Hiroshima Math. J.
40 (2010), 101–113

On meromorphic functions sharing two one-point sets and two three-point sets

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(Received March 25, 2009)
(Revised August 17, 2009)

Abstract. For two meromorphic functions sharing two one-point sets and two three-point sets CM, we consider when one of them is a Möbius transform of the other.

1. Introduction

For nonconstant meromorphic functions f and g on $\boldsymbol{C}$ and a finite set S in $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$, we say that f and g share S CM (counting multiplicities) if $f^{-1}(S) = g^{-1}(S)$ and if for each $z_0 \in f^{-1}(S)$ the two functions $f - f(z_0)$ and $g - g(z_0)$ have the same multiplicity of zero at z_0, where the notations $f - \infty$ and $g - \infty$ mean $1/f$ and $1/g$, respectively. In particular, if S is a one-point set $\{a\}$, then we say also that f and g share a CM.

In [N], R. Nevanlinna showed the following:

Theorem 1. *Let f and g be two distinct nonconstant meromorphic functions on $\boldsymbol{C}$ and let $a_1, \dots, a_4$ be four distinct points in $\hat{\boldsymbol{C}}$. If f and g share $a_1, \dots, a_4$ CM, then f is a Möbius transform of g, i.e., there exists a Möbius transformation T such that $f = T \circ g$, and there exists a permutation σ of $\{1, 2, 3, 4\}$ such that $a_{\sigma(3)}$, $a_{\sigma(4)}$ are Picard exceptional values of f and g and the cross ratio $(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}) = -1$.*

Also, in [7], Tohge considered two meromorphic functions sharing 1, -1, ∞ and a two-point set containing none of them.

Theorem 2. *Let f and g be two nonconstant meromorphic functions on $\boldsymbol{C}$ sharing 1, -1, ∞ and a two-point set $S = \{a, b\}$ CM, respectively, where $a, b \neq 1, -1, \infty$. If $a + b \neq 0$, $ab \neq 1$, $a + b \neq 2$, $a + b \neq -2$, $(a+1)(b+1) \neq 4$ and $(a-1)(b-1) \neq 4$, then $f = g$. Otherwise one of $f + g = 0$, $fg = 1$, $f + g = 2$, $f + g = -2$, $(f+1)(b+1) = 4$ and $(f-1)(g-1) = 4$ holds.*

2000 *Mathematics Subject Classification*. Primary 30D35.

Key words and phrases. Uniqueness theorem, Shared values, Nevanlinna theory.

By Tohge's result, we can get a uniqueness theorem of meromorphic functions sharing three values and one two-point set CM since given three points are mapped to 1, -1, ∞, respectively, by a suitable Möbius transformation. For a finite set S, we denote by $\# S$ the number of elements of S.

COROLLARY 1. *Let* $S_1, \dots, S_4$ *be pairwise disjoint subsets in* $\hat{\boldsymbol{C}}$ *with* $\# S_1 = \# S_2 = \# S_3 = 1$ *and* $\# S_4 = 2$. *If two nonconstant meromorphic functions* f *and* g *on* $\boldsymbol{C}$ *share* $S_1, \dots, S_4$ *CM, respectively, then* f *is a Möbius transform of* g.

Also, by Theorem 1.2 in [6] and its proof, we see

THEOREM 3. *Let* $S_1, \dots, S_4$ *be pairwise disjoint subsets in* $\hat{\boldsymbol{C}}$ *with* $\# S_1 = \# S_2 = 1$ *and* $\# S_3 = \# S_4 = 2$. *If two nonconstant meromorphic functions* f *and* g *on* $\boldsymbol{C}$ *share* S_1, S_2, S_3, S_4 *CM, respectively, then* f *is a Möbius transform of* g.

On the other hand, in [5], the second author gave two meromorphic functions sharing 0, 1, ∞ and a three-point set with a certain specific property which are not transformed to each other by any Möbius transformation.

EXAMPLE. Let α be an entire function without zeros, and consider the two polynomials; (i) $P(z) = z^2(z-1)$ and (ii) $P(z) = z(z-1)^2$. For (i) put $f = \dfrac{\alpha(\alpha+1)}{\alpha^2+\alpha+1}$ and $g = \dfrac{\alpha+1}{\alpha^2+\alpha+1}$, and for (ii) put $f = \dfrac{1}{\alpha^2+\alpha+1}$ and $g = \dfrac{\alpha^2}{\alpha^2+\alpha+1}$. It is easy to see that there exists no Möbius transformation T such that $f = T \circ g$. By simple calculation they share 0, 1 and ∞ CM, and we have $P(f) = P(g)$ in each cases. Hence f and g share the zero sets of $P(z) + c$ CM for any complex number c. The functions f and g share infinitely many such three-point sets, but the sets are very restricted.

How about two meromorphic functions sharing two one-point sets and two three-point sets? In this paper, we consider two meromorphic functions f and g on $\boldsymbol{C}$ sharing two one-point sets and two three-point sets CM. If we study whether there is a Möbius transformation T such that $f = T \circ g$, it is enough to consider the case where the one-point sets are $\{0\}$ and $\{\infty\}$.

