)$ be an element of $\mathbf{N}_{\#\mathbf{J}} \times \mathscr{C}_n\mathbf{J} \times W_{\mathbf{K}_\bullet} \times \mathscr{W}^{\infty}_{\mathbf{K}_\bullet}$ satisfying the conditions* (3.1) *and* (3.2) *(cf. Theorem* 3.2(2)*), and $\preceq$ the convex order on $\Delta_{\mathbf{J}}(w, -)$ associated with the $(n, \mathbf{K}_\bullet, y_\bullet, \mathbf{s}_\bullet)$. For each $\alpha \in \Delta_{\mathbf{J}}(w, -)$, we define a weight vector $E_\alpha = E_{\preceq, \alpha} \in U^+_{\mathbf{J}}$ with weight α by setting*

$$E_\alpha = E_{\preceq, \alpha} := T_{w^{\mathbf{K}_{i-1}}} \cdot {}_{\mathbf{J}}T_{y_{i-1}} \cdot {}_{\mathbf{J}}T_{\mathbf{s}_{i-1}(1)} \cdot \cdots \cdot {}_{\mathbf{J}}T_{\mathbf{s}_{i-1}(p-1)}(E_{\mathbf{s}_{i-1}(p)}), \tag{8.12}$$

where $\alpha = w^{\mathbf{K}_{i-1}} y_{i-1} \phi_{\mathbf{s}_{i-1}}(p)$ with $i \in \mathbf{N}_n$ and $p \in \mathbf{N}$. Then each of the sets $E^*_{\prec}(\Delta_{\mathbf{J}}(w, -))$ and $E^*_{\succ}(\Delta_{\mathbf{J}}(w, -))$ (see Definition 5.18*) is a convex basis of the $\mathbf{Q}(q)$-algebra $A_{\mathbf{J}}(w, -)$ and of the $\mathscr{A}_1$-algebra ${}_{\mathscr{A}_1}A_{\mathbf{J}}(w, -) := A_{\mathbf{J}}(w, -) \cap {}_{\mathscr{A}_1}U^+$. Moreover, if I is an initial section with respect to $\preceq$, then I is a real biconvex set in $\Delta_{\mathbf{J}+}$ and the following equalities hold:*

$$\langle E_\alpha \mid \alpha \in I \rangle_{\mathbf{Q}(q)\text{-}alg} = A_{\mathbf{J}}(I), \tag{8.13}$$

$$\langle E_\alpha \mid \alpha \in I^c \rangle_{\mathbf{Q}(q)\text{-}alg} = A_{\mathbf{J}}(I)^c \cap A_{\mathbf{J}}(w, -), \tag{8.14}$$

where $I^c := \Delta_{\mathbf{J}}(w, -) \backslash I$.

Proof. By (3.1)(3.2) in Theorem 3.2(2), we have

$$\Delta(w, -) = \coprod_{i=1}^{n} w^{\mathbf{K}_{i-1}} y_{i-1} \Phi^{\infty}_{\mathbf{K}_{i-1}}([\mathbf{s}_{i-1}]), \tag{8.15}$$

$$C_{i-1} \amalg w^{\mathbf{K}_{i-1}} y_{i-1} \Phi^{\infty}_{\mathbf{K}_{i-1}}([\mathbf{s}_{i-1}]) = C_i \in \mathfrak{B}^{\infty}_{\mathbf{J}} \qquad \text{for each } 1 \le i \le n, \tag{8.16}$$

where $y_0 := 1$ and $C_0 := \emptyset$. We set $B_{i-1} := \Phi^{\infty}_{\mathbf{K}_{i-1}}([s_{i-1}])$ for each $i \in \mathbf{N}_n$. Then, by (8.2) and (8.16), we see that the multiplication defines the following $\mathbf{Q}(q)$-linear isomorphism:

$$A_{\mathbf{J}}(C_{i-1}) \otimes T_{w^{\mathbf{K}_{i-1}}} \cdot {}_{\mathbf{J}}T_{y_{i-1}} A_{\mathbf{K}_{i-1}}(B_{i-1}) \xrightarrow{\sim} A_{\mathbf{J}}(C_i) \tag{8.17}$$

for each $i \in \mathbf{N}_n$. Since $C_n = \Delta_{\mathbf{J}}(w, -)$ and $C_1 = B_0$, the multiplication defines the following $\mathbf{Q}(q)$-linear isomorphisms:

$$\bigotimes_{j=1}^{n} T_{w^{\mathbf{K}_{j-1}}} \cdot {}_{\mathbf{J}}T_{y_{j-1}} A_{\mathbf{K}_{j-1}}(B_{j-1}) \xrightarrow{\sim} A_{\mathbf{J}}(w, -), \tag{8.18}$$

$$\bigotimes_{j=1}^{i} T_{w^{\mathbf{K}_{j-1}}} \cdot {}_{\mathbf{J}}T_{y_{j-1}} A_{\mathbf{K}_{j-1}}(B_{j-1}) \xrightarrow{\sim} A_{\mathbf{J}}(C_i), \tag{8.19}$$

$$\bigotimes_{j=i+1}^{n} T_{w^{\mathbf{K}_{j-1}}} \cdot {}_{\mathbf{J}}T_{y_{j-1}} A_{\mathbf{K}_{j-1}}(B_{j-1}) \xrightarrow{\sim} A_{\mathbf{J}}(C_i)^c \cap A_{\mathbf{J}}(w, -). \tag{8.20}$$

Here, $A_{j-1} := T_{w^{\mathbf{K}_{j-1}}} \cdot {}_{\mathbf{J}}T_{y_{j-1}} A_{\mathbf{K}_{j-1}}(B_{j-1})$ is located on the left side of $A_{j'-1}$ in the tensor products above if $j < j'$. By (8.18), we see that $E^*_{\prec}(\Delta_{\mathbf{J}}(w, -))$ is a basis of $A_{\mathbf{J}}(w, -)$. Moreover, by Lemma 4.4(1) we see that $E^*_{\prec}(\Delta_{\mathbf{J}}(w, -))$ is a subset of ${}_{\mathscr{A}_1}A_{\mathbf{J}}(w, -) \backslash (q-1)_{\mathscr{A}_1}A_{\mathbf{J}}(w, -)$, and hence that the set $E^*_{\prec}(\Delta_{\mathbf{J}}(w, -))$ is also a basis of ${}_{\mathscr{A}_1}A_{\mathbf{J}}(w, -)$ over $\mathscr{A}_1$ by Proposition 4.1 and Lemma 4.2.

