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Higher-order asymptotic expansions for a parabolic system modeling chemotaxis in the whole space

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ABSTRACT. We consider the initial value problem for a system of parabolic partial differential equations modeling chemotaxis in $\mathbf{R}^n (n \geq 1)$, and give the asymptotic profiles for a specific class of solutions by space-time higher-order asymptotic expansions.

1. Introduction

In this paper we are concerned with the large time behavior of solutions to the initial value problem for the system of parabolic partial differential equations:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla v), & x \in \mathbf{R}^n,\ t > 0, \\ \partial_t v = \Delta v - v + u, & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \mathbf{R}^n. \end{cases} \tag{P}$$

This system is a simple mathematical model to describe chemotaxis which is a biological phenomenon simulating the directed movement of an organism in response to gradients of a chemical attractant (see [11]).

Let us now recall the previous results for (P). It is well known that the possibility of blow-up of nonnegative solutions depends strongly on space dimension: The finite time blow-up never occurs in the case $n = 1$, while it can occur in the case $n \geq 2$. For related results to these studies, we refer to [2, 8, 6, 18, 7].

Concerning the large time behavior of solutions to (P), it is known in [19] that when $n \geq 2$, every bounded solution to (P) decays to zero as $t \to \infty$ and behaves like the heat kernel with the self-similarity. Furthermore, it was shown in [20] that the result obtained in [19] holds for the case $n = 1$, and the improved asymptotic profiles of bounded solution (u, v) to (P) are given

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as follows: *Let* $1 \leq q \leq \infty$. *Then the following assertions hold under suitable initial conditions.*

(i) *In the case* $n \geq 2$, *the integral* $\int_0^\infty \int_{\mathbf{R}^n} u\nabla v \, dyds$ *converges and*

$$\lim_{t\to\infty} t^{n(1-1/q)/2+1/2} \left\| u(t) - \int_{\mathbf{R}^n} u_0 \, dy G(t) + \left(\int_{\mathbf{R}^n} y u_0 \, dy + \int_0^\infty \int_{\mathbf{R}^n} u\nabla v \, dyds \right) \cdot \nabla G(t) \right\|_q = 0. \quad (1.1)$$

(ii) *In the case* $n = 1$, $|\int_0^t \int_{\mathbf{R}} u\partial_y v \, dyds| \leq C \log(1+t)$ $(t > 0)$ *and*

$$\lim_{t\to\infty} t^{(1-1/q)/2+1/2} (\log t)^{-1} \left\| u(t) - \int_{\mathbf{R}} u_0 \, dy G(t) + \left(\int_{\mathbf{R}} y u_0 \, dy + \int_0^t \int_{\mathbf{R}} u\partial_y v \, dyds \right) \partial_x G(t) \right\|_q = 0. \quad (1.2)$$

(iii) v *has the same asymptotic behavior as* u.

Here $\|\cdot\|_q$ *is the usual* $L^q(\mathbf{R}^n)$*-norm and* $G = G(x,t)$ *is the heat kernel, that is,*

$$G(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}. \quad (1.3)$$

As noted above, we see that the logarithmic term appears in the asymptotic rate of (1.2), because the L^q-estimates of the solutions for $n = 1$ might not decay faster than those for $n \geq 2$.

Kato [10] introduced a correction term to remove the logarithmic function appearing in the asymptotic rate of (1.2), and obtained the improved asymptotic profiles of bounded solution (u,v) to (P) for $n = 1$: *Let* $1 \leq q \leq \infty$. *Then, under suitable initial conditions, it holds that the integral* $\int_0^\infty \int_{\mathbf{R}} u\partial_y v \, dyds$ *converges and*

$$\lim_{t\to\infty} t^{(1-1/q)/2+1/2} \left\| u(t) - \int_{\mathbf{R}} u_0 \, dy G(t) + \left(\int_{\mathbf{R}} y u_0 \, dy + \int_0^\infty \int_{\mathbf{R}} u\partial_y v \, dyds \right) \partial_x G(t) + w(t) \right\|_q = 0, \quad (1.4)$$

where the correction term $w(x,t)$ *is defined by*

$$w(x,t) = \left(\int_{\mathbf{R}} u_0 \, dy \right)^2 \int_0^t \int_{\mathbf{R}} G(x-y, t-s) \partial_y (G(y, 1+s) \partial_y G(y, 1+s)) dyds,$$

and the upper and lower bounds of decay rates for $\|w(t)\|_\infty$ *are given by*

$$c_1(1+t)^{-1} \le \|w(t)\|_\infty \le C_1(1+t)^{-1}, \qquad t \ge 2$$

with some positive constants c_1, C_1. *Also,* v *has the same asymptotic behavior as* u.

For a further study on the asymptotic profiles of bounded solutions to (P) in the case $n = 1$, adjusting the center of the heat kernel by use of a shift which is suitably determined by the initial data and the nonlinear term, Nishihara [21] obtained the decay estimates of difference between the solution and the heat kernel whose center is adjusted. The decay estimates obtained in this result are rather sharp, though he imposed stronger assumptions on the initial data u_0 than ones in [10].

Such an asymptotic profile with self-similarity was observed for other nonlinear partial differential equations. We now refer to several works, closely related to our study. Escobedo-Zuazua [3] proved that the solutions to the heat convection-diffusion behave like the heat kernel as $t \to \infty$ if the diffusion term is more dominant than the convection one. Carpio [1] gave the asymptotic profiles of solutions to the incompressible Navier-Stokes equations up to the second order in terms of the heat kernel. Moreover, it was shown in [4] that under small initial data, a solution with the space-time decay properties admits the higher-order asymptotic expansion in terms of the space-time derivatives of Gaussian-like functions. This result improves essentially an earlier one obtained in [1]. Here we note the fact that the solution treated in [4] decays sufficiently faster because for the initial data in $L^1(\mathbf{R}^n)$ space, the average of initial data is naturally zero by virtue of the divergence free condition for the initial data (see [15]). This is the reason why the logarithmic term does not appear in the asymptotic rate of the expansion in contrast to (P) in one-dimensional space. Ogawa [22] and Luckhaus-Sugiyama [14] discussed that for a parabolic-elliptic system with degenerate diffusion modeling chemotaxis, a solution behaves like the Barenblatt solution as $t \to \infty$, where its solution is the self-similar one to the porous medium equation, and obtained the convergence rates for the difference between these solutions. In [17], it was proved that the solutions to generalized Burgers equations in the one-dimensional space tend to nonlinear waves at the rate $t^{-1}\log t$ in $L^\infty(\mathbf{R})$ as $t \to \infty$. This asymptotic rate in $L^\infty(\mathbf{R})$ space was improved to the rate t^{-1} by exactly giving the second asymptotic profile of the solution. For detail, we refer to [9]. Recently, Kobayashi-Kawashima [12] showed that when $n \ge 3$, the solutions to the drift-diffusion system closely related to (P) approach asymptotically to the heat kernel. These asymptotic profiles decay in $L^q(\mathbf{R}^n)$ $(1 \le q \le \infty)$ space at the rate $t^{-n(1-1/q)/2-1/2}$ as $t \to \infty$ if $n \ge 4$, and at the rate $t^{-3(1-1/q)/2-1/2}\log t$ as $t \to \infty$ if $n = 3$. As mentioned above, the logarithmic term appears in the asymptotic rate for the case $n = 3$. The situation is rather similar to that of (P) in the case $n = 1$. To remove the logarithmic function, Ogawa-Yamamoto [23] gave a correction term on the basis of the argument used in [10], and obtained the improved asymptotic profiles for the solutions.

Our aim of this paper is to give the higher-order asymptotic expansions of solutions to (P) in higher-dimensional case. More precisely, we shall prove the three assertions for the solutions to (P) with the following space-time decay properties:

$$\sup_{\substack{x\in\mathbf{R}^n,\, t>0\\ 0\le\mu\le\gamma}} (1+|x|)^{\gamma-\mu}(1+t)^{\mu/2}(|u(x,t)|+|v(x,t)|) < \infty, \tag{D}$$

where γ is either n or $n+1$. Under the appropriate moment conditions on the initial data,

(i) in the case $n \ge 2$, the solutions to (P) satisfying (D) with $\gamma = n$ admit asymptotic expansions up to n-th order,

(ii) in the case $n \ge 1$, the solutions to (P) satisfying (D) with $\gamma = n+1$ admit asymptotic expansions up to $(n+1)$-st order under the condition $\int_{\mathbf{R}^n} u_0\, dy = 0$,

(iii) there exists a unique solution to (P) with (D) for small initial data.

The proofs of the assertions (i) and (ii) are obtained by applying techniques in [4], that is, some decay estimates for the solutions and Taylor's formula for the heat kernel. For the proof of the assertion (iii), we use the contraction mapping principle. As mentioned in Section 2 below, in the assertion (i) (Theorem 1), we need to introduce a correction term on the basis of the method used in [10] because the logarithmic term appears in the asymptotic rates of the expansions, and the desired asymptotic rates can not be obtained if we do not add its term. Furthermore, under the influence of such a correction term, there appears a difference between the expansions in the odd and even dimensional cases. On the other hand, in the assertion (ii) (Theorem 2), the logarithmic term does not appear in the asymptotic rates of the expansions since the solutions decay sufficiently faster.

The plan of this paper is the following: In Section 2, we state our main results in this paper. In Section 3, we prepare several lemmas which will be used in this paper. In Section 4, we show some L^q-estimates for the solutions to (P) satisfying (D) with $\gamma = n$. In Sections 5–7, we give the proofs of Theorems 1, 2 and 3, respectively.

2. Main Theorems

We denote by $\|\cdot\|_q$ the usual $L^q(\mathbf{R}^n)$-norm, and by $W^{k,q}(\mathbf{R}^n)$ the usual Sobolev space. $\mathscr{B}(\mathbf{R}^n)$ is the Banach space of all bounded and uniformly continuous functions on $\mathbf{R}^n$ with the essential supremum norm. We define the weighted Lebesgue space $L_a^q(\mathbf{R}^n)$ by

$$L_a^q(\mathbf{R}^n) = \{f \in L^q(\mathbf{R}^n) \mid (1 + |\cdot|)^a |f| \in L^q(\mathbf{R}^n)\}.$$

The integer part of s is denoted by $[s]$. For simplicity, we use the notation

$$\mathbf{Z}_+ = \mathbf{N} \cup \{0\}, \qquad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n,$$

$$\partial_t = \frac{\partial}{\partial t}, \qquad \partial_j = \frac{\partial}{\partial x_j}, \qquad \partial_x^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}, \qquad \nabla = (\partial_1, \dots, \partial_n).$$

Throughout this paper, let γ be either n or $n+1$, and for the initial functions u_0, v_0, we always assume that

$$u_0, v_0, \partial_j v_0 \in L^1(\mathbf{R}^n) \cap \mathscr{B}(\mathbf{R}^n) \cap L_\gamma^\infty(\mathbf{R}^n) \qquad (1 \le j \le n).$$

To give the definition of solutions to (P), we define $e^{t\Delta} f(x)$ by

$$e^{t\Delta} f(x) = \int_{\mathbf{R}^n} G(x - y, t) f(y) dy,$$

where $G(x,t)$ is the heat kernel given by (1.3).

Definition 1. A function (u,v) on $\mathbf{R}^n \times [0,T]$ $(0 < T < \infty)$ is said to be a solution to (P) on $\mathbf{R}^n \times [0,T]$ if u, v satisfy

$$u, v, \partial_j v \in C([0,T]; L^1(\mathbf{R}^n)) \cap C([0,T]; \mathscr{B}(\mathbf{R}^n)) \qquad (1 \le j \le n),$$

and for all $0 < t \le T$,

$$u(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \nabla \cdot (u \nabla v)(s) ds, \tag{2.1}$$

$$v(t) = e^{-t} e^{t\Delta} v_0 + \int_0^t e^{-(t-s)} e^{(t-s)\Delta} u(s) ds. \tag{2.2}$$

Also, (u,v) is said to be a solution to (P) on $\mathbf{R}^n \times [0,\infty)$ if (u,v) is a solution to (P) on $\mathbf{R}^n \times [0,T]$ for all $0 < T < \infty$.