THEOREM 4. *Let* S_1 *and* S_2 *be two disjoint three-point subsets not containing* 0 *in* $\boldsymbol{C}$ *defined by* $P_1(z) = z^3 + a_1 z^2 + b_1 z + c_1 = 0$ *and* $P_2(z) = z^3 + a_2 z^2 + b_2 z + c_2 = 0$, *respectively. Assume* (C1) $a_1 \neq a_2$ *or both* $b_1 \neq b_2$ *and* $c_1 \neq c_2$, *and* (C2) $c_1 b_2 \neq b_1 c_2$ *or both* $c_1 a_2 \neq a_1 c_2$ *and* $c_1 \neq c_2$. *If two nonconstant meromorphic functions* f *and* g *on* $\boldsymbol{C}$ *share* 0, ∞, S_1, S_2 *CM, respectively, then* f *is a Möbius transform of* g.

Remark 1. *Take the transformation* $w = 1/z$ *which interchanges* 0 *and* ∞, *then* $P_j(z)$ *becomes* $c_j\{w^3 + (b_j/c_j)w^2 + (a_j/c_j)w + (1/c_j)\}$ $(j = 1, 2)$. *Hence,* (C2) *is the same as* (C1) *for these polynomials.*

Corollary 2. *Let* $S_1, \dots, S_4$ *be pairwise disjoint subsets in* $\hat{\boldsymbol{C}}$ *with* $\# S_1 = \# S_2 = 3$ *and* $\# S_3 = \# S_4 = 1$. *Assume that for any Möbius transformation* T *mapping* $S_3 \cup S_4$ *to* $\{0, \infty\}$, $\xi_1 + \eta_1 + \zeta_1 \neq \xi_2 + \eta_2 + \zeta_2$, *or both* $\xi_1\eta_1 + \eta_1\zeta_1 + \zeta_1\xi_1 \neq \xi_2\eta_2 + \eta_2\zeta_2 + \zeta_2\xi_2$ *and* $\xi_1\eta_1\zeta_1 \neq \xi_2\eta_2\zeta_2$, *where* $T(S_j) = \{\xi_j, \eta_j, \zeta_j\}$ $(j = 1, 2)$. *If two nonconstant meromorphic functions* f *and* g *on* $\boldsymbol{C}$ *share* S_1, S_2, S_3, S_4 *CM, respectively, then* f *is a Möbius transform of* g.

2. Representations of rank N and some lemmas

In this section we introduce the definition of representations of rank N. Let G be a torsion-free abelian multiplicative group, and consider a q-tuple $A = (a_1, \dots, a_q)$ of elements a_i in G.

Definition 1. Let N be a positive integer. We call integers μ_j *representations of rank* N of a_j if

$$\prod_{j=1}^{q} a_j^{\varepsilon_j} = \prod_{j=1}^{q} a_j^{\varepsilon_j'} \tag{2.1}$$

and

$$\sum_{j=1}^{q} \varepsilon_j \mu_j = \sum_{j=1}^{q} \varepsilon_j' \mu_j \tag{2.2}$$

are equivalent for any integers ε_j, ε_j' with $\sum_{j=1}^{q} |\varepsilon_j| \leq N$ and $\sum_{j=1}^{q} |\varepsilon_j'| \leq N$.

Remark 2. *For the existence of representations of rank N, see* [5]. *However, according to the construction of them in* [5], (2.1) *always implies* (2.2) *for any integers* ε_j, ε_j'. *Hence, in Definition* 2.1, *it is significant that* (2.2) *implies* (2.1) *for any integers* ε_j, ε_j' *with* $\sum_{j=1}^{q} |\varepsilon_j| \leq N$ *and* $\sum_{j=1}^{q} |\varepsilon_j'| \leq N$.

We introduce the following lemma due to Borel, whose proof can be found, for example, on p. 186 of [La].

Lemma 1. *If entire functions* $\alpha_0, \alpha_1, \dots, \alpha_n$ *without zeros satisfy*

$$\alpha_0 + \alpha_1 + \cdots + \alpha_n = 0,$$

then for each $j = 0, 1, \dots, n$ *there exists some* $k \neq j$ *such that* α_j/α_k *is constant.*

Now we investigate the torsion-free abelian multiplicative group $G = \mathscr{E}/\mathscr{C}$, where $\mathscr{E}$ is the abelian group of entire functions without zeros and $\mathscr{C}$ is the subgroup of all non-zero constant functions. We represent by $[\alpha]$ the element of $\mathscr{E}/\mathscr{C}$ with the representative $\alpha \in \mathscr{E}$. Let $\alpha_1, \dots, \alpha_q$ be elements in $\mathscr{E}$. Take representations μ_j of rank N of $[\alpha_j]$. For $\alpha = \prod_{j=1}^{q} \alpha_j^{\varepsilon_j}$ we define its index $\mathrm{Ind}(\alpha)$ by $\sum_{j=1}^{q} \varepsilon_j \mu_j$. The indices depend only on $\left[\prod_{j=1}^{q} \alpha_j^{\varepsilon_j}\right]$ under the condition $\sum_{j=1}^{q} |\varepsilon_j| \le N$. Trivially $\mathrm{Ind}(1) = 0$, and hence $\mathrm{Ind}(\alpha) = 0$ if and only if α is constant. Moreover, $\mathrm{Ind}(\alpha) = \mathrm{Ind}(\alpha')$ is equivalent to that α/α' is constant, where $\alpha = \prod_{j=1}^{q} \alpha_j^{\varepsilon_j}$ and $\alpha' = \prod_{j=1}^{q} \alpha_j^{\varepsilon_j'}$ with $\sum_{j=1}^{q} |\varepsilon_j| \le N$ and $\sum_{j=1}^{q} |\varepsilon_j'| \le N$.