We next prove (8.13)(8.14). Since I is an initial section, it is easy to see that I is a real biconvex set in $\Delta_{\mathbf{J}+}$. Let us consider the case where $I = (*, \alpha]_{\preceq}$, and let $i \in \mathbf{N}_n$ and $p \in \mathbf{N}$ be unique elements such that $\alpha = w^{\mathbf{K}_{i-1}} y_{i-1} \phi_{s_{i-1}}(p)$. We put $x := [s_{i-1}|_p]$ and $B'_{i-1} := x^{-1}\{B_{i-1} \backslash \Phi_{\mathbf{K}_{i-1}}(x)\}$. Then we see that

$$I = C_{i-1} \amalg w^{\mathbf{K}_{i-1}} y_{i-1} \Phi_{\mathbf{K}_{i-1}}(x), \qquad C_i = I \amalg w^{\mathbf{K}_{i-1}} y_{i-1} x B'_{i-1}. \tag{8.21}$$

By (8.2) and the left equality in (8.21), we see that the multiplication defines the following $\mathbf{Q}(q)$-linear isomorphism:

$$A_{\mathbf{J}}(C_{i-1}) \otimes T_{w^{\mathbf{K}_{i-1}}} \cdot {}_{\mathbf{J}}T_{y_{i-1}} A_{\mathbf{K}_{i-1}}(x) \xrightarrow{\sim} A_{\mathbf{J}}(I). \tag{8.22}$$

By (8.19) with i replaced by $i-1$ and (8.22), we see that $E_{\prec}(I)$ is a basis of $A_{\mathbf{J}}(I)$ and that (8.13) holds for $I = (*, \alpha]_{\preceq}$. By (8.1) and the right equality in (8.21), we have

$$A_{\mathbf{J}}(I)^c \cap A_{\mathbf{J}}(C_i) = T_{w^{\mathbf{K}_{i-1}}} \cdot {}_{\mathbf{J}}T_{y_{i-1}} \cdot {}_{\mathbf{J}}T_x A_{\mathbf{K}_{i-1}}(B'_{i-1}). \tag{8.23}$$

Since $I \subset C_i \subset \Delta_{\mathbf{J}}(w, -)$, it follows from (7.17) that the multiplication defines the following $\mathbf{Q}(q)$-linear isomorphism:

$$\{A_{\mathbf{J}}(I)^c \cap A_{\mathbf{J}}(C_i)\} \otimes \{A_{\mathbf{J}}(C_i)^c \cap A_{\mathbf{J}}(w, -)\} \xrightarrow{\sim} A_{\mathbf{J}}(I)^c \cap A_{\mathbf{J}}(w, -). \tag{8.24}$$

By (8.20)(8.23)(8.24), we see that $E^*_{\prec}(I^c)$ is a basis of $A_{\mathbf{J}}(I)^c \cap A_{\mathbf{J}}(w, -)$ and that (8.14) holds for $I = (*, \alpha]_{\preceq}$. The assertion for $I = (*, \alpha)_{\preceq}$ can be proved similarly.

For each $\alpha \preceq \beta$, we see that $E^*_{\prec}([\alpha,*)_{\preceq}) \cap E^*_{\prec}((*,\beta]_{\preceq}) = E^*_{\prec}([\alpha,\beta]_{\preceq})$ and

$$\langle E_\eta \mid \eta \in [\alpha,*)_{\preceq}\rangle_{\mathbf{Q}(q)\text{-}alg} \cap \langle E_\eta \mid \eta \in (*,\beta]_{\preceq}\rangle_{\mathbf{Q}(q)\text{-}alg} = \langle E_\eta \mid \eta \in [\alpha,\beta]_{\preceq}\rangle_{\mathbf{Q}(q)\text{-}alg}.$$

Thus $E^*_{\prec}(I)$ is a basis of $\langle E_\eta \mid \eta \in I\rangle_{\mathbf{Q}(q)\text{-}alg}$ for $I = [\alpha,\beta]_{\preceq}$. Similarly, we can prove the assertion for any sections with respect to $\preceq$. Therefore $E^*_{\prec}(\Delta_{\mathbf{J}}(w,-))$ is a convex basis of $A_{\mathbf{J}}(w,-)$, and hence the set is a convex basis of ${}_{\mathscr{A}_1}A_{\mathbf{J}}(w,-)$ over $\mathscr{A}_1$ by Proposition 4.1 and Lemma 4.2.

The proof of the assertions for $E^*_{\succ}(\Delta_{\mathbf{J}}(w,-))$ is quite similar. □

Proposition 8.5. *The following equalities hold:*

$$A_{\mathbf{J}}(1,+) = \langle x^+_{j,m} \mid j \in \mathbf{J}, m \in \mathbf{Z}_+\rangle_{\mathbf{Q}(q)\text{-}alg}, \tag{8.25}$$

$$A_{\mathbf{J}}(1,-) = \langle E_{\delta-\varepsilon}, x^-_{j,n} \mid \varepsilon \in \mathring{\Delta}_{\mathbf{J}+}, j \in \mathbf{J}, n \in \mathbf{N}\rangle_{\mathbf{Q}(q)\text{-}alg}, \tag{8.26}$$

where both $x^+_{j,m}$ and $x^-_{j,n}$ are introduced in Definition 6.1, and $E_{\delta-\varepsilon}$ is introduced in Definition 5.2. Moreover, for each $w \in \mathring{W}$, the following inclusions hold:

$$[A_{\mathbf{J}}(w,\pm), A_{\mathbf{J}}(w,0)] \subset A_{\mathbf{J}}(w,\pm). \tag{8.27}$$

Proof. Since the proof of (8.26) is similar to that of (8.25), we prove only (8.25). Set $X^+_{\mathbf{J}} := \langle x^+_{j,m} \mid j \in \mathbf{J}, m \in \mathbf{Z}_+\rangle_{\mathbf{Q}(q)\text{-}alg}$. Then, by Lemma 6.2(2) we have $A_{\mathbf{J}}(1,+) \supset X^+_{\mathbf{J}}$. To prove the opposite inclusion $A_{\mathbf{J}}(1,+) \subset X^+_{\mathbf{J}}$, let λ be an element of $\mathring{Q}^\vee_{\mathbf{J}}\backslash\{0\}$ such that $\lambda = \sum_{j\in\mathbf{J}} k_j\varepsilon_j$ with $k_j \in \mathbf{N}$ for all $j \in \mathbf{J}$, and $s_1, s_2, \ldots, s_n$ elements of $S_{\mathbf{J}}$ such that $s_1s_2\ldots s_n = t_{-\lambda}$ with $n = \ell_{\mathbf{J}}(t_{-\lambda})$. Here, we define an infinite sequence $\boldsymbol{s} = (\boldsymbol{s}(p))_{p\in\mathbf{N}} \in S^{\mathbf{N}}_{\mathbf{J}}$ by setting $\boldsymbol{s}(p) := s_{\bar{p}}$ for each $p \in \mathbf{N}$, where $\bar{p} \in \mathbf{N}_n$ such that $\bar{p} \equiv p \bmod n$. Then the sequence $\boldsymbol{s}$ is an element of $\mathscr{W}^\infty_{\mathbf{J}}$ such that $\Phi^\infty_{\mathbf{J}}([\boldsymbol{s}]) = \Delta_{\mathbf{J}}(w,+)$, and hence the convex order $\preceq$ on $\Delta_{\mathbf{J}}(w,+)$ associated with $\boldsymbol{s}$ is of 1-row type (see Theorem 3.2 and Remark 3.3). Since $A_{\mathbf{J}}(1,+) = \langle \Psi E_{\preceq,\alpha} \mid \alpha \in \Delta_{\mathbf{J}}(1,+)\rangle_{\mathbf{Q}(q)\text{-}alg}$, it suffices to show that $\Psi E_{\preceq,\alpha} \in X^+_{\mathbf{J}}$ for all $\alpha \in \Delta_{\mathbf{J}}(1,+)$. We use the induction on $\mathrm{ht}(\bar{\alpha})$. Firstly, we consider the case where $\mathrm{ht}(\bar{\alpha}) = 1$. Then $\alpha = m\delta + \alpha_j$ with $(j,m) \in \mathbf{J} \times \mathbf{Z}_+$. Hence, by Lemma 6.2(3) we see that $\Psi E_{\preceq,m\delta+\alpha_j} = x^+_{j,m} \in X^+_{\mathbf{J}}$. Secondly, we consider the case where $\mathrm{ht}(\bar{\alpha}) \geq 2$. Let $[\beta,\gamma]_{\preceq}$ be a minimal section of $\Delta_{\mathbf{J}}(1,+)$ satisfying $\alpha = \beta + \gamma$. By Theorem 8.4 and Proposition 4.1, we see that there exist elements $c_1, c_2 \in \mathscr{A}_1\backslash(q-1)\mathscr{A}_1$ such that $E_{\preceq,\gamma}E_{\preceq,\beta} = c_1E_{\preceq,\alpha} + c_2E_{\preceq,\beta}E_{\preceq,\gamma}$. Since $\mathrm{ht}(\bar{\beta}), \mathrm{ht}(\bar{\gamma}) < \mathrm{ht}(\bar{\alpha})$, by the hypothesis of the induction, we see that