Remark 1. Making use of the standard regularity argument for parabolic equations (for example, see [13]), we see that (u,v) is a classical solution to (P) on $\mathbf{R}^n \times (0,T]$, which satisfies

$$u, v \in C((0,T); W^{2,q}(\mathbf{R}^n)) \cap C^1((0,T); L^q(\mathbf{R}^n))$$

for all $1 < q < \infty$, and

$$\partial_j u, \Delta v \in C((0,T); L^\infty(\mathbf{R}^n)) \qquad (1 \le j \le n).$$

Before stating our main results, we introduce the following notation

$$M_0 = \int_{\mathbf{R}^n} u_0 \, dy,$$

and the correction term $R(t)$ mentioned in Introduction as

$$R(t) = \begin{cases} W(t) - \dfrac{M_0^2}{2(8\pi)} \Delta G(1+t) \log(1+t) \quad \text{if } n = 2, \\ W(t) - \dfrac{M_0^2}{2(8\pi)^{n/2}} \displaystyle\sum_{|\alpha|+p \le [(n-3)/2]} \dfrac{(-1)^p \Delta \partial_x^{2\alpha} \partial_t^p G(1+t)}{2^{|\alpha|-1} \alpha! p! \{(n-2) - 2(|\alpha|+p)\}} \\ \qquad \text{if } n \text{ is odd with } n \ge 3, \\ W(t) - \dfrac{M_0^2}{2(8\pi)^{n/2}} \displaystyle\sum_{|\alpha|+p \le [(n-3)/2]} \dfrac{(-1)^p \Delta \partial_x^{2\alpha} \partial_t^p G(1+t)}{2^{|\alpha|-1} \alpha! p! \{(n-2) - 2(|\alpha|+p)\}} \\ \quad - \dfrac{M_0^2}{2(8\pi)^{n/2}} \displaystyle\sum_{|\alpha|+p=(n-2)/2} \dfrac{(-1)^p}{2^{|\alpha|} \alpha! p!} \Delta \partial_x^{2\alpha} \partial_t^p G(1+t) \log(1+t) \\ \qquad \text{if } n \text{ is even with } n \ge 3, \end{cases} \tag{2.3}$$

where

$$W(t) = M_0^2 \int_0^t e^{(t-s)\Delta} \nabla \cdot (G \nabla G)(1+s) ds. \tag{2.4}$$

The first result gives us the space-time asymptotic expansions of the solutions to (P) on $\mathbf{R}^n \times [0, \infty)$ satisfying (D) with $\gamma = n$ under the condition $|x|^n u_0 \in L^1(\mathbf{R}^n)$.

Theorem 1. *Assume that* $n \ge 2$ *and* $1 \le q \le \infty$, *and let* (u, v) *be the solution to* (P) *on* $\mathbf{R}^n \times [0, \infty)$ *satisfying* (D) *with* $\gamma = n$. *Under the condition* $|x|^n u_0 \in L^1(\mathbf{R}^n)$, *the following assertions hold:*

(i) *If* n *is odd, then the integral*

$$\int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u \nabla v \, dyds$$

converges for $|\alpha| + 2p \le n - 1$, *and*

$$\lim_{t\to\infty} t^{n(1-1/q)/2+n/2} \Bigg\| u(t) - \sum_{|\alpha| \le n} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^\alpha G(1+t) \int_{\mathbf{R}^n} y^\alpha u_0 \, dy \\ + \sum_{|\alpha|+2p \le n-1} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \\ \cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u \nabla v \, dyds + R(t) \Bigg\|_q = 0. \tag{2.5}$$

(ii) *If n is even, then the integral*

$$\int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u \nabla v \, dyds$$

converges for $|\alpha| + 2p \le n-2$ *and the one*

$$\int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p \{(u\nabla v)(y,s) - M_0^2 (G\nabla G)(y,1+s)\} dyds$$

is well-defined for $|\alpha| + 2p = n-1$, *and*

$$\begin{aligned} \lim_{t\to\infty} t^{n(1-1/q)/2+n/2} \Bigg\| u(t) &- \sum_{|\alpha|\le n} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^\alpha G(1+t) \int_{\mathbf{R}^n} y^\alpha u_0 \, dy \\ &+ \sum_{|\alpha|+2p\le n-2} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u\nabla v \, dyds \\ &+ \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{|\alpha|+p=(n-2)/2} \frac{(-1)^p}{2^{|\alpha|}\alpha! p!} \Delta \partial_x^{2\alpha} \partial_t^p G(1+t) \log(1+t) \\ &+ \sum_{|\alpha|+2p=n-1} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p \\ &\times \{(u\nabla v)(y,s) - M_0^2 (G\nabla G)(y,1+s)\} dyds + R(t) \Bigg\|_q = 0. \end{aligned} \tag{2.6}$$

(iii) *The correction term $R(t)$ defined by* (2.3) *is estimated as*

$$\|R(t)\|_q \le C M_0^2 (1+t)^{-n(1-1/q)/2-n/2} \qquad \text{for } t > 0. \tag{2.7}$$

(iv) *v also has the same asymptotic profile as u.*

Remark 2. The decay estimate for (2.7) is shown in the assertion (ii) of Lemma 5 below. Moreover, in the case $n = 2$,

$$\|R(t)\|_\infty = \left\| W(t) - \frac{M_0^2}{2(8\pi)} \Delta G(1+t) \log(1+t) \right\|_\infty \ge c M_0^2 (1+t)^{-2} \qquad \text{for } t \ge 2,$$

where c is a positive constant. For details, see the assertion (iii) of Lemma 5 below.

Now, we explain why the correction term (2.3) is needed in Theorem 1. First of all, we see that when $|\alpha| + 2p \le n-2$, the integral $\int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u\nabla v \, dyds$ converges, but when $|\alpha| + 2p = n-1$, the estimate

$$\left|\int_0^t\int_{\mathbf{R}^n} y^\alpha(1+s)^p u\nabla v\, dyds\right| \le C\log(1+t) \qquad (t>0) \tag{2.8}$$

is satisfied because from the L^q-estimates of u and ∇v we have the following estimate (see (5.9))

$$\sup_{s>0}(1+s)^{n/2+1/2-|\alpha|/2}\left|\int_{\mathbf{R}^n} y^\alpha u\nabla v\, dy\right| < \infty \qquad \text{for } |\alpha|\le n.$$

If the asymptotic expansions up to n-th order are deduced without introducing the correction term, then the logarithmic term appears in the asymptotic rates of the expansions due to (2.8), and the asymptotic rate given in Theorem 1 can not be obtained. Therefore, to remove the logarithmic function, we have introduced (2.3) as the correction term.

Next, it is observed in Theorem 1 that a difference between the asymptotic expansions in the odd and even dimensional cases appears in the n-th order term of the expansions.

Indeed, in the odd dimensional case, the integral $\int_0^\infty\int_{\mathbf{R}^n} y^\alpha(1+s)^p u\nabla v\, dyds$ is well-defined for $|\alpha|+2p=n-1$ by means of (4.24), (4.25) (see Proposition 3) and the equalities

$$\int_0^\infty\int_{\mathbf{R}^n} y^\alpha(1+s)^p(G\nabla G)(y,1+s)dyds = 0 \qquad (|\alpha|+2p=n-1).$$

Hence the coefficients in the n-th order term of the asymptotic expansions are determined by the well-defined ones $\int_0^\infty\int_{\mathbf{R}^n} y^\alpha(1+s)^p u\nabla v\, dyds$ $(|\alpha|+2p=n-1)$ without adding an extra term.

On the other hand, in the even dimensional case, (2.8) does not assure the convergence of the integral $\int_0^\infty\int_{\mathbf{R}^n} y^\alpha(1+s)^p u\nabla v\, dyds$ for $|\alpha|+2p=n-1$, but

$$\int_0^\infty\int_{\mathbf{R}^n} y^\alpha(1+s)^p\{(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)\}dyds \tag{2.9}$$

converges for $|\alpha|+2p=n-1$ by giving the following estimate (see (5.5))

$$\left|\int_{\mathbf{R}^n} y^\alpha\{(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)\}dy\right| \le C(1+s)^{-n/2-1+|\alpha|/2}$$

for $s>0$, $|\alpha|\le n$. Therefore, in order to avoid the logarithmic term appearing in (2.8), it is necessary to add the extra term (see (5.15))

$$M_0^2\sum_{|\alpha|+2p=n-1}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot\int_0^t\int_{\mathbf{R}^n} y^\alpha(1+s)^p(G\nabla G)(y,1+s)dyds$$

$$= \frac{M_0^2}{2(8\pi)^{n/2}}\sum_{|\alpha|+p=(n-2)/2}\frac{(-1)^p}{2^{|\alpha|}\alpha!p!}\Delta\partial_x^{2\alpha}\partial_t^p G(1+t)\log(1+t)$$

to the asymptotic expansions. As a result, the coefficients in the n-th order term of the expansions are determined by the well-defined ones (2.9) under the influence of such an extra term.

The next result gives us space-time asymptotic expansions of the solutions to (P) on $\mathbf{R}^n \times [0, \infty)$ satisfying (D) with $\gamma = n+1$ under the conditions $M_0 = 0$ and $|x|^{n+1} u_0 \in L^1(\mathbf{R}^n)$.

THEOREM 2. *Assume that $n \geq 1$ and $1 \leq q \leq \infty$, and let (u, v) be the solution to* (P) *on $\mathbf{R}^n \times [0, \infty)$ satisfying* (D) *with $\gamma = n+1$. Then, under the conditions $M_0 = 0$ and $|x|^{n+1} u_0 \in L^1(\mathbf{R}^n)$, the integral*

$$\int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u \nabla v \, dyds$$

converges for $|\alpha| + 2p \leq n$, and

$$\lim_{t\to\infty} t^{n(1-1/q)/2+(n+1)/2} \Bigg\| u(t) - \sum_{1 \leq |\alpha| \leq n+1} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^\alpha G(1+t) \int_{\mathbf{R}^n} y^\alpha u_0 \, dy$$
$$+ \sum_{|\alpha|+2p \leq n} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u \nabla v \, dyds \Bigg\|_q = 0. \quad (2.10)$$

Also, v has the same asymptotic behavior as u.

In Theorem 2, we see that the integral $\int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u \nabla v \, dyds$ converges for $|\alpha| + 2p \leq n$ because the solutions to (P) treated in Theorem 2 decay faster than those in Theorem 1 (see (6.1) and (6.2)). Hence the logarithmic term does not appear in the asymptotic rate of (2.10). The situation is rather similar to that of the incompressible Navier-stokes equations (see [4]).

The final result gives us the existence of solutions to (P) satisfying (D).

THEOREM 3. *Let $n \geq 1$. Then the following assertions hold:*

(i) *In the case $\gamma = n$, there exists a unique solution (u, v) to* (P) *with* (D) *if $\|u_0\|_1$, $\|u_0\|_{L_\gamma^\infty}$, $\|\nabla v_0\|_{L_\gamma^\infty}$ are small enough.*

(ii) *In the case $\gamma = n+1$, let $|x| u_0 \in L^1(\mathbf{R}^n)$ and $\int_{\mathbf{R}^n} u_0 \, dy = 0$. Then there exists a unique solution (u, v) to* (P) *with* (D) *if $\|u_0\|_{L_1^1}$, $\|u_0\|_{L_\gamma^\infty}$, $\|\nabla v_0\|_{L_\gamma^\infty}$ are small enough.*

3. Preliminaries

In this section, we prepare several lemmas which will be used often in the proofs of Theorems 1, 2 and 3. We begin with mentioning the pointwise estimates and the L^q-estimates for the heat kernel $G(x, t) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$.