We use the following Lemma in the proof of Theorem 4 which is an application of Lemma 1 (for the proof see [6, Lemma 2.3]).

LEMMA 2. *Assume that there is a relation*

$$\Psi(\alpha_1, \dots, \alpha_q) \equiv 0$$

where $\Psi(X_1, \dots, X_q) \in \boldsymbol{C}[X_1, \dots, X_q]$ *is a nonconstant polynomial of degree at most* N *of* $X_1, \dots, X_q$. *Then each term* $aX_1^{\varepsilon_1} \dots X_q^{\varepsilon_q}$ *of* $\Psi(X_1, \dots, X_q)$ *has another term*

$$bX_1^{\varepsilon_1'} \dots X_q^{\varepsilon_q'}$$

such that $\alpha_1^{\varepsilon_1} \dots \alpha_q^{\varepsilon_q}$ *and* $\alpha_1^{\varepsilon_1'} \dots \alpha_q^{\varepsilon_q'}$ *have the same indices, where* a *and* b *are non-zero constants.*

We close this section by introducing the theorem of completely multiple values and a generalization of Theorem 1.

Let f be a nonconstant meromorphic function, and let c be a point in $\hat{\boldsymbol{C}}$. If each zero of $f - c$ has multiplicity greater than 1, then we call c a completely multiple value of f. For meromorphic functions defined on $\boldsymbol{C}$ we have from [4, Theorem E] the following:

LEMMA 3. (i) *A nonconstant meromorphic function on* $\boldsymbol{C}$ *has at most four completely multiple values in* $\hat{\boldsymbol{C}}$.

(ii) *A nonconstant entire function has at most two completely multiple values in* $\boldsymbol{C}$.

(iii) *A nonconstant entire function without zeros has no completely multiple values in* $\boldsymbol{C}\backslash\{0\}$.

We give a generalization of Theorem 1 which is a constant target version of Theorem 1 of [2].

LEMMA 4. *Let* f *and* g *be two nonconstant meromorphic functions on* $\boldsymbol{C}$. *Let* $a_1, \dots, a_4$ *be four distinct points in* $\hat{\boldsymbol{C}}$ *and let* $b_1, \dots, b_4$ *be four distinct* *points in* $\hat{\boldsymbol{C}}$. *If* $f - a_j$ *and* $g - b_j$ *share zero CM* $(j = 1, \dots, 4)$, *then* f *is a Möbius transform of* g.

3. Proof of Theorem 4

We give a proof by contradiction. Let us assume that

$$\text{(NM)} \quad f \text{ is not any Möbius transform of } g.$$

In particular, $f \neq g$.

By assumption there exist entire functions without zeros α_0, α_1, α_1 such that

$$f = \alpha_0 g \tag{3.1}$$

and

$$f^3 + a_j f^2 + b_j f + c_j = \alpha_j (g^3 + a_j g^2 + b_j g + c_j) \qquad (j = 1, 2). \tag{3.2}$$

By substituting (3.1) into (3.2) we have

$$(\alpha_0^3 - \alpha_j) g^3 + a_j (\alpha_0^2 - \alpha_j) g^2 + b_j (\alpha_0 - \alpha_j) g + c_j (1 - \alpha_j) = 0 \qquad (j = 1, 2).$$

Consider the resultant R_0 of these as polynomials of g;

$$R_0 = \begin{vmatrix} \alpha_0^3 - \alpha_1 & a_1(\alpha_0^2 - \alpha_1) & b_1(\alpha_0 - \alpha_1) & c_1(1 - \alpha_1) & 0 & 0 \\ 0 & \alpha_0^3 - \alpha_1 & a_1(\alpha_0^2 - \alpha_1) & b_1(\alpha_0 - \alpha_1) & c_1(1 - \alpha_1) & 0 \\ 0 & 0 & \alpha_0^3 - \alpha_1 & a_1(\alpha_0^2 - \alpha_1) & b_1(\alpha_0 - \alpha_1) & c_1(1 - \alpha_1) \\ \alpha_0^3 - \alpha_2 & a_2(\alpha_0^2 - \alpha_2) & b_2(\alpha_0 - \alpha_2) & c_2(1 - \alpha_2) & 0 & 0 \\ 0 & \alpha_0^3 - \alpha_2 & a_2(\alpha_0^2 - \alpha_2) & b_2(\alpha_0 - \alpha_2) & c_2(1 - \alpha_2) & 0 \\ 0 & 0 & \alpha_0^3 - \alpha_2 & a_2(\alpha_0^2 - \alpha_2) & b_2(\alpha_0 - \alpha_2) & c_2(1 - \alpha_2) \end{vmatrix}$$