$$\Psi(E_{\preceq,\alpha}) = \frac{1}{c_1}\Psi(E_{\preceq,\beta})\Phi(E_{\preceq,\gamma}) - \frac{c_2}{c_1}\Psi(E_{\preceq,\gamma})\Psi(E_{\preceq,\gamma}) \in X^+_{\mathbf{J}}.$$

By (8.25), Proposition 6.6, and Theorem 4.7 in [1], we have

$$[A_{\mathbf{J}}(1,+), A_{\mathbf{J}}(1,0)] \subset A_{\mathbf{J}}(1,+).$$

Let $w_\circ$ be the longest element of $\mathring{W}_{\mathbf{J}}$. Since $\Delta_{\mathbf{J}}(1,+) = \Delta_{\mathbf{J}}(w_\circ,-)$, by (5.27)(5.28) and the right equality in Lemma 5.17(2), we have

$$[A_{\mathbf{J}}(w_\circ,-), A_{\mathbf{J}}(w_\circ,0)] \subset A_{\mathbf{J}}(w_\circ,-). \tag{8.28}$$

Set $w' = w_\circ w^{-1}$. Then $T_{w_\circ} = T_w T_{w'}$ and $T_{w'} A_{\mathbf{J}}(w,0) = A_{\mathbf{J}}(w_\circ,0)$. Since the multiplication $U_<(w') \otimes T_{w'} U_{\mathbf{J},<}(w,-) \to U_{\mathbf{J},<}(w_\circ,-)$ is an isomorphism of $\mathbf{Q}(q)$-vector spaces, we have $T_{w'} A_{\mathbf{J}}(w,-) \subset A_{\mathbf{J}}(w_\circ,-)$. Therefore, by (8.28), we have

$$[T_{w'} A_{\mathbf{J}}(w,-), T_{w'} A_{\mathbf{J}}(w,0)] \subset A_{\mathbf{J}}(w_\circ,-).$$

Since $[T_{w'} A_{\mathbf{J}}(w,-), T_{w'} A_{\mathbf{J}}(w,0)] \subset A_{\mathbf{J}}(w_\circ,-) \cap A(w')^c$, we see that

$$[T_{w'} A_{\mathbf{J}}(w,-), T_{w'} A_{\mathbf{J}}(w,0)] \subset T_{w'} A_{\mathbf{J}}(w,-),$$

hence $[A_{\mathbf{J}}(w,-), A_{\mathbf{J}}(w,0)] \subset A_{\mathbf{J}}(w,-)$. By the left and right equalities in Lemma 5.17(2), we see that

$$[A_{\mathbf{J}}(w,+), A_{\mathbf{J}}(w,0)] = [A_{\mathbf{J}}(w,+), \Psi A_{\mathbf{J}}(ww_\circ,0)] \subset A_{\mathbf{J}}(w,+). \qquad \square$$

Theorem 8.6. *Let $\preceq$ be an arbitrary convex order on $\Delta_{\mathbf{J}+}$, and $w \in \mathring{W}_{\mathbf{J}}$ the unique element such that $\Delta_{\mathbf{J}}(w,-) \prec \Delta_+^{im} \prec \Delta_{\mathbf{J}}(w,+)$. We define $\preceq_-$, $\preceq_0$, and $\preceq_+$ to be the restriction of $\preceq$ to $\Delta_{\mathbf{J}}(w,-)$, Δ_+^{im}, and $\Delta_{\mathbf{J}}(w,+)$, respectively, and define a total order $\tilde{\preceq}_0$ on the set*

$$\tilde{\Delta}_{\mathbf{J}+}^{im} = \Delta_+^{im} \times \mathbf{J} = \{(n\delta, j) \mid n \in \mathbf{N}, j \in \mathbf{J}\}$$

by setting

$$(n\delta, j) \,\tilde{\prec}_0\, (n'\delta, j') \Leftrightarrow \begin{cases} n\delta \prec_0 n'\delta & \text{if } n \neq n', \\ j < j' & \text{if } n = n'. \end{cases} \tag{8.29}$$

In addition, we define a total order $\tilde{\preceq}$ on the set

$$\tilde{\Delta}_{\mathbf{J}+} = \Delta_{\mathbf{J}+}^{re} \amalg \tilde{\Delta}_{\mathbf{J}+}^{im} = \Delta_{\mathbf{J}}(w,-) \amalg \tilde{\Delta}_{\mathbf{J}+}^{im} \amalg \Delta_{\mathbf{J}}(w,+)$$

by extending $\preceq_-$, $\tilde{\preceq}_0$, and $\preceq_+$ so that $\Delta_{\mathbf{J}}(w,-) \,\tilde{\prec}\, \tilde{\Delta}_{\mathbf{J}+}^{im} \,\tilde{\prec}\, \Delta_{\mathbf{J}}(w,+)$. For each $\eta \in \tilde{\Delta}_{\mathbf{J}+}$, we set

$$E_\eta = E_{\preceq,\eta} := \begin{cases} E_{\preceq_-,\eta} & \text{if } \eta \in \Delta_{\mathbf{J}}(w,-), \\ T_w(I_{j,n}) & \text{if } \eta = (n\delta, j) \in \tilde{\Delta}_{\mathbf{J}+}^{im}, \\ \Psi(E_{\preceq_+^{op},\eta}) & \text{if } \eta \in \Delta_{\mathbf{J}}(w,+), \end{cases} \tag{8.30}$$

where $\preceq_+^{op}$ is the opposite order of $\preceq_+$. Then each of the sets $E_{\tilde{\prec}}^*(\tilde{\Delta}_{\mathbf{J}+})$ and $E_{\tilde{\succ}}^*(\tilde{\Delta}_{\mathbf{J}+})$ (see Definition 5.18) is a convex basis of the $\mathbf{Q}(q)$-algebra $U_{\mathbf{J}}^+$ and of the $\mathscr{A}_1$-algebra ${}_{\mathscr{A}_1}U_{\mathbf{J}}^+ := {}_{\mathscr{A}_1}U^+ \cap U_{\mathbf{J}}^+$. Moreover, for each $\eta, \zeta \in \tilde{\Delta}_{\mathbf{J}+}$ satisfying $\eta \,\tilde{\prec}\, \zeta$, the following equalities hold:

$$[E_\eta, E_\zeta]_q = \sum_{\mathrm{supp}(\mathbf{c}) \subset (\eta,\zeta)_{\tilde{\preceq}}} h_{\mathbf{c}} E^{\mathbf{c}}_{\prec}, \qquad [E_\eta, E_\zeta]_q = \sum_{\mathrm{supp}(\mathbf{c}) \subset (\eta,\zeta)_{\tilde{\preceq}}} g_{\mathbf{c}} E^{\mathbf{c}}_{\succ}, \qquad (8.31)$$

where $h_{\mathbf{c}}, g_{\mathbf{c}} \in \mathscr{A}_1$.