LEMMA 1. *Let $n \geq 1$. Then the following estimates hold:*
(i) *For each integer j with $1 \leq j \leq n$ and each $k = 0, 1$,*

$$|\partial_j^k G(x,t)| \leq C|x|^{-n-k} e^{-|x|^2/(8t)} \qquad (x \in \mathbf{R}^n, t > 0), \tag{3.1}$$

$$|\partial_j^k G(x,t)| \leq C t^{-n/2-k/2} e^{-|x|^2/(8t)} \qquad (x \in \mathbf{R}^n, t > 0), \tag{3.2}$$

where C is a positive constant depending on k, n.
(ii) *Let $\alpha_j, \beta \in \mathbf{Z}_+$ $(1 \leq j \leq n)$ and $1 \leq q \leq \infty$. Then*

$$\|\partial_x^\alpha \partial_t^\beta G(t)\|_q \leq C t^{-n(1-1/q)/2-|\alpha|/2-\beta} \qquad \text{for } t > 0, \tag{3.3}$$

where C is a positive constant depending on α, β, n, q.

The following lemma gives a version of Taylor's formula for the heat kernel $G(x,t)$ to prove Theorems 1 and 2.

LEMMA 2. *Let $m \geq 1$ be an arbitrary integer. Then, for each integer $1 \leq j \leq n$,*

$$\partial_j G(x-y, t-s) = \sum_{|\alpha|+2p \leq m-1} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} y^\alpha (1+s)^p \partial_j \partial_x^\alpha \partial_t^p G(x, 1+t) + R_m,$$

where

$$R_m = \sum_{\substack{|\alpha|+2p=m,\\ |\alpha|\geq 1}} \frac{|\alpha|(-1)^{|\alpha|+p}}{\alpha! p!} \int_0^1 (1-\theta)^{|\alpha|-1} y^\alpha (1+s)^p \partial_x^\alpha \partial_t^p \partial_j G(x-\theta y, 1+t) d\theta$$
$$+ \frac{(-1)^{[(m-1)/2]+1}}{([(m-1)/2])!} \int_0^1 (1-\tau)^{[(m-1)/2]} (1+s)^{[(m-1)/2]+1}$$
$$\times\ \partial_t^{[(m-1)/2]+1} \partial_j G(x-y, 1+t-\tau(1+s)) d\tau.$$

The following lemma is concerned with well-known L^r-L^q estimates of $e^{t\Delta} f$, which are proved by Young's inequality for convolution.

LEMMA 3. *Let $1 \leq q \leq r \leq \infty$ and $\alpha_j, \beta \in \mathbf{Z}_+$ $(1 \leq j \leq n)$. Then*

$$\|\partial_x^\alpha \partial_t^\beta e^{t\Delta} f\|_r \leq C t^{-n(1/q-1/r)/2-|\alpha|/2-\beta} \|f\|_q \qquad \text{for } f \in L^q(\mathbf{R}^n), \tag{3.4}$$

where C is a positive constant depending on α, β, n, r, q.

The following lemma gives the asymptotic behavior of $e^{t\Delta} f$.

Lemma 4. *Let $n \geq 1$ and $1 \leq q \leq \infty$. Then the following assertions hold:*
(i) *Under the condition $f \in L^1_1(\mathbf{R}^n)$,*

$$\sup_{t\geq 1} t^{n(1-1/q)/2+(k+1)/2}\left\| \partial_j^k e^{t\Delta} f - \partial_j^k G(1+t)\int_{\mathbf{R}^n} f(y)dy \right\|_q < \infty \tag{3.5}$$

for each integer j with $1 \leq j \leq n$ and each $k = 0, 1$.
(ii) *Let $m \in \mathbf{Z}_+$. Under the condition $f \in L^1_m(\mathbf{R}^n)$,*

$$\lim_{t\to\infty} t^{n(1-1/q)/2+m/2}\left\| e^{t\Delta} f - \sum_{|\alpha|\leq m} \frac{(-1)^{|\alpha|}}{\alpha!}\partial_x^\alpha G(1+t)\int_{\mathbf{R}^n} y^\alpha f(y)dy \right\|_q = 0. \tag{3.6}$$

Proof. By using Taylor's formula, we obtain

$$\begin{aligned}
&\partial_j^k e^{t\Delta} f(x) - \partial_j^k G(x, 1+t)\int_{\mathbf{R}^n} f(y)dy \\
&\quad = -\int_{\mathbf{R}^n} f(y)dy \int_0^1 y\cdot\nabla\partial_j^k G(x-\theta y, t)d\theta - \int_{\mathbf{R}^n} f(y)dy\int_0^1 \partial_t\partial_j^k G(x, \tau+t)d\tau.
\end{aligned}$$

Hence (3.5) follows from Minkowski's inequality and (3.3). For the proof of (3.6), see [4]. □

The following lemma gives some estimates for $W(t)$ and $R(t)$ introduced by (2.4) and (2.3), respectively. Here we recall $M_0 = \int_{\mathbf{R}^n} u_0\, dy$.

Lemma 5. *The following assertions hold:*
(i) *Let $n \geq 1$ and $1 \leq q \leq \infty$. Then*

$$\|\partial_j^k W(t)\|_q \leq CM_0^2(1+t)^{-n(1-1/q)/2-(k+1)/2} \qquad (t > 0) \tag{3.7}$$

for each integer j with $1 \leq j \leq n$ and each $k = 0, 1$.
(ii) *Assume that $n \geq 2$ and $1 \leq q \leq \infty$. Then*

$$\|R(t)\|_q \leq CM_0^2(1+t)^{-n(1-1/q)/2-n/2} \qquad (t > 0). \tag{3.8}$$

(iii) *Let $n = 2$. Then*

$$\begin{aligned}
\|R(t)\|_\infty &= \left\| W(t) - \frac{M_0^2}{2(8\pi)}\Delta G(1+t)\log(1+t)\right\|_\infty \\
&\geq cM_0^2(1+t)^{-2} \qquad \text{for } t \geq 2,
\end{aligned} \tag{3.9}$$

where c is a positive constant.

PROOF. First of all, we shall show (3.7). We write $W(t)$ as follows:

$$W(t) = \frac{M_0^2}{2(8\pi)^{n/2}} \int_0^t (1+s)^{-n/2} \Delta G(t - s/2 + 1/2) ds. \tag{3.10}$$

Here we have used the fact that

$$e^{(t-s)\Delta} G^2(s) = \{(8\pi)^{n/2}\}^{-1} (1+s)^{-n/2} G(t - s/2 + 1/2).$$

Therefore, making use of Minkowski's inequality and (3.3), we have

$$\begin{aligned} \|\partial_j^k W(t)\|_q &\le CM_0^2 \int_0^t \|\partial_j^k \Delta G(t - s/2 + 1/2)\|_q (1+s)^{-n/2} ds \\ &\le CM_0^2 \int_0^t (t - s/2 + 1/2)^{-n(1-1/q)/2-(k+2)/2} (1+s)^{-n/2} ds \\ &\le CM_0^2 (1+t)^{-n(1-1/q)/2-(k+2)/2} \int_0^t (1+s)^{-n/2} ds \\ &\le CM_0^2 (1+t)^{-n(1-1/q)/2-(k+1)/2} \end{aligned}$$

for each integer j with $1 \le j \le n$ and each $k = 0, 1$. Hence we get the assertion (i) of Lemma 5.

Next, we are going to prove (3.8). Firstly, we use Taylor's formula to get

$$W(t) =: W_1(t) + W_2(t),$$

where

$$\begin{aligned} W_1(t) &= \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{k=0}^{[(n-2)/2]} \frac{(-1)^k}{2^k k!} \partial_t^k \Delta G(1+t) \int_0^t (1+s)^{-n/2+k} ds, \\ W_2(t) &= \frac{M_0^2 (-1)^{[(n-2)/2]+1}}{2^{[(n-2)/2]+2} (8\pi)^{n/2} ([(n-2)/2])!} \int_0^t (1+s)^{-n/2+[(n-2)/2]+1} ds \\ &\quad \times \int_0^1 (1-\theta)^{[(n-2)/2]} \partial_t^{[(n-2)/2]+1} \Delta G(1 + t - \theta(1+s)/2) d\theta. \end{aligned}$$

Minkowski's inequality and (3.3) yield that

$$\begin{aligned} \|W_2(t)\|_q \le CM_0^2 &\int_0^t (1+s)^{-n/2+[(n-2)/2]+1} ds \\ &\times \int_0^1 \|\partial_t^{[(n-2)/2]+1} \Delta G(1 + t - \theta(1+s)/2)\|_q d\theta \end{aligned}$$

$$
\begin{aligned}
&\le CM_0^2 \int_0^t (1+s)^{-n/2+[(n-2)/2]+1} ds \\
&\quad \times \int_0^1 \{2(1+t) - \theta(1+s)\}^{-n(1-1/q)/2-[(n-2)/2]-2} d\theta \\
&\le CM_0^2 \int_0^t (2t-s+1)^{-n(1-1/q)/2-[(n-2)/2]-2} (1+s)^{-n/2+[(n-2)/2]+1} ds \\
&\le CM_0^2 (1+t)^{-n(1-1/q)/2-[(n-2)/2]-2} \int_0^t (1+s)^{-n/2+[(n-2)/2]+1} ds \\
&\le CM_0^2 (1+t)^{-n(1-1/q)/2-n/2}.
\end{aligned}
$$

Hence this estimate implies that

$$
\|W(t) - W_1(t)\|_q \le CM_0^2 (1+t)^{-n(1-1/q)/2-n/2}. \tag{3.11}
$$

Next, noting that for every integer satisfying $0 \le k \le [(n-2)/2]$,

$$
\left(-\frac{1}{2}\right)^k = \sum_{p=0}^{k} \binom{k}{p} \frac{(-1)^p}{2^{k-p}},
$$

we obtain

$$
\begin{aligned}
W_1(t) &= \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{k=0}^{[(n-2)/2]} \sum_{p=0}^{k} \binom{k}{p} \frac{(-1)^p}{2^{k-p}k!} \Delta \partial_t^{(k-p)+p} G(1+t) \\
&\quad \times \int_0^t (1+s)^{-n/2+(k-p)+p} ds \\
&= \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{k=0}^{[(n-2)/2]} \sum_{p=0}^{k} \sum_{|\alpha|=k-p} \frac{(k-p)!}{\alpha!} \binom{k}{p} \frac{(-1)^p}{2^{k-p}k!} \\
&\quad \times \Delta \partial_x^{2\alpha} \partial_t^p G(1+t) \int_0^t (1+s)^{-n/2+p+(k-p)} ds \\
&= \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{|\alpha|+p \le [(n-2)/2]} \frac{(-1)^p}{2^{|\alpha|}\alpha! p!} \Delta \partial_x^{2\alpha} \partial_t^p G(1+t) \\
&\quad \times \int_0^t (1+s)^{-n/2+|\alpha|+p} ds.
\end{aligned} \tag{3.12}
$$

Thus substituting (3.12) into (3.11), we have

$$\Bigg\| W(t) - \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{|\alpha|+p \le [(n-2)/2]} \frac{(-1)^p}{2^{|\alpha|}\alpha!p!} \Delta\partial_x^{2\alpha}\partial_t^p G(1+t) \times \int_0^t (1+s)^{-n/2+|\alpha|+p} ds \Bigg\|_q \le CM_0^2(1+t)^{-n(1-1/q)/2-n/2}, \tag{3.13}$$

from which (3.8) for $n = 2$ directly follows since for $|\alpha| + 2p \le [(n-2)/2]$,

$$-\frac{n}{2} + |\alpha| + p \le -\frac{n}{2} + \left[\frac{n-2}{2}\right] = \begin{cases} -\dfrac{3}{2} & \text{if } n \text{ is odd,} \\ -1 & \text{if } n \text{ is even.} \end{cases} \tag{3.14}$$