$$\begin{aligned} = & \sum_{\substack{0 \le k+l \le 3 \\ 0 \le k, l \le 3}} A_{9kl} \alpha_0^9 \alpha_1^k \alpha_2^l + \sum_{\substack{1 \le k+l \le 3 \\ 0 \le k, l \le 3}} A_{8kl} \alpha_0^8 \alpha_1^k \alpha_2^l \\ & + \sum_{\substack{1 \le k+l \le 3 \\ 0 \le k, l \le 3}} A_{7kl} \alpha_0^7 \alpha_1^k \alpha_2^l + \sum_{\substack{1 \le k+l \le 4 \\ 0 \le k, l \le 3}} A_{6kl} \alpha_0^6 \alpha_1^k \alpha_2^l \\ & + \sum_{\substack{2 \le k+l \le 4 \\ 0 \le k, l \le 3}} A_{5kl} \alpha_0^5 \alpha_1^k \alpha_2^l + \sum_{\substack{2 \le k+l \le 4 \\ 0 \le k, l \le 3}} A_{4kl} \alpha_0^4 \alpha_1^k \alpha_2^l \\ & + \sum_{\substack{2 \le k+l \le 5 \\ 0 \le k, l \le 3}} A_{3kl} \alpha_0^3 \alpha_1^k \alpha_2^l + \sum_{\substack{3 \le k+l \le 5 \\ 0 \le k, l \le 3}} A_{2kl} \alpha_0^2 \alpha_1^k \alpha_2^l \\ & + \sum_{\substack{3 \le k+l \le 5 \\ 0 \le k, l \le 3}} A_{1kl} \alpha_0 \alpha_1^k \alpha_2^l + \sum_{\substack{3 \le k+l \le 6 \\ 0 \le k, l \le 3}} A_{0kl} \alpha_1^k \alpha_2^l \equiv 0, \end{aligned} \tag{3.3}$$

where A_{jkl} are complex coefficients. In particular, any of the coefficients $A_{030} = -c_2^3$ of α_1^3, $A_{003} = c_1^3$ of α_1^3, $A_{930} = c_1^3$ of $\alpha_0^9\alpha_1^3$ and $A_{903} = -c_2^3$ of $\alpha_0^9\alpha_1^3$ are not zero, and the coefficients A_{900} of α_0^9 and A_{033} of $\alpha_1^3\alpha_2^3$ are the resultant of P_1 and P_2 which is not zero by assumption.

Let μ_0, μ_1, μ_2 be representations of $[\alpha_0]$, $[\alpha_1]$, $[\alpha_1]$ of rank 12. We see $\mu_0 \neq 0$ by (NM) and assume that $3\mu_0$, μ_1, μ_2 and 0 are distinct.

If $\mu_0 < 0$ and $\mu_1, \mu_2 \geq 0$, then in (3.3), $A_{900}\alpha_0^9$ is the unique term with the minimal index, which contradicts Lemma 2. If $\mu_1 < 0$ and $\mu_0, \mu_2 \geq 0$, then $A_{030}\alpha_1^3$ is the unique term with the minimal index, which is a contradiction. In the case that $\mu_2 < 0$ and $\mu_0, \mu_1 \geq 0$ we get the same contradiction. Hence we may assume that all μ_0, μ_1, μ_2 are non-negative by taking $-\mu_j$ in place of μ_j if they all are non-positive.

Consider the case where $0 < 3\mu_0 < \mu_1, \mu_2$. Note that in (3.3) the ranges of k, l of the summation symbols of the terms containing α_0^j $(j = 0, 1, \dots, 9)$ are $[(11-j)/3] \leq k + l \leq 3 + [(9-j)/3]$, where $[x]$ is the maximal integer not greater than x for a real number x. For such k, l except $k = l = 0$, $\mathrm{Ind}(\alpha_0^j\alpha_1^k\alpha_1^l) = j\mu_0 + k\mu_1 + l\mu_2 > (j + 3k + 3l)\mu_0 \geq (j + 3[(11-j)/3])\mu_0 \geq 9\mu_0$. Hence the term $A_{900}\alpha_0^9$ is the unique one with the minimal index, which is a contradiction.

If $0 < \mu_1 < 3\mu_0, \mu_2$ or $0 < \mu_2 < 3\mu_0, \mu_1$, then only $A_{030}\alpha_1^3$ or $A_{003}\alpha_2^3$, respectively, has the minimal index, which is a contradiction.

Therefore we conclude that one of $\mu_1 = 3\mu_0$, $\mu_2 = 3\mu_0$, $\mu_1 = \mu_2$, $\mu_1 = 0$ and $\mu_2 = 0$ holds.

(I) The case where $\mu_1 = 0$ or $\mu_2 = 0$.

First we show that $\mu_1 = 0$ and $\mu_2 = 0$ are equivalent.

Assume $\mu_1 = 0$. Then α_1 is constant. In (3.3), the term $A_{030}\alpha_1^3$ is a nonzero constant and is the unique term containing neither α_0 nor α_2. Hence there exists another constant term $\alpha_0^j\alpha_2^l$. Since $\mathrm{Ind}(\alpha_0^j\alpha_2^l) = j\mu_0 + l\mu_2 > 0$ for $j > 0$, such term must be of $j = 0$ and $\mu_2 = 0$. Therefore $\mu_1 = 0$ and $\mu_2 = 0$ are equivalent.

Now we put $\alpha_1 = C$.

(i) The case where $C = 1$.

It follows from $P_1(f) = CP_1(g)$ that

$$f^2 + fg + g^2 + a_1(f + g) + b_1 = 0. \tag{3.4}$$

Put $E(w_1, w_2) := \{z \in \boldsymbol{C} : (f(z), g(z)) = (w_1, w_2) \text{ or } (f(z), g(z)) = (w_2, w_1)\}$ for $w_1, w_2 \in \boldsymbol{C}$, and set $S_j = \{\xi_j, \eta_j, \zeta_j\}$ $(j = 1, 2)$.