Proof. By Proposition 6.6, Proposition 7.1(1), Theorem 8.4, and Proposition 7.2(2), we see that $E^*_{\tilde{\prec}}(\tilde{\Delta}_{\mathbf{J}+})$ is a basis of $U^+_{\mathbf{J}}$. Moreover, by Lemma 4.4(1) and Lemma 6.4(1), we see that $E^*_{\tilde{\prec}}(\tilde{\Delta}_{\mathbf{J}+})$ is a subset of ${}_{\mathscr{A}_1}U^+_{\mathbf{J}} \backslash (q-1)_{\mathscr{A}_1}U^+_{\mathbf{J}}$. Hence, it follows from Proposition 4.1 and Lemma 4.2 that the set $E^*_{\tilde{\prec}}(\tilde{\Delta}_{\mathbf{J}+})$ is also a basis of the $\mathscr{A}_1$-algebra ${}_{\mathscr{A}_1}U^+_{\mathbf{J}}$.

Suppose that $\eta \in \Delta_{\mathbf{J}}(w,-)$. Since $(*,\eta]_{\tilde{\preceq}} = (*,\eta]_{\preceq_-}$, by (8.13), we see that $E^*_{\tilde{\prec}}((*,\eta]_{\tilde{\preceq}})$ is a basis of $A_{\mathbf{J}}((*,\eta]_{\tilde{\preceq}})$. By (8.14), (7.17), and the left equality in (7.14), we see that $E^*_{\tilde{\prec}}((\eta,*)_{\tilde{\preceq}})$ is a basis of $A_{\mathbf{J}}((*,\eta]_{\tilde{\preceq}})^c$.

We next suppose that $\eta \in \Delta_{\mathbf{J}}(w,+)$. Here we remark that Ψ is an anti-automorphism of the $\mathbf{Q}(q)$-algebra $U^+_{\mathbf{J}}$. Since $(\eta,*)_{\tilde{\preceq}} = (\eta,*)_{\preceq_+}$, by (8.13), we see that $E^*_{\tilde{\prec}}((\eta,*)_{\tilde{\preceq}})$ is a basis of $\Psi A_{\mathbf{J}}((\eta,*)_{\tilde{\preceq}})$. By (8.14), (7.17), and the right equality in (7.14), we see that $E^*_{\tilde{\prec}}((*,\eta]_{\tilde{\preceq}})$ is a basis of $\Psi A_{\mathbf{J}}((\eta,*)_{\tilde{\preceq}})^c$.

We next suppose that $\eta \in \tilde{\Delta}^{im}_{\mathbf{J}+}$. By Proposition 6.6, (7.2), (7.13), Theorem 8.4, and Proposition 8.5(2), we see that $E^*_{\tilde{\prec}}(I)$ is a basis of the subalgebra $\langle E_\eta \,|\, \eta \in I \rangle_{\mathbf{Q}(q)\text{-}alg}$ in the case where $I = (*,\eta]_{\tilde{\preceq}}$ or $I = (\eta,*)_{\tilde{\preceq}}$.

Therefore we see that $E^*_{\tilde{\prec}}(I)$ is a basis of $\langle E_\eta \,|\, \eta \in I \rangle_{\mathbf{Q}(q)\text{-}alg}$ in the cases where $I = (*,\eta]_{\tilde{\preceq}}$ or $I = (\eta,*)_{\tilde{\preceq}}$ for each $\eta \in \Delta_{\mathbf{J}+}$. Similarly, we can prove that $E^*_{\tilde{\prec}}(I)$ is a basis of $\langle E_\eta \,|\, \eta \in I \rangle_{\mathbf{Q}(q)\text{-}alg}$ in the case where $I = (*,\eta)_{\tilde{\preceq}}$ or $I = [\eta,*)_{\tilde{\preceq}}$ for each $\eta \in \Delta_{\mathbf{J}+}$.

For each $\eta \tilde{\preceq} \zeta$, we see that $E^*_{\tilde{\prec}}([\eta,*)_{\tilde{\preceq}}) \cap E^*_{\tilde{\prec}}((*,\zeta]_{\tilde{\preceq}}) = E^*_{\tilde{\prec}}([\eta,\zeta]_{\tilde{\preceq}})$ and

$$\langle E_\eta \,|\, \eta \in [\eta,*)_{\tilde{\preceq}} \rangle_{\mathbf{Q}(q)\text{-}alg} \cap \langle E_\eta \,|\, \eta \in (*,\zeta]_{\tilde{\preceq}} \rangle_{\mathbf{Q}(q)\text{-}alg} = \langle E_\eta \,|\, \eta \in [\eta,\zeta]_{\tilde{\preceq}} \rangle_{\mathbf{Q}(q)\text{-}alg}.$$

Thus $E^*_{\tilde{\prec}}(I)$ is a basis of $\langle E_\eta \,|\, \eta \in I \rangle_{\mathbf{Q}(q)\text{-}alg}$ in the case where $I = [\eta,\zeta]_{\tilde{\preceq}}$. Similarly, we can prove that $E^*_{\tilde{\prec}}(I)$ is a basis of $\langle E_\eta \,|\, \eta \in I \rangle_{\mathbf{Q}(q)\text{-}alg}$ for each section I with respect to $\tilde{\preceq}$. Therefore $E^*_{\tilde{\prec}}(\tilde{\Delta}_{\mathbf{J}+})$ is a convex basis of $U^+_{\mathbf{J}}$. Since $E^*_{\tilde{\prec}}(\tilde{\Delta}_{\mathbf{J}+})$ is also a basis of the $\mathscr{A}_1$-algebra ${}_{\mathscr{A}_1}U^+_{\mathbf{J}}$, it is easy to see that $E^*_{\tilde{\prec}}(\tilde{\Delta}_{\mathbf{J}+})$ is a convex basis of ${}_{\mathscr{A}_1}U^+_{\mathbf{J}}$. The proof of (8.31) is similar to that of Proposition 7 in [2]. □

9. Dual convex bases with respect to the q-Killing form

Firstly, we introduce a well-known standard $\mathbf{Q}(q)$-bilinear form between $U^{\geq 0}$ and $U^{\leq 0}$, which is called the q-Killing form since it can be regarded as a q-analogue of the Killing form on $\mathfrak{g}$. Secondly, we introduce Damiani's work concerning detailed computation of values of the q-Killing form on the subalgebras generated by the imaginary root vectors. Thirdly, we will construct the dual convex bases of U^+ and U^- with respect to the q-Killing form. Finally, we will present the multiplicative formula for the R-matrix of $U_q(\mathfrak{g})$ associated with an arbitrary convex order on Δ_+.