We now consider (3.8) in the case $n \ge 3$ in order to complete the proof of assertion (ii) of Lemma 5. If n is odd with $n \ge 3$, then the convergence of integral $\int_0^\infty (1+s)^{-n/2-1+|\alpha|+p} ds$ is assured for $|\alpha| + p \le (n-3)/2$ due to (3.14). Hence, it holds from (3.3), (3.13) and

$$\int_0^\infty (1+s)^{-n/2+|\alpha|+p} ds = 2\{(n-2) - 2(|\alpha|+p)\}^{-1} \tag{3.15}$$

for $|\alpha| + p \le (n-3)/2$ that

$$\begin{aligned} \|R(t)\|_q &= \Bigg\| W(t) - \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{|\alpha|+p \le (n-3)/2} \frac{(-1)^p \Delta\partial_x^{2\alpha}\partial_t^p G(1+t)}{2^{|\alpha|-1}\alpha!p!\{(n-2) - 2(|\alpha|+p)\}} \Bigg\|_q \\ &\le \Bigg\| \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{|\alpha|+p \le (n-3)/2} \frac{(-1)^p}{2^{|\alpha|}\alpha!p!} \Delta\partial_x^{2\alpha}\partial_t^p G(1+t) \int_t^\infty (1+s)^{-n/2+|\alpha|+p} ds \Bigg\|_q \\ &\quad + \Bigg\| W(t) - \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{|\alpha|+p \le (n-3)/2} \frac{(-1)^p}{2^{|\alpha|}\alpha!p!} \Delta\partial_x^{2\alpha}\partial_t^p G(1+t) \\ &\quad \times \int_0^t (1+s)^{-n/2+|\alpha|+p} ds \Bigg\|_q \\ &\le CM_0^2(1+t)^{-n(1-1/q)/2-n/2}. \end{aligned}$$

On the other hand, if n is even with $n \ge 3$, then we find that the convergence of integral $\int_0^\infty (1+s)^{-n/2+|\alpha|+p} ds$ is assured for $|\alpha| + p \le (n-4)/2$, but is not assured for $|\alpha| + p = (n-2)/2$ because the equality $\int_0^t (1+s)^{-n/2+|\alpha|+p} ds =$

$\log(1+t)$ is satisfied for $|\alpha|+p=(n-2)/2$. Therefore, from (3.3) and (3.13) we have

$$\begin{aligned}\|R(t)\|_q &= \Bigg\| W(t) - \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{|\alpha|+p\le(n-4)/2} \frac{(-1)^p \Delta\partial_x^{2\alpha}\partial_t^p G(1+t)}{2^{|\alpha|-1}\alpha!p!\{(n-2)-2(|\alpha|+p)\}} \\ &\quad - \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{|\alpha|+p=(n-2)/2} \frac{(-1)^p}{2^{|\alpha|}\alpha!p!} \Delta\partial_x^{2\alpha}\partial_t^p G(1+t)\log(1+t) \Bigg\|_q \\ &\le \Bigg\| \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{|\alpha|+p\le(n-4)/2} \frac{(-1)^p}{2^{|\alpha|}\alpha!p!} \Delta\partial_x^{2\alpha}\partial_t^p G(1+t) \int_t^\infty (1+s)^{-n/2+|\alpha|+p} ds \Bigg\|_q \\ &\quad + \Bigg\| W(t) - \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{|\alpha|+p\le(n-2)/2} \frac{(-1)^p}{2^{|\alpha|}\alpha!p!} \Delta\partial_x^{2\alpha}\partial_t^p G(1+t) \\ &\quad \times \int_0^t (1+s)^{-n/2+|\alpha|+p} ds \Bigg\|_q \\ &\le CM_0^2(1+t)^{-n(1-1/q)/2-n/2}.\end{aligned}$$

Here we also have used the fact that (3.15) holds for $|\alpha|+p\le(n-4)/2$. As a consequence, we obtain the desired estimate (3.8).

Finally, we shall show (3.9). Let $t\ge 2$. A direct calculation gives

$$\partial_t\Delta G(x,t) = t^{-2}G(x,t)\left(\sum_{i=1}^{2} P_4\left(\frac{x_i}{2\sqrt{t}}\right) + 2\prod_{i=1}^{2} P_2\left(\frac{x_i}{2\sqrt{t}}\right)\right), \tag{3.16}$$

where $P_2(z)=z^2-1/2$ and $P_4(z)=z^4-3z^2+3/4$. Hence it follows from (3.16) that

$$|W_2(0,t)| = cM_0^2\int_0^t ds\int_0^1 \{2(1+t)-\theta(1+s)\}^{-3}d\theta \ge cM_0^2(1+t)^{-2},$$

which implies that

$$\begin{aligned}\|R(t)\|_\infty &= \left\| W(t) - \frac{M_0^2}{2(8\pi)}\Delta G(1+t)\log(1+t)\right\|_\infty \\ &= \|W(t)-W_1(t)\|_\infty = \|W_2(t)\|_\infty \ge cM_0^2(1+t)^{-2} \qquad \text{for } t\ge 2. \quad \square\end{aligned}$$

Let $X = L^q(\mathbf{R}^n)$ for $1 \le q < \infty$ or $X = \mathscr{B}(\mathbf{R}^n)$. The following lemma is needed to get decay estimates of ∇v (see [20]).

Lemma 6. *Define $v(t)$ by*

$$v(t) = e^{-t}e^{t\Delta}v_0 + \int_0^t e^{-(t-s)}e^{(t-s)\Delta}u(s)ds \quad (t > 0), \qquad v(0) = v_0.$$

Then the following assertions hold:

(i) *If v_0, $\partial_j v_0 \in X$ and $u \in C([0,\infty); X)$, then v, $\partial_j v \in C([0,\infty); X)$.*

(ii) *Let $1 \le r \le q \le \infty$ and $1/r - 1/q < 1/n$, and assume that v_0, $|\nabla v_0| \in L^q(\mathbf{R}^n)$ and $u \in C([0,\infty); L^r(\mathbf{R}^n))$. Then,*

$$\|\nabla v(t)\|_q \le e^{-t}\|\nabla v_0\|_q + C\Gamma(\delta_1) \sup_{0<s<t} \|u(s)\|_r, \tag{3.17}$$

$$\|\nabla v(t)\|_q \le e^{-t}\|\nabla v_0\|_q + Ct^{\delta_1 - 1}e^{-t/2} \sup_{0<s<t/2} \|u(s)\|_r + C\Gamma(\delta_1) \sup_{t/2<s<t} \|u(s)\|_r. \tag{3.18}$$

Furthermore, under the additional condition $|\nabla u| \in C((0,\infty); L^r(\mathbf{R}^n))$,

$$\|\nabla v(t)\|_q \le e^{-t}\|\nabla v_0\|_q + Ct^{\delta_1 - 1}e^{-t/2} \sup_{0<s<t/2} \|u(s)\|_r + C\Gamma(\delta_2) \sup_{t/2<s<t} \|\nabla u(s)\|_r. \tag{3.19}$$

Here C is a positive constant depending only on q and r, $\Gamma(z)$ is the gamma function and $\delta_1 = 1/2 - n(1/r - 1/q)/2$, $\delta_2 = \delta_1 + 1/2$.

Finally, we need the following lemma to show Theorem 3. This lemma is obtained by an argument similar to that in the proof of Theorem 2 of [16].

Lemma 7. *Let γ be either n or $n+1$, and assume that*

$$f \in L^\infty_\gamma(\mathbf{R}^n).$$

Then the following assertions hold:

(i) $|e^{-t}e^{t\Delta}f(x)| \le C\|f\|_{L^\infty_\gamma}(1+|x|)^{-\gamma}$, $|e^{-t}e^{t\Delta}f(x)| \le C\|f\|_{L^\infty_\gamma}e^{-t}$.

(ii) *In the case $\gamma = n$, if $f \in L^1(\mathbf{R}^n)$, then*

$$|e^{t\Delta}f(x)| \le C(\|f\|_1 + \|f\|_{L^\infty_n})(1+|x|)^{-n},$$

$$|e^{t\Delta}f(x)| \le C(\|f\|_1 + \|f\|_{L^\infty_n})(1+t)^{-n/2}.$$

(iii) *In the case* $\gamma = n+1$, *if* $\int_{\mathbf{R}^n} f(x)dx = 0$ *and* $f \in L_1^1(\mathbf{R}^n)$, *then*

$$|e^{t\Delta} f(x)| \le C(\|f\|_{L_1^1} + \|f\|_{L_{n+1}^\infty})(1+|x|)^{-n-1},$$

$$|e^{t\Delta} f(x)| \le C(\|f\|_{L_1^1} + \|f\|_{L_{n+1}^\infty})(1+t)^{-n/2-1/2}.$$

4. Decay estimates of solutions in the case $\gamma = n$

In this section, let (u, v) be the solution to (P) satisfying (D) with $\gamma = n$. Firstly, we begin with some L^q-estimates for the solution.

Proposition 1. *Let* $n \ge 1$ *and* $1 < q \le \infty$. *Then the following estimates hold.*

$$\sup_{t\ge 1}(1+t)^{n(1-1/q)/2+1/2}\|\nabla u(t)\|_q < \infty, \tag{4.1}$$

$$\sup_{t>0}(1+t)^{n(1-1/q)/2+1/2}\|\nabla v(t)\|_q < \infty, \tag{4.2}$$

$$\sup_{t>0}(1+t)^{n(1-1/q)/2}\|u(t)\|_q < \infty. \tag{4.3}$$

Proposition 1 can be obtained by an argument similar to that in Section 4 of [20], but we give the outline of proof for reader's convenience. We here note that for every $1 < q \le \infty$, the estimate

$$\sup_{t>0}(\|u(t)\|_q + \|\nabla v(t)\|_q) < \infty \tag{4.4}$$

follows from (D) with $\gamma = n$ and (3.17). Furthermore, the following decay estimates of u and ∇v are given by (D) with $\gamma = n$, (3.18) and (4.4):

$$\sup_{t>0}(1+t)^{n/2}(\|u(t)\|_\infty + \|\nabla v(t)\|_\infty) < \infty. \tag{4.5}$$

The following claim is a key one to show Proposition 1.