First we show that $E(\xi_2, \eta_2) \neq \emptyset$ implies $E(\xi_2, \zeta_2) = \emptyset$ and $E(\eta_2, \zeta_2) = \emptyset$. Indeed, if $E(\xi_2, \eta_2) \neq \emptyset$, then we have

$$\xi_2^2 + \xi_2\eta_2 + \eta_2^2 + a_1(\xi_2 + \eta_2) + b_1 = 0, \tag{3.5}$$

and if $E(\xi_2, \zeta_2) \neq \emptyset$, then we get

$$\xi_2^2 + \xi_2\zeta_2 + \zeta_2^2 + a_1(\xi_2 + \zeta_2) + b_1 = 0. \tag{3.6}$$

From (3.5) and (3.6) we obtain $a_1 = -(\xi_2 + \eta_2 + \zeta_2) = a_2$. Together with (3.5), this yields $b_1 = b_2$, which contradicts (C1). Hence at least two of $E(\xi_2, \eta_2)$, $E(\eta_2, \zeta_2)$, $E(\zeta_2, \xi_2)$ are empty. We may assume $E(\xi_2, \eta_2) = E(\zeta_2, \xi_2) = \emptyset$ by rearranging the elements if necessary. Then f and g share ξ_2 and $\{\eta_2, \zeta_2\}$ CM, and hence by Corollary 1, f is a Möbius transform of g, which contradicts (NM).

(ii) The case where $C \neq \pm 1$.

In this case we have $P_1(f) = CP_1(g)$ and $E(\xi_2, \xi_2) = E(\eta_2, \eta_2) = E(\zeta_2, \zeta_2) = \emptyset$. We put $E_0(w_1, w_2) := \{z \in \boldsymbol{C} : (f(z), g(z)) = (w_1, w_2)\}$ for $w_1, w_2 \in \boldsymbol{C}$. If any of $E_0(\xi_2, \eta_2)$, $E_0(\xi_2, \zeta_2)$, $E_0(\eta_2, \zeta_2)$ are not empty, then we have $P_1(\xi_2) = CP_1(\eta_2) = CP_1(\zeta_2)$ and $P_1(\eta_2) = CP_1(\zeta_2)$, which deduce a contradiction $C = 1$. By the same way at least one of $E_0(w_1, w_2)$ and $E_0(w_2, w_1)$ are empty for distinct $w_1, w_2 \in \boldsymbol{C}$.

First assume that $E_0(\xi_2, \eta_2) \neq \emptyset$, $E_0(\xi_2, \zeta_2) \neq \emptyset$. Then all of $E_0(\eta_2, \zeta_2)$, $E_0(\eta_2, \xi_2)$, $E_0(\zeta_2, \xi_2)$ are empty. If $E_0(\zeta_2, \eta_2) \neq \emptyset$, then we can get a contradiction $C = 1$ by the same way as above. Hence in this case, f omits η_2 and ζ_2, and we see from $P_1(f) = CP_1(g)$ that f omits also zero. It is impossible by the little Picard theorem.

Next we assume that $E_0(\xi_2, \eta_2) \neq \emptyset$, $E_0(\eta_2, \zeta_2) \neq \emptyset$. Then $E_0(\xi_2, \zeta_2) = E_0(\eta_2, \xi_2) = E_0(\zeta_2, \eta_2) = \emptyset$. Therefore $f^{-1}(\xi_2) = g^{-1}(\eta_2)$, $f^{-1}(\eta_2) = g^{-1}(\zeta_2)$, $f^{-1}(\zeta_2) = g^{-1}(\xi_2)$, and hence, by Lemma 4 we see that f is a Möbius transform of g, which contradicts (NM).

In all other cases we can deduce contradictions.

(iii) The case where $C = -1$.

In this case we have $P_1(f) = -P_1(g)$ and $E(\xi_2, \xi_2) = E(\eta_2, \eta_2) = E(\zeta_2, \zeta_2) = \emptyset$. If any of $E(\xi_2, \eta_2)$, $E(\eta_2, \zeta_2)$ and $E(\zeta_2, \xi_2)$ are not empty, then we have $P_1(\xi_2) = -P_1(\eta_2) = P_1(\zeta_2) = -P_1(\xi_2)$, which is a contradiction. Hence we may assume that $E(\xi_2, \eta_2) = \emptyset$. Now we have

$$f^{-1}(\zeta_2) = g^{-1}(\xi_2) \cup g^{-1}(\eta_2), \qquad g^{-1}(\zeta_2) = f^{-1}(\xi_2) \cup f^{-1}(\eta_2).$$

As we have shown above $\mu_2 = 0$ and $\alpha_2 \equiv -1$ in this case. So, similarly we may assume

$$f^{-1}(\zeta_1) = g^{-1}(\xi_1) \cup g^{-1}(\eta_1), \qquad g^{-1}(\zeta_1) = f^{-1}(\xi_1) \cup f^{-1}(\eta_1).$$

Since we see that f and g omit 0 by $P_1(f) = -P_1(g)$, we get by using the second main theorem and the first main theorem of the value distribution theory

$$3T(r,f) \le \sum_{j=1,2}\left(N\left(r,\frac{1}{f-\xi_j}\right)+N\left(r,\frac{1}{f-\eta_j}\right)\right)+N\left(r,\frac{1}{f}\right)+S(r,f)$$
$$=\sum_{j=1,2}N\left(r,\frac{1}{g-\zeta_j}\right)+S(r,f)\le 2T(r,g)+S(r,f)$$

and $3T(r,g) \le 2T(r,f)+S(r,g)$ by the same way. They immediately lead to a contradiction.