Let m_0 be a positive integer such that $m_0(P \mid P) \subset \mathbf{Z}$, and $\mathbf{F}$ an extension field of $\mathbf{Q}(q)$ such that $\mathbf{F}$ contains an m_0-th root q^{1/m_0} of q.

Theorem 9.1 ([15]). *There exists a unique non-degenerate $\mathbf{Q}(q)$-bilinear form $(\,|\,) : U^{\geq 0} \times U^{\leq 0} \to \mathbf{F}$ which satisfies the following equalities:*

$$(x \mid y_1 y_2) = (\Delta(x) \mid y_1 \otimes y_2), \qquad (x_1 x_2 \mid y) = (x_2 \otimes x_1 \mid \Delta(y)),$$

$$(K_\mu \mid K_\nu) = q^{-(\mu|\nu)}, \qquad (E_i \mid K_\mu) = (K_\mu \mid F_i) = 0, \qquad (E_i \mid F_j) = \delta_{ij}/(q_i^{-1} - q_i),$$

where $x, x_1, x_2 \in U^{\geq 0}$, $y, y_1, y_2 \in U^{\leq 0}$, $i, j \in \mathbf{I}$, $\mu, \nu \in P$, and Δ is the coproduct of U defined by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \qquad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \qquad \Delta(K_\mu) = K_\mu \otimes K_\mu.$$

Here we use the notation $(\,|\,)$ also for the $\mathbf{Q}(q)$-bilinear form $(\,|\,) : (U^{\geq 0})^{\otimes 2} \times (U^{\leq 0})^{\otimes 2} \to \mathbf{F}$ induced by $(x_1 \otimes x_2 \mid y_1 \otimes y_2) := (x_1 \mid y_1)(x_2 \mid y_2)$.

Lemma 9.2 ([10]). (1) *For each $\mu, \nu \in Q_+$, $x \in U_\mu^+$, $y \in U_{-\nu}^-$, and $\xi, \eta \in P$, the following equality holds: $(xK_\xi \mid yK_\eta) = \delta_{\mu\nu} q^{-(\xi|\eta)} (x \mid y)$. Moreover, the restriction of the form $(\,|\,)$ to $U_\mu^+ \times U_{-\mu}^-$ is non-degenerate.*

(2) *For each $(x, y) \in U^+ \times U^-$, the following equality holds: $(\Psi(x) \mid \Psi(y)) = (x \mid y)$.*

(3) *For each $i \in \mathbf{I}$, $x \in A(s_i)^c$, $y \in A^-(s_i)^c$, and $m, n \in \mathbf{Z}_{\geq 0}$, the following equality holds:*

$$(xE_i^m \mid yF^n) = \delta_{mn}(x \mid y)(E_i^m \mid F_i^m) = \delta_{mn}(x \mid y)(m)_{q_i}!/(q_i^{-1} - q_i)^m. \tag{9.1}$$

Proof. Although in [10] the assertions are proved in the case where $\mathfrak{g}$ is an arbitrary finite dimensional simple Lie algebra, the proof can be applied to the untwisted affine case. □

Proposition 9.3. *For each $y \in W$, $a \in \Psi(A(y)^c)$, and $b \in \Psi(A^-(y)^c)$, the following equality holds: $(T_y(a) \mid T_y(b)) = (a \mid b)$.*

Proof. Since the equality is clear in the case where $y = 1$, we may assume that $\ell(y) > 1$. We use the induction on $l = \ell(y)$. In the case where $l = 1$, we can apply the proof of Proposition 8.28 in [10] to this case. In the case where $l \geq 2$, there exist $i \in \mathbf{I}$ and $y' \in W$ such that $y = y' s_i$ and $\ell(y') = l - 1$. Then we see that $T_y = T_{y'} T_i$, $T_i(a) \in \Psi(A(y')^c)$, and $T_i(b) \in \Psi(A^-(y')^c)$. Hence, by the inductive assumption, we have the following equalities:

$$(T_y(a) \mid T_y(b)) = (T_{y'} T_i(a) \mid T_{y'} T_i(b)) = (T_i(a) \mid T_i(b)) = (a \mid b). \quad \square$$

Proposition 9.4. *Let B be an arbitrary element of $\mathfrak{B}^*$. If $x_1 \in A(B)^c$, $x_2 \in A(B)$, $y_1 \in A^-(B)^c$, $y_2 \in A^-(B)$, then the following equality holds: $(x_1x_2 \,|\, y_1y_2) = (x_1 \,|\, y_1)(x_2 \,|\, y_2)$.*

Proof. We first prove the claim in the case where $B = \Phi(y)$ with $y \in W$. By Proposition 7.2(2), we have $A(B) = A(y) = U_>(y)$ and $A^-(B) = A^-(y) = U^-_>(y)$. We use the induction on $l = \ell(y)$. In the case where $l = 0$, since $y = 1$, we have $U_>(y) = U^-_>(y) = \mathbf{Q}(q)$, which implies the equality. In the case where $l > 0$, there exist $i \in \mathbf{I}$ and $y' \in W$ such that $y = s_iy'$ and $\ell(y') = l - 1$. By Proposition 5.20(2), there exist $m, n \in \mathbf{Z}_{\geq 0}$, $a'_2 \in T_iU_>(y')$, and $b'_2 \in T_iU^-_>(y')$ such that $a_2 = a'_2E_i^m$ and $b_2 = b'_2F_i^n$. Then $T_i^{-1}(a'_2) \in U_>(y')$ and $T_i^{-1}(b'_2) \in U^-_>(y')$. In addition, we have $T_i^{-1}(a_1) \in A(y')^c$ and $T_i^{-1}(b_1) \in A^-(y')^c$. By Lemma 9.2(3), Proposition 9.3, and the inductive assumption, we see that

$$
\begin{aligned}
(a_1a_2 \,|\, b_1b_2) &= (a_1a'_2E_i^m \,|\, b_1b'_2F_i^n) = (a_1a'_2 \,|\, b_1b'_2)(E_i^m \,|\, F_i^n) \\
&= (T_i^{-1}(a_1)T_i^{-1}(a'_2) \,|\, T_i^{-1}(b_1)T_i^{-1}(b'_2))(E_i^m \,|\, F_i^n) \\
&= (T_i^{-1}(a_1) \,|\, T_i^{-1}(b_1))(T_i^{-1}(a'_2) \,|\, T_i^{-1}(b'_2))(E_i^m \,|\, F_i^n) \\
&= (a_1 \,|\, b_1)(a'_2 \,|\, b'_2)(E_i^m \,|\, F_i^n) = (a_1 \,|\, b_1)(a_2 \,|\, b_2).
\end{aligned}
$$

We next prove the claim in the case where $B \in \mathfrak{B}^\infty$. By Proposition 7.2(2), we have $A(B) = U_>(B)$ and $A^-(B) = U^-_>(B)$. Then, by Proposition 5.20(2), there exists $y \in W(B)$ such that $x_2 \in A(y)$ and $y_2 \in A^-(y)$, and hence $x_1 \in A(y)^c$ and $y_1 \in A^-(y)^c$. Therefore the equality is still valid in this case. □