Claim 1. *Under the assumption of Proposition* 1, *the following estimates hold:*

$$\|\nabla u(t)\|_q \le C(1+t)^{-n(1-1/q)/2-1/2}B(t;\beta) \qquad (t \ge 1), \tag{4.6}$$

$$\|\nabla v(t)\|_q \le C(1+t)^{-n(1-1/q)/2-1/2}B(t;\beta) \qquad (t > 0), \tag{4.7}$$

where $\beta \in (1,2]$ *and*

$$B(t;\beta) = \begin{cases} 1 & \text{if } n \ge 2, \\ t^{-1/2+\beta/2} & \text{if } n = 1. \end{cases} \tag{4.8}$$

Proof. Fix $t \geq 4$ and $\varepsilon \in (0, 1/2)$, and take β such that $\beta \in (1, 2]$. Then it follows from (4.4) and (4.5) that

$$\sup_{t>0}(1+t)^{n(1-\beta/2)/2}(\|u(t)\|_2 + \|\nabla v(t)\|_2) < \infty. \tag{4.9}$$

To estimate $\|\nabla u(t)\|_q$, by (2.1), we write $\nabla u(t)$ as

$$\begin{aligned}\nabla u(t) &= \nabla e^{t\Delta}u_0 - \int_0^{(1-\varepsilon)t} \nabla\nabla \cdot e^{(t-s)\Delta}(u\nabla v)(s)ds \\ &\quad - \int_{(1-\varepsilon)t}^{t} \nabla e^{(t-s)\Delta}(\nabla u \cdot \nabla v)(s)ds - \int_{(1-\varepsilon)t}^{t} \nabla e^{(t-s)\Delta}(u\Delta v)(s)ds \\ &= \nabla e^{t\Delta}u_0 - I_1^{\varepsilon}(t) - I_2^{\varepsilon}(t) - I_3^{\varepsilon}(t).\end{aligned}$$

By (3.4) and (4.9), we have

$$\begin{aligned}\|I_1^{\varepsilon}(t)\|_q &\leq C\int_0^{(1-\varepsilon)t}(t-s)^{-n(1-1/q)/2-1}\|u(s)\|_2\|\nabla v(s)\|_2 ds \\ &\leq C\varepsilon^{-n(1-1/q)/2-1}t^{-n(1-1/q)/2-1}\int_0^{(1-\varepsilon)t}(1+s)^{-n(1-\beta/2)}ds.\end{aligned}$$

Now we estimate the integral $\int_0^{(1-\varepsilon)t}(1+s)^{-n(1-\beta/2)}ds$ appearing in the right-hand side of $\|I_1^{\varepsilon}(t)\|_q$. For $n \geq 2$, take β again such that

$$\beta \in (1, 3/2] \quad \text{if } n = 2, \qquad \beta \in (1, 2(n-1)/n) \quad \text{if } n \geq 3.$$

Then a simple calculation yields that

$$\int_0^{(1-\varepsilon)t}(1+s)^{-n(1-\beta/2)}ds \leq C \times \begin{cases} 1 & \text{if } n \geq 3, \\ t^{1/2} & \text{if } n = 2. \end{cases}$$

Also, for $n = 1$,

$$\int_0^{(1-\varepsilon)t}(1+s)^{-(1-\beta/2)}ds \leq Ct^{\beta/2}.$$

Therefore,

$$\|I_1^{\varepsilon}(t)\|_q \leq C\varepsilon^{-n(1-1/q)/2-1}t^{-n(1-1/q)/2-1/2}B(t;\beta), \tag{4.10}$$

where $B(t;\beta)$ is the one given by (4.8).

By use of (3.4) and (4.5), the estimates of $\|I_2^{\varepsilon}(t)\|_q$ and $\|I_3^{\varepsilon}(t)\|_q$ are obtained as follows:

$$\|I_2^\varepsilon(t)\|_q \le C\int_{(1-\varepsilon)t}^t (t-s)^{-1/2}s^{-n/2}\|\nabla u(s)\|_q ds, \tag{4.11}$$

$$\|I_3^\varepsilon(t)\|_q \le C\int_{(1-\varepsilon)t}^t (t-s)^{-1/2}s^{-n/2}\|\Delta v(s)\|_q ds. \tag{4.12}$$

Therefore summing up (4.10), (4.11) and (4.12) implies that

$$\begin{aligned}\|\nabla u(t)\|_q &\le \|\nabla e^{t\Delta}u_0\|_q + C\varepsilon^{-n(1-1/q)/2-1}t^{-n(1-1/q)/2-1/2}B(t;\beta)\\ &\quad + C\int_{(1-\varepsilon)t}^t (t-s)^{-1/2}s^{-n/2}\|\nabla u(s)\|_q ds\\ &\quad + C\int_{(1-\varepsilon)t}^t (t-s)^{-1/2}s^{-n/2}\|\Delta v(s)\|_q ds. \end{aligned}\tag{4.13}$$

Define here $M(t)$ by

$$M(t) = \sup_{1\le s\le t} s^{n(1-1/q)/2+1/2}\|\nabla u(s)\|_q.$$

Then the third term on the right-hand side of (4.13) is estimated as

$$\int_{(1-\varepsilon)t}^t (t-s)^{-1/2}s^{-n/2}\|\nabla u(s)\|_q ds \le C\varepsilon^{1/2}t^{-n(1-1/q)/2-n/2}M(t). \tag{4.14}$$

We next consider the estimate of $\|\Delta v(s)\|_q$ on the right-hand side of (4.13). Fix $s \ge 2$. To estimate it, by (2.2), we represent $\Delta v(s)$ as follows:

$$\begin{aligned}\Delta v(s) &= e^{-s}\nabla\cdot e^{s\Delta}\nabla v_0 + \int_0^{(1-\varepsilon)s} e^{-(s-\tau)}\Delta e^{(s-\tau)\Delta}u(\tau)d\tau\\ &\quad + \int_{(1-\varepsilon)s}^s e^{-(s-\tau)}\nabla\cdot e^{(s-\tau)\Delta}\nabla u(\tau)d\tau\\ &= e^{-s}\nabla\cdot e^{s\Delta}\nabla v_0 + J_1^\varepsilon(s) + J_2^\varepsilon(s).\end{aligned}$$

(3.4) and (4.4) imply that

$$\begin{aligned}\|J_1^\varepsilon(s)\|_q &\le Ce^{-\varepsilon s}\int_0^{(1-\varepsilon)s}(s-\tau)^{-n(1/r-1/q)/2-1}\|u(\tau)\|_r d\tau\\ &\le Ce^{-\varepsilon s}\varepsilon^{-n(1/r-1/q)/2-1}s^{-n(1/r-1/q)/2}\\ &\le C\varepsilon^{-n(1-1/q)/2-3/2}s^{-n(1-1/q)/2-1/2}.\end{aligned}$$

Here we have taken r such that $1 < r \le q \le \infty$. By an argument similar to that in (4.14), we also have

$$\|J_2^{\varepsilon}(s)\|_q \le Cs^{-n(1-1/q)/2-1/2}M(s).$$

Therefore we find that

$$\|\Delta v(s)\|_q \le Ce^{-s}\|\nabla v_0\|_q + C\varepsilon^{-n(1-1/q)/2-3/2}s^{-n(1-1/q)/2-1/2} + Cs^{-n(1-1/q)/2-1/2}M(s). \tag{4.15}$$

Substituting (4.14) and (4.15) into the right-hand side of (4.13) gives

$$t^{n(1-1/q)/2+1/2}\|\nabla u(t)\|_q \le C + C\varepsilon^{1/2} + C\varepsilon^{-n(1-1/q)/2-1}(1 + B(t;\beta)) + C\varepsilon^{1/2}M(t),$$

where a positive constant C is taken as $C > 1$. Here take ε such that $\varepsilon = 1/(4C^2)$. Then, by $C > 1$, we easily see that $\varepsilon \in (0, 1/2)$. Hence we have

$$t^{n(1-1/q)/2+1/2}\|\nabla u(t)\|_q \le C + CB(t;\beta) + \frac{1}{2}M(t),$$

from which it follows that $M(t) \le C + CB(t;\beta)$. As a consequence, we obtain

$$\|\nabla u(t)\|_q \le Ct^{-n(1-1/q)/2-1/2} + Ct^{-n(1-1/q)/2-1/2}B(t;\beta) \qquad \text{for } t \ge 4, \tag{4.16}$$

which yields (4.6). Also, (4.7) follows from (3.19), (4.4) and (4.6). □

(4.1) and (4.2) for $n \ge 2$ follow from Claim 1. Before proving them for $n = 1$, we need the following estimate of $\|u(t)\|_q$ for $1 < q \le \infty$. This yields the desired estimate (4.3).

CLAIM 2. *Under the assumption of Proposition* 1, *the following holds:*

$$\sup_{t>0}(1+t)^{n(1-1/q)/2}\|u(t)\|_q < \infty. \tag{4.17}$$

PROOF. Let $1 < q \le \infty$ and $t \ge 2$. (4.7) reads as follows: For $n \ge 1$,

$$\sup_{t>0}(1+t)^{n(1-1/q)/2+1/2-\tilde{\beta}}\|\nabla v(t)\|_q < \infty, \tag{4.18}$$

where

$$\tilde{\beta} = \beta/2 - 1/2 \in (0, 1/2] \quad \text{if } n = 1, \qquad \tilde{\beta} = 0 \quad \text{if } n \ge 2.$$

We now put $I(t) = \int_0^t e^{(t-s)\Delta}\nabla \cdot (u\nabla v)(s)ds$, and split this integral as follows:

$$I(t) = \int_{t/2}^{t} e^{(t-s)\Delta}\nabla \cdot (u\nabla v)(s)ds + \int_0^{t/2} e^{(t-s)\Delta}\nabla \cdot (u\nabla v)(s)ds =: I_1(t) + I_2(t).$$

Then (3.4), (4.5) and (4.18) give

$$\begin{aligned}\|I_1(t)\|_q &\le C\int_{t/2}^{t}(t-s)^{-1/2}\|u(s)\|_\infty\|\nabla v(s)\|_q ds\\ &\le C\int_{t/2}^{t}(t-s)^{-1/2}(1+s)^{-n(1-1/q)/2-n/2-1/2+\tilde{\beta}}ds\\ &\le Ct^{-n(1-1/q)/2-n/2+\tilde{\beta}} \le Ct^{-n(1-1/q)/2}.\end{aligned}$$

We next evaluate $\|I_2(t)\|_q$. Using (3.4), (4.9) and (4.18) implies that

$$\begin{aligned}\|I_2(t)\|_q &\le C\int_0^{t/2}(t-s)^{-n(1-1/q)/2-1/2}\|u(s)\|_2\|\nabla v(s)\|_2 ds\\ &\le Ct^{-n(1-1/q)/2-1/2}\int_0^{t/2}(1+s)^{-1/2-n(3/2-\beta/2)/2+\tilde{\beta}}ds.\end{aligned}$$

For $n \ge 2$, noting that $-n(3/2-\beta/2)/2+\tilde{\beta} < 0$, we have

$$\|I_2(t)\|_q \le Ct^{-n(1-1/q)/2-1/2}\int_0^{t/2}(1+s)^{-1/2}ds \le Ct^{-n(1-1/q)/2}.$$

On the other hand, for $n = 1$, by taking β again such that $\beta \in (1, 5/3]$, we get

$$\begin{aligned}\|I_2(t)\|_q &\le Ct^{-(1-1/q)/2-1/2}\int_0^{t/2}(1+s)^{-1+3(\beta-1)/4}ds\\ &\le Ct^{-(1-1/q)/2-1/2+3(\beta-1)/4} \le Ct^{-(1-1/q)/2}.\end{aligned}$$

Hence

$$\|I(t)\|_q \le Ct^{-n(1-1/q)/2} \qquad \text{for } t \ge 2,$$

which together with (2.1) and (3.4) yields that

$$\sup_{t\ge 2}(1+t)^{n(1-1/q)/2}\|u(t)\|_q < \infty.$$

Consequently, the desired estimate (4.17) follows from this estimate and (4.4). □

Proof of Proposition 1. We show only (4.1) and (4.2) for $n = 1$ due to Claims 1 and 2. Using (3.18) and (4.3) for $n = 1$ yields that

$$\sup_{t>0}(1+t)^{(1-1/q)/2}\|\partial_x v(t)\|_q < \infty$$

for $1 < q \leq \infty$. Therefore, repeating arguments in Claim 1, we obtain the desired estimates (4.1) and (4.2) for $n = 1$. □

The following proposition gives the decay estimates of $\|u(t)\|_q$ and $\|v(t)\|_q$ for $1 \leq q \leq \infty$.

PROPOSITION 2. *For $n \geq 1$ and $1 \leq q \leq \infty$, the following holds:*

$$\sup_{t>0}(1+t)^{n(1-1/q)/2}(\|u(t)\|_q + \|v(t)\|_q) < \infty. \tag{4.19}$$

REMARK 3. Once Proposition 2 is proved, we easily see that $\|u(t)\|_1$ and $\|v(t)\|_1$ are bounded on $[0, \infty)$.

To obtain Proposition 2, we need to show the following lemma by making use of the decay estimates (4.2) and (4.3) of Proposition 1. The proof can be obtained by applying an argument similar to that in the proof of Proposition 5.1 of [20].