(II) The case where $\mu_1 = 3\mu_0$ or $\mu_2 = 3\mu_0$.

First we show that $\mu_1 = 3\mu_0$ and $\mu_2 = 3\mu_0$ are equivalent.

Assume $\mu_1 = 3\mu_0$. Then we have $\mu_2 \neq 0$, otherwise $3\mu_0 = \mu_1 = \mu_2 = 0$ by the case (I), which is a contradiction.

We have denied $0 < \mu_2 < 3\mu_0, \mu_1$ and hence $\mu_2 \ge \mu_1 = 3\mu_0$. If $\mu_2 > 3\mu_0$, $A_{903}\alpha_0^9\alpha_2^3$ is the unique term with the maximal index, which is a contradiction. So we get also $\mu_2 = 3\mu_0$. Therefore $\mu_1 = 3\mu_0$ and $\mu_2 = 3\mu_0$ are equivalent, and we can deduce contradictions as in the case (I).

(III) The case where $\mu_1 = \mu_2$.

In this case $0 < \mu_1 = \mu_2 < 3\mu_0$ by what we have shown, and α_2/α_1 is a constant. Put $C = \alpha_2/\alpha_1$, then

$$\begin{aligned}
&(1-C)f^3g^3 + (a_1 - Ca_2)f^3g^2 + (a_2 - Ca_1)f^2g^3 \\
&\quad + (b_1 - Cb_2)f^3g + a_1a_2(1-C)f^2g^2 + (b_2 - Cb_1)fg^3 \\
&\quad + (c_1 - Cc_2)f^3 + (b_1a_2 - Ca_1b_2)f^2g + (a_1b_2 - Cb_1a_2)fg^2 + (c_2 - Cc_1)g^3 \\
&\quad + (c_1a_2 - Ca_1c_2)f^2 + b_1b_2(1-C)fg + (a_1c_2 - Cc_1a_2)g^2 \\
&\quad + (c_1b_2 - Cb_1c_2)f + (b_1c_2 - Cc_1b_2)g + c_1c_2(1-C) = 0.
\end{aligned}$$

If $C \neq 1$, then we see from this equation that f and g have neither zeros nor poles. If $C = 1$, then the above equation reduces to

$$\begin{aligned}
&(a_1 - a_2)f^2g^2 + (b_1 - b_2)fg(f+g) + (c_1 - c_2)(f^2 + fg + g^2) \\
&\quad + (b_1a_2 - a_1b_2)fg + (c_1a_2 - a_1c_2)(f+g) + (c_1b_2 - b_1c_2) = 0. \qquad (3.7)
\end{aligned}$$

Then if $a_1 \neq a_2$ and $b_1c_2 \neq c_1b_2$, f and g have neither zeros nor poles. In both cases where $C \neq 1$ and where $C = 1$, $a_1 \neq a_2$, $b_1c_2 \neq b_2c_1$, by Lemma 1 one of f^mg^n is constant, where m and n are integers with $0 \le |m|, |n| \le 3$. Since f and g are not constant, $mn \neq 0$, and we have $|m| \neq |n|$ by the assumption (NM). Without loss of generality we may assume that $1 \le |m| < |n| \le 3$. Then we get

$|m|T(r,f) = |n|T(r,g) + O(1)$. On the other hand by the second fundamental theorem

$$
\begin{aligned}
6T(r,f) &\le \sum_{j=1,2}\left(N\left(r,\frac{1}{f-\xi_j}\right) + N\left(r,\frac{1}{f-\eta_j}\right) + N\left(r,\frac{1}{f-\zeta_j}\right)\right) \\
&\quad + N(r,1/f) + N(r,f) + S(r,f) \\
&= \sum_{j=1,2}\left(N\left(r,\frac{1}{g-\xi_j}\right) + N\left(r,\frac{1}{g-\eta_j}\right) + N\left(r,\frac{1}{g-\zeta_j}\right)\right) \\
&\quad + N(r,1/g) + N(r,g) + S(r,f) \\
&\le 8T(r,g) + S(r,f).
\end{aligned}
$$

These yield $6|n| \le 8|m|$ which does not hold for any $(|m|,|n|) = (1,2),(1,3),(2,3)$.

Hence $C = 1$, and at least one of $a_1 = a_2$ and $b_1c_2 = b_2c_1$ hold in the case.