Proposition 9.5. (1) *Let w be an arbitrary element of $\mathring{W}$. If $X_+ \in A(w,+)$, $Y_+ \in A^-(w,+)$, $X_0 \in A(w,0)$, $Y_0 \in A^-(w,0)$, $X_- \in A(w,-)$, $Y_- \in A^-(w,-)$, then $(X_+X_0X_- \,|\, Y_+Y_0Y_-) = (X_+ \,|\, Y_+)(X_0 \,|\, Y_0)(X_- \,|\, Y_-)$.*

(2) *Let $\preceq$ be an arbitrary convex order on $\Delta(w,-)$. Set $F^{\mathbf{c}}_{\succ} := \Omega(E^{\mathbf{c}}_{\prec})$. Then*

$$
(E^{\mathbf{c}}_{\succ} \,|\, F^{\mathbf{c}'}_{\succ}) = \delta_{\mathbf{c},\mathbf{c}'} \prod\nolimits_{\alpha \in \Delta(w,-)} (\mathbf{c}(\alpha))_{q_\alpha}!/(q_\alpha^{-1} - q_\alpha)^{\mathbf{c}(\alpha)}. \tag{9.2}
$$

Proof. (1) Since $X_+X_0 \in A(w,-)^c$ and $Y_+Y_0 \in A^-(w,-)^c$, by Proposition 9.4, we see that $(X_+X_0X_- \,|\, Y_+Y_0Y_-) = (X_+X_0 \,|\, Y_+Y_0)(X_- \,|\, Y_-)$. Moreover, since $A(w,0) \subset A(w,+)^c$ and $A^-(w,0) \subset A^-(w,+)^c$, by Lemma 9.2(2) and Proposition 9.4, we see that $(X_+X_0 \,|\, Y_+Y_0) = (X_+ \,|\, Y_+)(X_0 \,|\, Y_0)$. Thus (1) is valid.

(2) We assume that $\mathrm{supp}(\mathbf{c}) \cup \mathrm{supp}(\mathbf{c}') = \{\beta_1, \beta_2, \dots, \beta_m\}$ with $\beta_1 \prec \beta_2 \prec \cdots \prec \beta_m$, and put $I = (*, \beta_1]_{\preceq}$. By (8.13)(8.14) in Theorem 8.4, we see that $E^{\mathbf{c}(\beta_m)}_{\preceq,\beta_m} \dots E^{\mathbf{c}(\beta_2)}_{\preceq,\beta_2} \in A(I)^c$, $E^{\mathbf{c}(\beta_1)}_{\preceq,\beta_1} \in A(I)$, $F^{\mathbf{c}'(\beta_m)}_{\preceq,\beta_m} \dots F^{\mathbf{c}'(\beta_2)}_{\preceq,\beta_2} \in A^-(I)^c$, and $F^{\mathbf{c}'(\beta_1)}_{\preceq,\beta_1} \in A^-(I)$, where $F^{\mathbf{c}(\alpha)}_{\preceq,\alpha} := \Omega(E^{\mathbf{c}(\alpha)}_{\preceq,\alpha})$ for each $\alpha \in \Delta(w,-)$. Hence, by Lemma 9.2(1), Proposition 9.4, and the induction on m, we see that

$$\begin{aligned}(E^{\mathbf{c}}_{\succ} \mid F^{\mathbf{c}'}_{\succ}) &= (E^{\mathbf{c}(\beta_m)}_{\preceq,\beta_m} \dots E^{\mathbf{c}(\beta_2)}_{\preceq,\beta_2} E^{\mathbf{c}(\beta_1)}_{\preceq,\beta_1} \mid F^{\mathbf{c}'(\beta_m)}_{\preceq,\beta_m} \dots F^{\mathbf{c}'(\beta_2)}_{\preceq,\beta_2} F^{\mathbf{c}'(\beta_1)}_{\preceq,\beta_1}) \\ &= (E^{\mathbf{c}(\beta_m)}_{\preceq,\beta_m} \dots E^{\mathbf{c}(\beta_2)}_{\preceq,\beta_2} \mid F^{\mathbf{c}'(\beta_m)}_{\preceq,\beta_m} \dots F^{\mathbf{c}'(\beta_2)}_{\preceq,\beta_2})(E^{\mathbf{c}(\beta_1)}_{\preceq,\beta_1} \mid F^{\mathbf{c}'(\beta_1)}_{\preceq,\beta_1}) \\ &= \prod_{k=1}^{m} (E^{\mathbf{c}(\beta_k)}_{\preceq,\beta_k} \mid F^{\mathbf{c}'(\beta_k)}_{\preceq,\beta_k}) = \delta_{\mathbf{c},\mathbf{c}'} \prod_{\alpha \in \Delta(w,-)} (E^{\mathbf{c}(\alpha)}_{\preceq,\alpha} \mid F^{\mathbf{c}(\alpha)}_{\preceq,\alpha}).\end{aligned}$$

The equality (9.2) follows from Lemma 9.2(3) and Proposition 9.3. □

Thanks to Proposition 9.5, to complete the computation of values of the q-Killing form, it suffices to compute the values on $A(w,0) \times A^-(w,0)$. For the completion of the task, we refer to the following work of I. Damiani concerning detailed computation of the values of the q-Killing form on the subalgebras generated by the imaginary root vectors.

Proposition 9.6 ([6]). (1) *For each* $n \in \mathbf{N}$ *and* $i, j \in \mathring{\mathbf{I}}$ *with* $i < j$, *there is a solution* $\{A^{(n)}_{il} \in \mathbf{Q}(q) \mid i \le l \in \mathring{\mathbf{I}}\}$ *of the following system of linear equations:* $\sum_{i \le l} A^{(n)}_{il} (\mathrm{sgn}(\mathrm{A}_{lj}))^n [n\mathrm{A}_{lj}]_{q_l}/n = 0$ *under the condition* $A^{(n)}_{ii} \neq 0$.

(2) *For each* $(i,n) \in \mathring{\mathbf{I}} \times \mathbf{N}$, *one set* $\tilde{I}_{i,n} := \sum_{i \le l} A^{(n)}_{il} I_{l,n}$ *and* $\tilde{J}_{i,n} := \Omega(\tilde{I}_{i,n})$. *Then the elements* $\{\tilde{I}_{i,n} \mid n \in \mathbf{N}, i \in \mathring{\mathbf{I}}\}$ *satisfy the following conditions* (i)(ii):

(i) *for each* $n \in \mathbf{N}$, *the sets* $\{\tilde{I}_{i,n} \mid i \in \mathring{\mathbf{I}}\}$ *and* $\{I_{i,n} \mid i \in \mathring{\mathbf{I}}\}$, *respectively, are bases of the same* $\mathbf{Q}(q)$*-vector subspace of* U^+;