LEMMA 8. *Let $n \geq 1$ and $1 \leq q \leq \infty$. Then*

$$\sup_{t>0}(1+t)^{n(1-1/q)/2+1/2}D(t)\left\|\int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)ds\right\|_q < \infty, \tag{4.20}$$

where

$$D(t) = \begin{cases} 1 & \text{if } n \geq 2, \\ (\log(2+t))^{-1} & \text{if } n = 1. \end{cases} \tag{4.21}$$

PROOF OF PROPOSITION 2. Let $n \geq 1$ and $1 \leq q \leq \infty$. To obtain the L^q-estimate of u, we prove only $\sup_{t>0}\|u(t)\|_1 < \infty$ due to (4.3). Using (2.1), (3.4) and (4.20) with $q = 1$, we have

$$\begin{aligned} \|u(t)\|_1 &\leq \|e^{t\Delta}u_0\|_1 + \left\|\int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)ds\right\|_1 \\ &\leq \|u_0\|_1 + C(1+t)^{-1/2}D(t)^{-1}, \end{aligned}$$

where $D(t)$ is the one defined by (4.21). Therefore this estimate implies that

$$\sup_{t>0}\|u(t)\|_1 < \infty. \tag{4.22}$$

Next we consider the L^q-estimate of v. Using (D) with $\gamma = n$ yields that $\sup_{t>0}(1+t)^{n/2}\|v(t)\|_\infty < \infty$. $\|v(t)\|_1$ is also estimated as follows:

$$\|v(t)\|_1 \le e^{-t}\|e^{t\Delta}v_0\|_1 + \left\|\int_0^t e^{-(t-s)}e^{(t-s)\Delta}u(s)ds\right\|_1$$
$$\le e^{-t}\|v_0\|_1 + \int_0^t e^{-(t-s)}\|u(s)\|_1 ds \le e^{-t}\|v_0\|_1 + C\int_0^\infty e^{-z}\,dz \le C.$$

Here we have used (2.2), (3.4) and (4.22). Hence interpolating between the L^1 and L^∞ estimates of v yields that $\sup_{t>0}(1+t)^{n(1-1/q)/2}\|v(t)\|_q < \infty$. □

The following proposition is a key one to show Proposition 4 below. Firstly we remark that (3.17) and (4.19) imply

$$\sup_{t>0}\|\nabla v(t)\|_q < \infty \qquad \text{for } 1 \le q \le \infty. \tag{4.23}$$

Proposition 3. *Let $n \ge 1$ and $1 \le q \le \infty$. Then, under the condition $|x|u_0 \in L^1(\mathbf{R}^n)$, the following estimates hold:*

$$\sup_{t>0}(1+t)^{n(1-1/q)/2+1/2}\|u(t) - M_0G(1+t)\|_q < \infty, \tag{4.24}$$

$$\sup_{t>0}(1+t)^{n(1-1/q)/2+1}\|\nabla v(t) - M_0\nabla G(1+t)\|_q < \infty, \tag{4.25}$$

$$\sup_{t>0}(1+t)^{n(1-1/q)/2+(n+2)/2}\|(u\nabla v)(t) - M_0^2(G\nabla G)(1+t)\|_q < \infty, \tag{4.26}$$

where $M_0 = \int_{\mathbf{R}^n} u_0\,dy$.

Once the estimates (4.24) and (4.25) are proved, from these estimates, (3.3) and (4.19) we see that

$$\|(u\nabla v)(t) - M_0^2(G\nabla G)(1+t)\|_q \le \|u(t)\|_\infty\|\nabla v(t) - M_0\nabla G(1+t)\|_q$$
$$+ C\|\nabla G(1+t)\|_\infty\|u(t) - M_0G(1+t)\|_q$$
$$\le C(1+t)^{-n(1-1/q)/2-(n+2)/2},$$

which implies the desired estimate (4.26). Hence we show only the estimates (4.24) and (4.25) in Proposition 3. The proof can be done by calculations similar to those in [10], but we give the outline of proof for reader's convenience. Now, we begin with the following claim in order to show these estimates.

Claim 3. *Under the assumption of Proposition 3, we have*

$$\sup_{t>0}(1+t)^{n(1-1/q)/2+1/2}D(t)\|u(t) - M_0G(1+t)\|_q < \infty, \tag{4.27}$$

where $D(t)$ is the one given by (4.21), and $M_0 = \int_{\mathbf{R}^n} u_0\,dy$.

PROOF. First of all, we use (2.1) to get

$$u(t) - M_0 G(1+t) = \{e^{t\Delta}u_0 - M_0 G(1+t)\} - \int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)ds. \tag{4.28}$$

Since we easily see that

$$\sup_{t>0} \|e^{t\Delta}u_0 - M_0 G(1+t)\|_q < \infty,$$

it follows from (3.5) with $k=0$ that

$$\sup_{t>0}(1+t)^{n(1-1/q)/2+1/2}\|e^{t\Delta}u_0 - M_0 G(1+t)\|_q < \infty, \tag{4.29}$$

which together with (4.20) and (4.28) implies (4.27). □

Therefore we see that (4.24) for $n \geq 2$ follows from Claim 3. Next claim is to give the L^q-estimate of $\nabla v - M_0 \nabla G$.

CLAIM 4. *Under the assumption of Proposition* 3, *we have*

$$\sup_{t>0}(1+t)^{n(1-1/q)/2+1}D(t)\|\nabla v(t) - M_0\nabla G(1+t)\|_q < \infty, \tag{4.30}$$

where $D(t)$ *is the one given by* (4.21), *and* $M_0 = \int_{\mathbf{R}^n} u_0\,dy$.

PROOF. Let $t \geq 4$. To prove this claim, by (2.1) and (2.2), we write $\nabla v(t) - M_0\nabla G(1+t)$ as

$$\begin{aligned}
&\nabla v(t) - M_0\nabla G(1+t)\\
&\quad = e^{-t}\nabla e^{t\Delta}(v_0 - u_0) + \{\nabla e^{t\Delta}u_0 - M_0\nabla G(1+t)\}\\
&\qquad - \int_0^{t/2} e^{-(t-s)}\nabla e^{(t-s)\Delta}\left(\int_0^s e^{(s-\tau)\Delta}\nabla\cdot(u\nabla v)(\tau)d\tau\right)ds\\
&\qquad - \int_{t/2}^{t} e^{-(t-s)}\nabla e^{(t-s)\Delta}\left(\int_0^s e^{(s-\tau)\Delta}\nabla\cdot(u\nabla v)(\tau)d\tau\right)ds\\
&\quad =: e^{-t}\nabla e^{t\Delta}(v_0 - u_0) + \{\nabla e^{t\Delta}u_0 - M_0\nabla G(1+t)\} - J_1(t) - J_2(t).
\end{aligned}$$

By making use of (3.4) and (4.20), we have

$$\begin{aligned}
\|J_1(t)\|_q &\leq Ce^{-t/2}\int_0^{t/2}(t-s)^{-n(1-1/q)/2-1/2}\left\|\int_0^s e^{(s-\tau)\Delta}\nabla\cdot(u\nabla v)(\tau)d\tau\right\|_1 ds\\
&\leq Ce^{-t/2}t^{-n(1-1/q)/2-1/2}\int_0^{t/2}(1+s)^{-1/2}(D(s))^{-1}ds\\
&\leq Ct^{-n(1-1/q)/2}(D(t))^{-1}e^{-t/2} \leq Ce^{-t/4}.
\end{aligned} \tag{4.31}$$

We next estimate $\|J_2(t)\|_q$. Let $\varepsilon \in (0, 1/2)$ be a constant to be specified later. Then we rewrite $J_2(t)$ as follows.

$$\begin{aligned}
J_2(t) &= \int_{t/2}^{t} e^{-(t-s)} e^{(t-s)\Delta} K_1(s) ds + \int_{t/2}^{t} e^{-(t-s)} \nabla e^{(t-s)\Delta} K_2(s) ds \\
&\quad + \int_{t/2}^{t} e^{-(t-s)} e^{(t-s)\Delta} \nabla \left(M_0^2 \int_0^s e^{(s-\tau)\Delta} \nabla \cdot (G\nabla G)(1+\tau) d\tau \right) ds \\
&=: J_{21}(t) + J_{22}(t) + J_{23}(t),
\end{aligned}$$

where

$$\begin{aligned}
K_1(s) &= \int_0^{(1-\varepsilon)s} \nabla\nabla \cdot e^{(s-\tau)\Delta} \{(u\nabla v)(\tau) - M_0^2 (G\nabla G)(1+\tau)\} d\tau, \\
K_2(s) &= \int_{(1-\varepsilon)s}^{s} \nabla \cdot e^{(s-\tau)\Delta} \{(u\nabla v)(\tau) - M_0^2 (G\nabla G)(1+\tau)\} d\tau.
\end{aligned}$$

We here need to consider the estimates of $\|K_k(s)\|_q$ $(k = 1, 2)$ to obtain the estimate of $\|J_2(t)\|_q$. To give them, we prepare the following claim.

Claim 5. *Let $n \geq 1$ and $1 \leq q \leq \infty$. Then it holds that for $t > 0$,*

$$\|(u\nabla v)(t) - M_0^2 (G\nabla G)(1+t)\|_q \leq C(1+t)^{-n(1-1/q)/2-(n+2)/2} (D(t))^{-1} + C(1+t)^{-n(1-1/q)/2-(n+1)/2}, \tag{4.32}$$

$$\|(u\nabla v)(t) - M_0^2 (G\nabla G)(1+t)\|_q \leq C(1+t)^{-n(1-1/q)/2-(n+2)/2} (D(t))^{-1} + C(1+t)^{-n/2} \|\nabla v(t) - M_0 \nabla G(t)\|_q, \tag{4.33}$$

where $D(t)$ is the one given by (4.21), *and $M_0 = \int_{\mathbf{R}^n} u_0 \, dy$.*

Proof. We show only (4.32) because a calculation similar to that in the proof of (4.32) gives the desired estimate (4.33). By using (3.3), (4.2), (4.19) and (4.27), we have

$$\begin{aligned}
&\|(u\nabla v)(t) - M_0^2 (G\nabla G)(1+t)\|_q \\
&\quad \leq C\|\nabla G(1+t)\|_q \|u(t) - M_0 G(1+t)\|_\infty + \|u(t)\|_q \|\nabla v(t) - M_0 \nabla G(1+t)\|_\infty \\
&\quad \leq C(1+t)^{-n(1-1/q)/2-(n+2)/2} (D(t))^{-1} + C(1+t)^{-n(1-1/q)/2-(n+1)/2}.
\end{aligned}$$

Hence we obtain the desired estimate (4.32). □

PROOF OF CLAIM 4, CONTINUED. Fix $s \geq 2$. By applying (3.4) and (4.32) with $q = 1$, the estimate of $\|K_1(s)\|_q$ can be achieved as follows:

$$\begin{aligned} \|K_1(s)\|_q &\leq C\varepsilon^{-n(1-1/q)/2-1} s^{-n(1-1/q)/2-1} \\ &\quad \times \int_0^{(1-\varepsilon)s} \|(u\nabla v)(\tau) - M_0^2(G\nabla G)(1+\tau)\|_1 d\tau \\ &\leq C\varepsilon^{-n(1-1/q)/2-1} s^{-n(1-1/q)/2-1}(D(s))^{-1}. \end{aligned} \tag{4.34}$$

Similarly, we see from (3.4) and (4.33) that

$$\begin{aligned} \|K_2(s)\|_q &\leq C\int_{(1-\varepsilon)s}^{s} (s-\tau)^{-1/2} \|(u\nabla v)(\tau) - M_0^2(G\nabla G)(1+\tau)\|_q d\tau \\ &\leq C\varepsilon^{1/2} s^{-n(1-1/q)/2-(n+1)/2}(D(s))^{-1} \\ &\quad + C\int_{(1-\varepsilon)s}^{s} (s-\tau)^{-1/2}\tau^{-n/2}\|\nabla v(\tau) - M_0\nabla G(1+\tau)\|_q d\tau. \end{aligned} \tag{4.35}$$

The second term on the right-hand side of (4.35) is here estimated as follows.