By symmetricity we consider only the case where $C = 1$ and $a_1 = a_2$. In this case, we have $\alpha_1 = \alpha_2$ and

$$
R_0 = \begin{vmatrix}
\alpha_0^3-\alpha_1 & a_1(\alpha_0^2-\alpha_1) & b_1(\alpha_0-\alpha_1) & c_1(1-\alpha_1) & 0 & 0 \\
0 & \alpha_0^3-\alpha_1 & a_1(\alpha_0^2-\alpha_1) & b_1(\alpha_0-\alpha_1) & c_1(1-\alpha_1) & 0 \\
0 & 0 & \alpha_0^3-\alpha_1 & a_1(\alpha_0^2-\alpha_1) & b_1(\alpha_0-\alpha_1) & c_1(1-\alpha_1) \\
\alpha_0^3-\alpha_1 & a_1(\alpha_0^2-\alpha_1) & b_2(\alpha_0-\alpha_1) & c_2(1-\alpha_1) & 0 & 0 \\
0 & \alpha_0^3-\alpha_1 & a_1(\alpha_0^2-\alpha_1) & b_2(\alpha_0-\alpha_1) & c_2(1-\alpha_1) & 0 \\
0 & 0 & \alpha_0^3-\alpha_1 & a_1(\alpha_0^2-\alpha_1) & b_2(\alpha_0-\alpha_1) & c_2(1-\alpha_1)
\end{vmatrix}
$$

$$
= \begin{vmatrix}
\alpha_0^3-\alpha_1 & a_1(\alpha_0^2-\alpha_1) & b_1(\alpha_0-\alpha_1) & c_1(1-\alpha_1) & 0 & 0 \\
0 & \alpha_0^3-\alpha_1 & a_1(\alpha_0^2-\alpha_1) & b_1(\alpha_0-\alpha_1) & c_1(1-\alpha_1) & 0 \\
0 & 0 & \alpha_0^3-\alpha_1 & a_1(\alpha_0^2-\alpha_1) & b_1(\alpha_0-\alpha_1) & c_1(1-\alpha_1) \\
0 & 0 & b_0(\alpha_0-\alpha_1) & c_0(1-\alpha_1) & 0 & 0 \\
0 & 0 & 0 & b_0(\alpha_0-\alpha_1) & c_0(1-\alpha_1) & 0 \\
0 & 0 & 0 & 0 & b_0(\alpha_0-\alpha_1) & c_0(1-\alpha_1)
\end{vmatrix}
$$

$$
= (\alpha_0^3-\alpha_1)^2 \begin{vmatrix}
\alpha_0^3-\alpha_1 & a_1(\alpha_0^2-\alpha_1) & b_1(\alpha_0-\alpha_1) & c_1(1-\alpha_1) \\
b_0(\alpha_0-\alpha_1) & c_0(1-\alpha_1) & 0 & 0 \\
0 & b_0(\alpha_0-\alpha_1) & c_0(1-\alpha_1) & 0 \\
0 & 0 & b_0(\alpha_0-\alpha_1) & c_0(1-\alpha_1)
\end{vmatrix}
$$

$$= (\alpha_0^3 - \alpha_1)^2 \times$$
$$\begin{vmatrix} \alpha_0^3 - \alpha_1 & a_1(\alpha_0^2 - \alpha_1) & b_1(\alpha_0 - \alpha_1) & c_1(1 - \alpha_1) \\ b_0(\alpha_0 - \alpha_1) & c_0(1 - \alpha_1) & 0 & 0 \\ 0 & b_0(\alpha_0 - \alpha_1) & c_0(1 - \alpha_1) & 0 \\ -c_0/c_1(\alpha_0^3 - \alpha_1) & -a_1(c_0/c_1)(\alpha_0^2 - \alpha_1) & (b_0 - b_1 c_0/c_1)(\alpha_0 - \alpha_1) & 0 \end{vmatrix}$$
$$= -(\alpha_0^3 - \alpha_1)^2(1 - \alpha_1) \begin{vmatrix} b_0(\alpha_0 - \alpha_1) & c_0(1 - \alpha_1) & 0 \\ 0 & b_0(\alpha_0 - \alpha_1) & c_0(1 - \alpha_1) \\ -c_0(\alpha_0^3 - \alpha_1) & -a_1 c_0(\alpha_0^2 - \alpha_1) & (c_1 b_0 - b_1 c_0)(\alpha_0 - \alpha_1) \end{vmatrix}$$
$$\equiv 0,$$

where $b_0 = b_2 - b_1$, $c_0 = c_2 - c_1$. Since $\alpha_0^3 \not\equiv \alpha_1$ and $\alpha_1 \not\equiv 1$, the final determinant is identically equal to zero. It is expanded as

$$\begin{aligned} & b_0^2(c_1 b_0 - b_1 c_0)(\alpha_0 - \alpha_1)^3 - c_0^3(1 - \alpha_1)^2(\alpha_0^3 - \alpha_1) \\ & \qquad + a_1 b_0 c_0^2(1 - \alpha_1)(\alpha_0 - \alpha_1)(\alpha_0^2 - \alpha_1) \\ & \quad = b_0^2(c_1 b_0 - b_1 c_0)(\alpha_0^3 - 3\alpha_0^2\alpha_1 + 3\alpha_0\alpha_1^2 - \alpha_1^3) \\ & \qquad - c_0^3(\alpha_0^3 - \alpha_1 - 2\alpha_0^3\alpha_1 + 2\alpha_1^2 + \alpha_0^3\alpha_1^2 - \alpha_1^3) \\ & \qquad + a_1 b_0 c_0^2(-\alpha_1^3 + \alpha_1^2 + \alpha_0\alpha_1^2 + \alpha_0^2\alpha_1^2 - \alpha_0\alpha_1 - \alpha_0^2\alpha_1 - \alpha_0^3\alpha_1 + \alpha_0^3) \\ & \quad \equiv 0. \end{aligned}$$

Since $0 < \mu_1 < 3\mu_0$, among all terms which appear in the above the term $\alpha_0^3\alpha_1^2$ is the unique one with the maximal index. Hence its coefficient $c_0 = 0$, i.e., $c_1 = c_2$, which contradicts (C1).