(ii) *for each pair* $(\mathbf{c}, \mathbf{c}')$ *of finitely supported* $\mathbf{Z}_+$*-valued functions on* $\mathring{\mathbf{I}} \times \mathbf{N}$, *the following equality holds:*

$$\Big(\prod_{(i,n) \in \mathring{\mathbf{I}} \times \mathbf{N}} \tilde{I}^{\mathbf{c}(i,n)}_{i,n} \mid \prod_{(i,n) \in \mathring{\mathbf{I}} \times \mathbf{N}} \tilde{J}^{\mathbf{c}'(i,n)}_{i,n}\Big) = \delta_{\mathbf{c},\mathbf{c}'} \prod_{(i,n) \in \mathring{\mathbf{I}} \times \mathbf{N}} (\mathbf{c}(i,n))! (\tilde{I}_{i,n} \mid \tilde{J}_{i,n})^{\mathbf{c}(i,n)},$$

where the value of $(\tilde{I}_{i,n} \mid \tilde{J}_{i,n})$ *is given by*

$$(\tilde{I}_{i,n} \mid \tilde{J}_{i,n}) = A^{(n)}_{ii} \sum_{i \le j} A^{(n)}_{ij} (\mathrm{sgn}(\mathrm{A}_{ji}))^n [n\mathrm{A}_{ji}]_{q_j} / \{n(q_i^{-1} - q_i)\}. \tag{9.3}$$

Remark 9.7. *For each* $n \in \mathbf{N}$ *and* $i, j \in \mathring{\mathbf{I}}$ *with* $i < j$, *a solution* $\{A^{(n)}_{il} \in \mathbf{Q}(q) \mid i \le l \in \mathring{\mathbf{I}}\}$ *of the system of the linear equations in the part* (1) *of Proposition* 9.6 *is given in Proposition* 7.4.3 *of* [6].

Theorem 9.8. *Let us use the notations as in Theorem* 8.6 *and Proposition* 9.6, *and assume that* $\mathbf{J} = \mathring{\mathbf{I}}$. *For each* $\eta \in \tilde{\Delta}_+$, *we set*

$$\tilde{E}_\eta = \tilde{E}_{\preceq,\eta} := \begin{cases} E_{\preceq_-,\eta} & \text{if } \eta \in \Delta(w,-), \\ T_w(\tilde{I}_{i,n}) & \text{if } \eta = (n\delta, i) \in \tilde{\Delta}_+^{im}, \\ \Psi(E_{\preceq_+^{op},\eta}) & \text{if } \eta \in \Delta(w,+), \end{cases} \tag{9.4}$$

and set $\tilde{F}_\eta = \tilde{F}_{\preceq,\eta} := \Omega(\tilde{E}_{\preceq,\eta})$. *Then the sets* $\tilde{E}_{\succ}^{*}(\tilde{\Delta}_+)$ *and* $\tilde{F}_{\succ}^{*}(\tilde{\Delta}_+)$ *(see Definition* 5.18*) are convex bases of* U^+ *and of* U^- *respectively satisfying*

$$(\tilde{E}_{\succ}^{\mathbf{c}} \,|\, \tilde{F}_{\succ}^{\mathbf{c}'}) = \delta_{\mathbf{c},\mathbf{c}'} \prod\nolimits_{\eta \in \tilde{\Delta}_+} (\mathbf{c}(\eta))_{q_\eta}!(\tilde{E}_{\preceq,\eta} \,|\, \tilde{F}_{\preceq,\eta})^{\mathbf{c}(\eta)} \tag{9.5}$$

$$= \delta_{\mathbf{c},\mathbf{c}'} \prod\nolimits_{\alpha \in \Delta_+^{re}} (\mathbf{c}(\alpha))_{q_\alpha}!/(q_\alpha^{-1} - q_\alpha)^{\mathbf{c}(\alpha)}$$

$$\times \prod\nolimits_{(n\delta,i) \in \tilde{\Delta}_+^{im}} (\mathbf{c}(n\delta,i))!(\tilde{I}_{i,n} \,|\, \tilde{J}_{i,n})^{\mathbf{c}(n\delta,i)}, \tag{9.6}$$

where the value of $(\tilde{I}_{i,n} \,|\, \tilde{J}_{i,n})$ *is given by* (9.3). *Therefore, the convex basis* $\tilde{E}_{\succ}^{*}(\tilde{\Delta}_+)$ *of* U^+ *and the convex basis*

$$\{\tilde{F}_{\succ}^{\mathbf{c}}/(\tilde{E}_{\succ}^{\mathbf{c}} \,|\, \tilde{F}_{\succ}^{\mathbf{c}}) \,|\, \mathbf{c} : \tilde{\Delta}_+ \to \mathbf{Z}_+ \ \ s.t.\ \#\mathrm{supp}(\mathbf{c}) < \infty\}$$

of U^- *form a pair of dual bases with respect to the q-Killing form* $(\,|\,)$.

Proof. By Proposition 6.6, (7.2), and the definition of $\tilde{I}_{i,n}$, we see that the set $\tilde{E}_{\succ}^{*}(\tilde{\Delta}_+^{im})$ is also a basis of the commutative subalgebra $A(w,0)$. So, in the same manner as in the proof of Theorem 8.6, it is easy to see that the first assertion is valid. By Proposition 9.6(2) and Proposition 9.3, we see that

$$(\tilde{E}_{\succ}^{\mathbf{c}} \,|\, \tilde{F}_{\succ}^{\mathbf{c}'}) = \delta_{\mathbf{c},\mathbf{c}'} \prod\nolimits_{(n\delta,i) \in \tilde{\Delta}_+^{im}} (\mathbf{c}(n\delta,i))!(\tilde{I}_{i,n} \,|\, \tilde{J}_{i,n})^{\mathbf{c}(n\delta,i)} \tag{9.7}$$

for each pair $(\mathbf{c},\mathbf{c}')$ satisfying $\mathrm{supp}(\mathbf{c}), \mathrm{supp}(\mathbf{c}') \subset \tilde{\Delta}_+^{im}$. Let $w_\circ$ be the longest element of $\mathring{W}_{\mathbf{J}}$. Then $\Delta(w,+) = \Delta(ww_\circ,-)$. Hence, by Proposition 9.5(2) and Lemma 9.2(2), we see that

$$(\tilde{E}_{\succ}^{\mathbf{c}} \,|\, \tilde{F}_{\succ}^{\mathbf{c}'}) = \delta_{\mathbf{c},\mathbf{c}'} \prod\nolimits_{\alpha \in \Delta(w,+)} (\mathbf{c}(\alpha))_{q_\alpha}!/(q_\alpha^{-1} - q_\alpha)^{\mathbf{c}(\alpha)}, \tag{9.8}$$

for each pair $(\mathbf{c},\mathbf{c}')$ satisfying $\mathrm{supp}(\mathbf{c}), \mathrm{supp}(\mathbf{c}') \subset \Delta(w,+)$. Therefore the equalities (9.5) and (9.6) follow from Proposition 9.5(1)(2), (9.7), and (9.8). The last assertion follows from the first assertion and the equality (9.5). □