$$\begin{aligned} &\int_{(1-\varepsilon)s}^{s} (s-\tau)^{-1/2}\tau^{-n/2}\|\nabla v(\tau) - M_0\nabla G(1+\tau)\|_q d\tau \\ &\quad \leq C\varepsilon^{1/2} s^{-n(1-1/q)/2-(n+1)/2} \sup_{1\leq\tau\leq s} \tau^{n(1-1/q)/2+1}\|\nabla v(\tau) - M_0\nabla G(1+\tau)\|_q, \end{aligned}$$

which yields that

$$\begin{aligned} \|K_2(s)\|_q &\leq C\varepsilon^{1/2} s^{-n(1-1/q)/2-(n+1)/2}(D(s))^{-1} + C\varepsilon^{1/2} s^{-n(1-1/q)/2-(n+1)/2} \\ &\quad \times \sup_{1\leq\tau\leq s} \tau^{n(1-1/q)/2+1}\|\nabla v(\tau) - M_0\nabla G(1+\tau)\|_q. \end{aligned}$$

Therefore the estimates of $\|K_k(s)\|_q$ $(k = 1, 2)$ and (3.7) with $k = 1$ give

$$\|J_{21}(t)\|_q \leq \int_{t/2}^{t} e^{-(t-s)}\|K_1(s)\|_q ds \leq C\varepsilon^{-n(1-1/q)/2-1} t^{-n(1-1/q)/2-1}(D(t))^{-1},$$

$$\begin{aligned} \|J_{22}(t)\|_q &\leq C\int_{t/2}^{t} e^{-(t-s)}(t-s)^{-1/2}\|K_2(s)\|_q ds \\ &\leq C\varepsilon^{1/2} t^{-n(1-1/q)/2-(n+1)/2}(D(t))^{-1}\int_{t/2}^{t} e^{-(t-s)}(t-s)^{-1/2} ds \\ &\quad + C\varepsilon^{1/2} t^{-n(1-1/q)/2-(n+1)/2}\int_{t/2}^{t} e^{-(t-s)}(t-s)^{-1/2} ds \\ &\quad \times \sup_{1\leq\tau\leq t} \tau^{n(1-1/q)/2+1}\|\nabla v(\tau) - M_0\nabla G(1+\tau)\|_q \end{aligned}$$

$$\begin{aligned}
&\le C\varepsilon^{1/2} t^{-n(1-1/q)/2-(n+1)/2} (D(t))^{-1} \int_0^\infty e^{-z} z^{-1/2}\, dz \\
&\quad + C\varepsilon^{1/2} t^{-n(1-1/q)/2-(n+1)/2} \int_0^\infty e^{-z} z^{-1/2}\, dz \\
&\quad \times \sup_{1\le\tau\le t} \tau^{n(1-1/q)/2+1} \|\nabla v(\tau) - M_0 \nabla G(1+\tau)\|_q \\
&\le C\varepsilon^{1/2} t^{-n(1-1/q)/2-(n+1)/2} (D(t))^{-1} + C\varepsilon^{1/2} t^{-n(1-1/q)/2-(n+1)/2} \\
&\quad \times \sup_{1\le\tau\le t} \tau^{n(1-1/q)/2+1} \|\nabla v(\tau) - M_0 \nabla G(1+\tau)\|_q,
\end{aligned}$$

$$\begin{aligned}
\|J_{23}(t)\|_q &\le C \int_{t/2}^t e^{-(t-s)} \left\| \nabla \left(M_0^2 \int_0^s e^{(s-\tau)\Delta} \nabla \cdot (G \nabla G)(1+\tau)\, d\tau \right) \right\|_q ds \\
&\le C \int_{t/2}^t e^{-(t-s)} s^{-n(1-1/q)/2-1}\, ds \le C t^{-n(1-1/q)/2-1} \int_0^\infty e^{-z}\, dz \\
&\le C t^{-n(1-1/q)/2-1},
\end{aligned}$$

which yield that for $t \ge 4$,

$$\begin{aligned}
t^{n(1-1/q)/2+1} \|J_2(t)\|_q &\le C + C\varepsilon^{1/2} (D(t))^{-1} + C\varepsilon^{-n(1-1/q)/2-1} (D(t))^{-1} \\
&\quad + C\varepsilon^{1/2} \sup_{1\le\tau\le t} \tau^{n(1-1/q)/2+1} \|\nabla v(\tau) - M_0 \nabla G(1+\tau)\|_q.
\end{aligned}$$

This together with (3.4), (3.5) with $k = 1$ and (4.31) implies that for $t \ge 4$,

$$\begin{aligned}
&t^{n(1-1/q)/2+1} \|\nabla v(t) - M_0 \nabla G(1+t)\|_q \\
&\quad \le C + C\varepsilon^{1/2} (D(t))^{-1} + C\varepsilon^{-n(1-1/q)/2-1} (D(t))^{-1} \\
&\qquad + C\varepsilon^{1/2} \sup_{1\le\tau\le t} \tau^{n(1-1/q)/2+1} \|\nabla v(\tau) - M_0 \nabla G(1+\tau)\|_q. \qquad (4.36)
\end{aligned}$$

In (4.36) we take a positive constant C as $C > 1$, and ε such as $\varepsilon = 1/(4C^2)$. Then, by $C > 1$, we find that $\varepsilon \in (0, 1/2)$. Thus (4.36) gives

$$t^{n(1-1/q)/2+1} \|\nabla v(t) - M_0 \nabla G(1+t)\|_q \le C + C(D(t))^{-1}$$

for $t \ge 4$. Consequently, by noting that $\sup_{t>0} \|\nabla v(t) - M_0 \nabla G(1+t)\|_q < \infty$ because of (4.23), the desired estimate (4.30) is obtained. □

Hence we find that (4.25) for $n \ge 2$ follows from Claim 4. Now, we shall prove (4.24) and (4.25) for $n = 1$. First of all, we give an improvement of estimate (4.32) for $n = 1$. The proof is given by applying (3.3), (4.19), (4.27) and (4.30).

Claim 6. *Let* $n = 1$ *and* $1 \le q \le \infty$. *Then*

$$\|(u\partial_x v)(t) - M_0^2(G\partial_x G)(1+t)\|_q \le C(1+t)^{-(1-1/q)/2-3/2}\log(2+t) \tag{4.37}$$

for $t > 0$, *where* $M_0 = \int_{\mathbf{R}^n} u_0\,dy$.

By using the estimate (4.37), we can remove the logarithmic function in $D(t)$ from (4.27) for $n = 1$, and get the desired estimate (4.24) for $n = 1$.

Claim 7. *Let* $n = 1$ *and* $1 \le q \le \infty$. *Then*

$$\sup_{t>0}(1+t)^{(1-1/q)/2+1/2}\|u(t) - M_0 G(1+t)\|_q < \infty, \tag{4.38}$$

where $M_0 = \int_{\mathbf{R}^n} u_0\,dy$.

Proof. Let $t \ge 2$. We now show that

$$\sup_{t>0}(1+t)^{(1-1/q)/2+1/2}\left\|\int_0^t e^{(t-s)\Delta}\partial_x(u\partial_x v)(s)ds\right\|_q < \infty. \tag{4.39}$$

Once (4.39) is shown, (4.38) is obtained by making use of (4.28), (4.29) and (4.39). To show this, we divide $\int_0^t e^{(t-s)\Delta}\partial_x(u\partial_x v)(s)ds$ into three parts:

$$\begin{aligned}
&\int_0^t e^{(t-s)\Delta}\partial_x(u\partial_x v)(s)ds \\
&\quad = \int_0^{t/2} e^{(t-s)\Delta}\partial_x((u\partial_x v)(s) - M_0^2(G\partial_x G)(1+s))ds \\
&\qquad + \int_{t/2}^t e^{(t-s)\Delta}\partial_x((u\partial_x v)(s) - M_0^2(G\partial_x G)(1+s))ds + W(t) \\
&\quad =: I_1(t) + I_2(t) + W(t),
\end{aligned}$$

where $W(t) = M_0^2\int_0^t e^{(t-s)\Delta}\partial_x(G\partial_x G)(1+s)ds$. The estimates of $\|I_1(t)\|_q$ and $\|I_2(t)\|_q$ can be achieved as follows.

$$\begin{aligned}
\|I_1(t)\|_q &\le C\int_0^{t/2}(t-s)^{-(1-1/q)/2-1/2}\|(u\partial_x v)(s) - M_0^2(G\partial_x G)(1+s)\|_1 ds \\
&\le Ct^{-(1-1/q)/2-1/2}, \\
\|I_2(t)\|_q &\le C\int_{t/2}^t (t-s)^{-1/2}\|(u\partial_x v)(s) - M_0^2(G\partial_x G)(1+s)\|_q ds \\
&\le Ct^{-(1-1/q)/2-1/2}.
\end{aligned}$$

Here we have used the estimates (3.4) and (4.37). As a consequence, (4.39) is obtained by using these estimates and (3.7) with $k = 0$ and noting that

$$\sup_{0<t\le 2}\left\|\int_0^t e^{(t-s)\Delta}\partial_x(u\partial_x v)(s)ds\right\|_q < \infty. \qquad \square$$

Finally, to finish the proof of Proposition 3, we need to show (4.25) for $n = 1$. Before proving this, we give the following claim which is an improvement of (4.33) for $n = 1$.

Claim 8. *Let* $n = 1$ *and* $1 \le q \le \infty$. *Then*

$$\|(u\partial_x v)(t) - M_0^2(G\partial_x G)(1+t)\|_q \le C(1+t)^{-(1-1/q)/2-3/2} + C(1+t)^{-1/2}\|\partial_x v(t) - M_0\partial_x G(1+t)\|_q \tag{4.40}$$

for $t > 0$, *where* $M_0 = \int_{\mathbf{R}^n} u_0\, dy$.

Proof. (4.40) easily follows from (3.3), (4.19) and (4.24). $\square$

Proof of Proposition 3. We shall show (4.25) only for $n = 1$. Let $n = 1$ and $1 \le q \le \infty$, and use the same notation as in Claim 4.

Fix $\varepsilon \in (0, 1/2)$ and $t \ge 4$. Using (4.37) to remove the logarithmic function in (4.34), we obtain

$$\begin{aligned}\|K_1(s)\|_q &\le C\int_0^{(1-\varepsilon)s}(s-\tau)^{-(1-1/q)/2-1}(1+\tau)^{-3/2}\log(2+\tau)d\tau \\ &\le C\varepsilon^{-(1-1/q)/2-1}s^{-(1-1/q)/2-1}\int_0^\infty (1+\tau)^{-3/2}\log(2+\tau)d\tau \\ &\le C\varepsilon^{-(1-1/q)/2-1}s^{-(1-1/q)/2-1}.\end{aligned} \tag{4.41}$$

Also, it follows from (4.40) that

$$\|K_2(s)\|_q \le C\varepsilon^{1/2}s^{-(1-1/q)/2-1} + C\varepsilon^{1/2}s^{-(1-1/q)/2-1} \times \sup_{1\le\tau\le t}\tau^{(1-1/q)/2+1}\|\partial_x v(\tau) - M_0\partial_x G(1+\tau)\|_q. \tag{4.42}$$

Therefore, by (4.41) and (4.42), calculations similar to those in Claim 4 yield the following estimate of $\|J_2(t)\|_q$.

$$\begin{aligned}\|J_2(t)\|_q &\le Ct^{-(1-1/q)/2-1} + C\varepsilon^{1/2}t^{-(1-1/q)/2-1} + C\varepsilon^{-(1-1/q)/2-1}t^{-(1-1/q)/2-1} \\ &\quad + C\varepsilon^{1/2}t^{-(1-1/q)/2-1}\sup_{1\le\tau\le t}\tau^{(1-1/q)/2+1}\|\partial_x v(\tau) - M_0\partial_x G(1+\tau)\|_q.\end{aligned}$$

This estimate together with (3.4), (3.5) with $k = 1$ and (4.31) implies that for $t \geq 4$,

$$t^{(1-1/q)/2+1}\|\partial_x v(t) - M_0\partial_x G(1+t)\|_q$$
$$\leq C + C\varepsilon^{1/2} + C\varepsilon^{-(1-1/q)/2-1}$$
$$+ C\varepsilon^{1/2} \sup_{1\leq\tau\leq t} \tau^{(1-1/q)/2+1}\|\partial_x v(\tau) - M_0\partial_x G(1+\tau)\|_q.$$

As a consequence, we obtain (4.25) for $n = 1$ by repeating arguments in Claim 4. □

5. Proof of Theorem 1

The aim of this section is to prove Theorem 1. We begin with the following decay estimate which is needed to get the asymptotic behavior of v.