Now we have completed the proof.

4. Exceptional cases

In this section we treat the cases which are excluded by Theorem 4; (a) $a_1 = a_2$, $b_1 = b_2$; (b) $a_1 = a_2$, $c_1 = c_2$; (c) $c_1 b_2 = b_1 c_2$, $c_1 a_2 = a_1 c_2$; (d) $c_1 b_2 = b_1 c_2$, $c_1 = c_2$. The final case is equivalent to that $c_1 = c_2$, $b_1 = b_2$, and we treat only the cases (a) and (b) since the case (c) is equivalent to the case (a) by symmetricity. For simplicity we write $\alpha = \alpha_0$.

(a) The case of $a_1 = a_2$, $b_1 = b_2$.

In the proof we obtained these on treating $\alpha_1 \equiv 1$ as a contradiction. In that case we have

$$(f^2 + fg + g^2) + a_1(f + g) + b_1 = 0. \tag{4.1}$$

By substituting (3.1) into this we get $(\alpha^2+\alpha+1)g^2+a_1(\alpha+1)g+b_1=0$, and rewrite as

$$\begin{aligned}\{2(\alpha^2+\alpha+1)g+a_1(\alpha+1)\}^2 &= a_1^2(\alpha+1)^2-4b_1(\alpha^2+\alpha+1)\\ &= (a_1^2-4b_1)\alpha^2+2(a_1^2-2b_1)\alpha+(a_1^2-4b_1).\end{aligned}$$

The entire function α without zeros is nonconstant by (NM). So, since it has no completely multiple values by Lemma 3, $a_1^2-4b_1=0$ or the final side above is a perfect square of α which implies $(a_1^2-2b_1)^2-(a_1^2-4b_1)^2=0$, i.e., $b_1=0$ or $b_1=a_1^2/3$.

(1) The case of $b_1=a_1^2/4$.

Take an entire function β such that $\beta^2=\alpha$, and let

$$g=-\frac{a_1}{2(\beta^2+\beta+1)} \qquad \text{and} \qquad f=-\frac{a_1\beta^2}{2(\beta^2+\beta+1)}.$$

They satisfy (4.1), but we can see that one of them is not any Möbius transform of the other. In this case the defining polynomials of S_j are $z^3+a_1z^2+\dfrac{a_1^2}{4}z+c_j$ $(j=1,2)$.

(2) The case where $b_1=0$.

Let

$$g=-\frac{a_1(\alpha+1)}{\alpha^2+\alpha+1} \qquad \text{and} \qquad f=-\frac{a_1\alpha(\alpha+1)}{\alpha^2+\alpha+1}.$$

They satisfy (4.1), but one of them is not any Möbius transformation of the other. In this case the defining polynomials of S_j are $z^3+a_1z^2+c_j$ $(j=1,2)$.

(3) The case where $b_1=a_1^2/3$.

Let

$$g=\frac{a_1(\omega_1\alpha+\omega_2)}{\alpha^2+\alpha+1} \qquad \text{and} \qquad f=\frac{a_1\alpha(\omega_1\alpha+\omega_2)}{\alpha^2+\alpha+1},$$

where ω_1 and ω_2 are the two roots of $3z^2+3z+1=0$. Then f and g satisfy (4.1) and there is no Möbius transformation T such that $f=T\circ g$. In this case the defining polynomials of S_j are $z^3+a_1z^2+\dfrac{a_1^2}{3}z+c_j$ $(j=1,2)$.

(b) The case where $a_1=a_2$, $c_1=c_2$.

In the proof we obtained these on treating $\alpha_2/\alpha_1\equiv 1$ as a contradiction. Moreover note that f and g have no zeros since otherwise $b_1c_2=c_1b_2$ by (3.7), and hence $P_1=P_2$, which is a contradiction. Then we have from (3.7)

$$fg(f+g)+a_1fg-c_1=0. \tag{4.2}$$

Rewrite this as $gf^2 + (g^2 + a_1 g)f - c_1 = 0$ and

$$\{2gf + (g^2 + a_1 g)\}^2 = (g^2 + a_1 g)^2 + 4c_1 g = g(g^3 + 2a_1 g^2 + a_1^2 g + 4c_1).$$

Since g omits 0, it has at most two completely multiple values by Lemma 3. Hence the cubic polynomial $z^3 + 2a_1 z^2 + a_1^2 z + 4c_1$ has a multiple zero. We can obtain $c_1 = \frac{a_1^3}{27}$ by simple calculation. Take an entire function β such that $\beta^3 = \alpha$ and put

$$f = \frac{a_1 \beta^2}{3(\beta + 1)} \qquad \text{and} \qquad g = \frac{a_1}{3\beta(\beta + 1)}.$$

Then f and g satisfy (4.2), and there exists no Möbius transformation T such that $f = T \circ g$. In this case the defining polynomials of S_j are $z^3 + a_1 z^2 + b_j z + \frac{a_1^3}{27}$ $(j = 1, 2)$.

Acknowledgement

The authors would like to express thanks to the referee for careful comments.

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