Corollary 9.9. *Let us use the notations as in Proposition* 9.6 *and Theorem* 9.8. *For each convex order* $\preceq$ *on* Δ_+, *the universal R-matrix* $\mathscr{R}$ *of* $U_q(\mathfrak{g})$ *can be expressed as follows:*

$$\mathscr{R} = \Big(\prod\nolimits_{\alpha \in \Delta(w,+)}^{\succ} \Theta_{\preceq,\alpha}\Big)\Big(\prod\nolimits_{\alpha \in \Delta_+^{im}}^{\succ} \Theta_{\preceq,\alpha}\Big)\Big(\prod\nolimits_{\alpha \in \Delta(w,-)}^{\succ} \Theta_{\preceq,\alpha}\Big) q^{-T},$$

where $T \in \mathfrak{h}^* \otimes \mathfrak{h}^*$ *is the canonical element of the inner product* $(\,|\,)$ *on* $\mathfrak{h}^*$ *and*

$$\Theta_{\preceq,\alpha} := \begin{cases} \exp_{q_\alpha}((q_\alpha^{-1} - q_\alpha)E_{\preceq,\alpha} \otimes F_{\preceq,\alpha}) & \text{for } \alpha \in \Delta_+^{re}, \\ \exp(\sum_{i=1}^r T_w(\tilde{I}_{i,n}) \otimes T_w(\tilde{J}_{i,n})/(\tilde{I}_{i,n} \mid \tilde{J}_{i,n})) & \text{for } \alpha = n\delta \in \Delta_+^{im}. \end{cases}$$

Here, $\Theta_{\preceq,\alpha'}$ *is located on the left side of* $\Theta_{\preceq,\alpha}$ *in the product above if* $\alpha' \succ \alpha$, *and* $\exp_{q_\alpha}(x) = \sum_{m=0}^{\infty} x^m/(m)_{q_\alpha}!$.

Proof. Let Θ be the canonical element of the restriction of the q-Killing form to $U_q^+ \times U_q^-$. Then it is known that the universal R-matrix $\mathscr{R}$ of $U_q(\mathfrak{g})$ can be expressed as follows (cf. [15]): $\mathscr{R} = \Theta \cdot q^{-T}$. By Theorem 9.8, we see that

$$\Theta = \sum_{\mathbf{c}} \frac{\tilde{E}_{\tilde{\succ}}^{\mathbf{c}} \otimes \tilde{F}_{\tilde{\succ}}^{\mathbf{c}}}{(\tilde{E}_{\tilde{\succ}}^{\mathbf{c}} \mid \tilde{F}_{\tilde{\succ}}^{\mathbf{c}})} = \sum_{\mathbf{c}} \prod_{\eta \in \tilde{\Delta}_+}^{\tilde{\succ}} \frac{\tilde{E}_{\preceq,\eta}^{\mathbf{c}(\eta)} \otimes \tilde{F}_{\preceq,\eta}^{\mathbf{c}(\eta)}}{(\mathbf{c}(\eta))_{q_\eta}!(\tilde{E}_{\preceq,\eta} \mid \tilde{F}_{\preceq,\eta})^{\mathbf{c}(\eta)}}$$

$$= \prod_{\eta \in \tilde{\Delta}_+}^{\tilde{\succ}} \sum_{m=0}^{\infty} \frac{1}{(m)_{q_\eta}!} \left(\frac{\tilde{E}_{\preceq,\eta} \otimes \tilde{F}_{\preceq,\eta}}{(\tilde{E}_{\preceq,\eta} \mid \tilde{F}_{\preceq,\eta})} \right)^m = \prod_{\eta \in \tilde{\Delta}_+}^{\tilde{\succ}} \exp_{q_\eta} \left(\frac{\tilde{E}_{\preceq,\eta} \otimes \tilde{F}_{\preceq,\eta}}{(\tilde{E}_{\preceq,\eta} \mid \tilde{F}_{\preceq,\eta})} \right),$$

$$= \left(\prod_{\alpha \in \Delta(w,+)}^{\succ} \Theta_{\preceq,\alpha} \right) \left(\prod_{\eta \in \tilde{\Delta}_+^{im}}^{\tilde{\succ}} \Theta_{\preceq,\eta} \right) \left(\prod_{\alpha \in \Delta(w,-)}^{\succ} \Theta_{\preceq,\alpha} \right)$$

where $\Theta_{\preceq,\eta} := \exp(T_w(\tilde{I}_{i,n}) \otimes T_w(\tilde{J}_{i,n})/(\tilde{I}_{i,n} \mid \tilde{J}_{i,n}))$ for $\eta = (n\delta, i) \in \tilde{\Delta}_+^{im}$. Since the elements of $\{T_w(\tilde{I}_{i,n}) \otimes T_w(\tilde{J}_{i,n}) \mid (i,n) \in \mathring{\mathbf{I}} \times \mathbf{N}\}$ are commutative with each other, the factor $(\prod_{\eta \in \tilde{\Delta}_+^{im}}^{\tilde{\succ}} \Theta_{\preceq,\eta})$ of Θ can be written as

$$\prod_{\eta \in \tilde{\Delta}_+^{im}}^{\tilde{\succ}} \Theta_{\preceq,\eta} = \prod_{n\delta \in \Delta_+^{im}}^{\succ} \prod_{i=1}^{r} \Theta_{\preceq,(n\delta,i)} = \prod_{n\delta \in \Delta_+^{im}}^{\succ} \Theta_{\preceq,n\delta}. \quad \square$$

References

[1] J. Beck, Braid group action and quantum affine algebras, Comm. Math. Phys., **165** (1994), 555–568.

[2] J. Beck, Convex bases of PBW-type for quantum affine algebras, Comm. Math. Phys., **165** (1994), 193–199.

[3] J. Beck, V. Chari, and A. Pressley, An algebraic characterization of the canonical basis, Duke Math. J., **99(3)** (1999), 455–487.

[4] N. Bourbaki, Groupes et algebres de Lie, Ch. 4–6, Hermann, Paris, 1968.

[5] C. De Contini and V. G. Kac, Representations of quantum groups at roots of 1, Progress in Math., **92** (1990), 471–506.

[6] I. Damiani, The R-matrix for (twisted) affine Quantum algebras, Proc. of the Int. Conference on Rep. Theory Representations and quantizations (Shanghai, 1998), China High. Educ. Press, Beijing, (2000), 89–144.

[7] K. Ito, Parametrizations of infinite biconvex sets in affine root systems, Hiroshima Math. J., **35(3)** (2005), 425–451.

[8] K. Ito, The classification of convex orders on affine root systems, Comm. Alg., **29** (2001), 5605–5630.
[9] N. Iwahori and H. Matsumoto, On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups, Inst. Hautes Études Sci. Publ. Math., **25** (1965), 5–48.
[10] J. C. Jantzen, Lectures on Quantum Groups, A.M.S., US, 1996.
[11] V. G. Kac, Infinite dimensional Lie algebras, third ed., Cambridge Univ. Press, Cambridge, UK, 1990.
[12] A. N. Kirillov and N. Reshetikhin, q-Weyl group and a multiplicative formula for universal R-matrices, Comm. Math. Phys., **134** (1990), 421–431.
[13] S. M. Khoroshkin and V. N. Tolstoy, The Cartan-Weyl basis and the universal R-matrix for quantum Kac-Moody algebras and superalgebras, Proc. of the Int. Workshop on Math. Phys. Quantum symmetries (Clausthal, 1991), World Sci. Publ., River Edge, NJ, (1993), 336–351.
[14] G. Lusztig, Introduction to Quantum Groups, Birkhäuser, Boston, 1993.
[15] T. Tanisaki, Killing forms, Harish-Chandra isomorphisms, and universal R-matrices for quantum algebras, Infinite Analysis Part B, Adv. Series in Math. Phys. **16** (1992), 941–962.

Ken Ito
Science Division
Center for General Education
Aichi Institute of Technology
Toyota 470-0392, *Japan*
E-mail: ken-ito@aitech.ac.jp