Lemma 9. *Let $n \geq 1$ and $1 \leq q \leq \infty$. Then*

$$\sup_{t>0}(1+t)^{n(1-1/q)/2+(n+2)/2}\|u(t) - v(t)\|_q < \infty. \tag{5.1}$$

Proof. This lemma can be proved by using arguments similar to those in [10], but we give the outline of proof for reader's convenience.

Let $t \geq 2$ and $1 \leq q \leq \infty$, and put $w(x,t) = u(x,t) - v(x,t)$. Since (u, v) is the classical solution to (P) on $(x,t) \in \mathbf{R}^n \times (0,\infty)$, $w(x,t)$ satisfies the following equation:

$$\begin{cases} \partial_t w = \Delta w - w - \nabla\cdot(u\nabla v), & x \in \mathbf{R}^n,\ t > 0, \\ w(x,0) = u_0(x) - v_0(x), & x \in \mathbf{R}^n. \end{cases}$$

Then we represent $w(t)$ as

$$w(t) = e^{-t}e^{t\Delta}(u_0 - v_0) - w_1(t) - w_2(t) - w_3(t),$$

where

$$w_1(t) = \int_0^{t/2} e^{-(t-s)}\nabla\cdot e^{(t-s)\Delta}(u\nabla v)(s)ds,$$

$$w_2(t) = \int_{t/2}^{t} e^{-(t-s)}\nabla\cdot e^{(t-s)\Delta}\{(u\nabla v)(s) - M_0^2(G\nabla G)(1+s)\}ds,$$

$$w_3(t) = M_0^2\int_{t/2}^{t} e^{-(t-s)}e^{(t-s)\Delta}\nabla\cdot(G\nabla G)(1+s)ds,$$

where $M_0 = \int_{\mathbf{R}^n} u_0\, dy$. By (3.4), (4.2), (4.19), (4.26) and

$$\sup_{s>0}(1+s)^{n(1-1/q)/2+(n+2)/2}\|\nabla\cdot(G\nabla G)(1+s)\|_q<\infty,$$

the estimates of $\|w_k(t)\|_q$ $(k=1,2,3)$ can be achieved as

$$\|w_1(t)\|_q\le Ce^{-t/2}\int_0^{t/2}(t-s)^{-n(1-1/q)/2-1/2}\|u(s)\|_2\|\nabla v(s)\|_2ds\le Ce^{-t/4},$$

$$\begin{aligned}\|w_2(t)\|_q&\le C\int_{t/2}^t e^{-(t-s)}(t-s)^{-1/2}\|(u\nabla v)(s)-M_0^2(G\nabla G)(1+s)\|_q ds\\&\le Ct^{-n(1-1/q)/2-(n+2)/2}\int_{t/2}^t e^{-(t-s)}(t-s)^{-1/2}ds\\&\le Ct^{-n(1-1/q)/2-(n+2)/2}\int_0^\infty e^{-z}z^{-1/2}\,dz\\&\le Ct^{-n(1-1/q)/2-(n+2)/2},\end{aligned}$$

$$\begin{aligned}\|w_3(t)\|_q&\le C\int_{t/2}^t e^{-(t-s)}\|\nabla\cdot(G\nabla G)(1+s)\|_q ds\\&\le C\int_{t/2}^t e^{-(t-s)}s^{-n(1-1/q)/2-(n+2)/2}\,ds\\&\le Ct^{-n(1-1/q)/2-(n+2)/2}.\end{aligned}$$

Hence, using these estimates and $\|e^{-t}e^{t\Delta}(u_0-v_0)\|_q\le Ce^{-t}$, we obtain

$$\sup_{t\ge 2}\,t^{n(1-1/q)/2+(n+2)/2}\|w(t)\|_q<\infty,$$

which together with $\sup_{t>0}\|w(t)\|_q<\infty$ yields the desired estimate (5.1). □

The following proposition is a key one to show Theorem 1.

Proposition 4. *Let $n\ge 2$ and $1\le q\le\infty$. Then the following holds:*

$$\begin{aligned}\lim_{t\to\infty}\,t^{n(1-1/q)/2+n/2}\Bigg\|&\int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)ds\\&-\sum_{|\alpha|+2p\le n-1}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^pG(1+t)\cdot\int_0^t\int_{\mathbf{R}^n}y^\alpha(1+s)^p\{(u\nabla v)(y,s)\\&-M_0^2(G\nabla G)(y,1+s)\}dyds-W(t)\Bigg\|_q=0,\end{aligned}\tag{5.2}$$

where $M_0=\int_{\mathbf{R}^n}u_0\,dy$ and $W(t)=M_0^2\int_0^t e^{(t-s)\Delta}\nabla\cdot(G\nabla G)(1+s)ds$ as before.

Proof. Let $t \geq 2$. Then we split $\int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)ds$ as follows:

$$\begin{aligned}
&\int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)ds \\
&\quad = \int_{t/2}^t e^{(t-s)\Delta}\nabla\cdot\{(u\nabla v)(s) - M_0^2(G\nabla G)(1+s)\}ds \\
&\qquad + \int_0^{t/2} e^{(t-s)\Delta}\nabla\cdot\{(u\nabla v)(s) - M_0^2(G\nabla G)(1+s)\}ds \\
&\qquad + M_0^2\int_0^t e^{(t-s)\Delta}\nabla\cdot(G\nabla G)(1+s)ds \\
&\quad =: I_1(t) + I_2(t) + W(t).
\end{aligned} \tag{5.3}$$

Applying Lemma 2, for all integer $m \geq 1$, we can rewrite $I_2(t)$ as follows:

$$\begin{aligned}
I_2(t) = &\sum_{|\alpha|+2p\leq m-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot \\
&\cdot\int_0^{t/2}\int_{\mathbf{R}^n} y^\alpha(1+s)^p\{(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)\}dyds \\
&+ \sum_{\substack{|\alpha|+2p=m,\\ |\alpha|\geq 1}} \frac{|\alpha|(-1)^{|\alpha|+p}}{\alpha!p!}\int_0^{t/2}\int_{\mathbf{R}^n} y^\alpha(1+s)^p\{(u\nabla v)(y,s) \\
&- M_0^2(G\nabla G)(y,1+s)\}dyds\cdot\int_0^1(1-\theta)^{|\alpha|-1}\nabla\partial_x^\alpha\partial_t^p G(\cdot-\theta y,1+t)d\theta \\
&+ A_m\int_0^{t/2}\int_{\mathbf{R}^n}(1+s)^{[(m-1)/2]+1}\{(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)\}dyds\cdot \\
&\cdot\int_0^1(1-\tau)^{[(m-1)/2]}\nabla\partial_t^{[(m-1)/2]+1}G(\cdot-y,1+t-\tau(1+s))d\tau,
\end{aligned}$$

where $A_m = \dfrac{(-1)^{[(m-1)/2]+1}}{([(m-1)/2])!}$. Substituting this equality into (5.3) implies that for all integer $m \geq 1$,

$$\int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)ds - \sum_{|\alpha|+2p\le m-1}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot$$

$$\cdot\int_0^t\int_{\mathbf{R}^n} y^\alpha(1+s)^p\{(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)\}dyds - W(t)$$

$$= I_1(t) - \sum_{|\alpha|+2p\le m-1}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot\int_{t/2}^t\int_{\mathbf{R}^n} y^\alpha(1+s)^p\{(u\nabla v)(y,s)$$

$$- M_0^2(G\nabla G)(y,1+s)\}dyds$$

$$+ \sum_{\substack{|\alpha|+2p=m,\\ |\alpha|\ge 1}}\frac{|\alpha|(-1)^{|\alpha|+p}}{\alpha!p!}\int_0^{t/2}\int_{\mathbf{R}^n} y^\alpha(1+s)^p\{(u\nabla v)(y,s)$$

$$- M_0^2(G\nabla G)(y,1+s)\}dsdy\cdot\int_0^1(1-\theta)^{|\alpha|-1}\nabla\partial_x^\alpha\partial_t^p G(\cdot-\theta y,1+t)d\theta$$

$$+ A_m\int_0^{t/2}\int_{\mathbf{R}^n}(1+s)^{[(m-1)/2]+1}\{(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)\}dyds\cdot$$

$$\cdot\int_0^1(1-\tau)^{[(m-1)/2]}\nabla\partial_t^{[(m-1)/2]+1}G(\cdot-y,1+t-\tau(1+s))d\tau$$

$$=: I_1(t) + I_{21}^m(t) + I_{22}^m(t) + I_{23}^m(t),$$

Hence using this representation with $m = n$, we obtain

$$\int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)ds - \sum_{|\alpha|+2p\le n-1}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot$$

$$\cdot\int_0^t\int_{\mathbf{R}^n} y^\alpha(1+s)^p\{(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)\}dyds - W(t)$$

$$=: I_1(t) + I_{21}^n(t) + I_{22}^n(t) + I_{23}^n(t).$$

By using Lemma 3 and (4.26), we have

$$\|I_1(t)\|_q \le C\int_{t/2}^t (t-s)^{-1/2}\|(u\nabla v)(s) - M_0^2(G\nabla G)(1+s)\|_q ds$$

$$\le C\int_{t/2}^t (t-s)^{-1/2}s^{-n(1-1/q)/2-(n+2)/2}\,ds \le Ct^{-n(1-1/q)/2-(n+1)/2},$$

which implies that

$$\lim_{t\to\infty} t^{n(1-1/q)/2+n/2}\|I_1(t)\|_q = 0. \tag{5.4}$$

To estimate $\|I_{2k}^n(t)\|_q$ $(k = 1, 2, 3)$, we here claim that for $|\alpha| \le n$,

$$\left|\int_{\mathbf{R}^n} y^\alpha\{(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)\}dy\right| \le C(1+s)^{-n/2-1+|\alpha|/2} \qquad (s>0). \tag{5.5}$$

Indeed, we prove (5.5) only for $|\alpha| > 0$ since the estimate (5.5) for $|\alpha| = 0$ follows from (4.26). For this purpose, we write $|y|^{|\alpha|}|(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)|$ as follows:

$$\begin{aligned}
&|y|^{|\alpha|}|(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)| \\
&\quad\le |y|^{|\alpha|}|u(y,s)|\,|\nabla v(y,s) - M_0\nabla G(y,1+s)| \\
&\qquad + |y|^{|\alpha|}|M_0\nabla G(y,1+s)|\,|u(y,s) - M_0 G(y,1+s)| \\
&\quad=: L_1(y,s) + L_2(y,s).
\end{aligned}$$

Then, by (D) with $\gamma = n$, $L_1(y,s)$ is estimated as

$$\begin{aligned}
L_1(y,s) &\le \{|y|^n|u(y,s)|\}^{|\alpha|/n}|u(y,s)|^{1-|\alpha|/n}|\nabla v(y,s) - M_0\nabla G(y,1+s)| \\
&\le C|u(y,s)|^{1-|\alpha|/n}|\nabla v(y,s) - M_0\nabla G(y,1+s)|.
\end{aligned}$$

Similarly, by (3.1) with $k = 1$,

$$L_2(y,s) \le C|\nabla G(y,1+s)|^{1-|\alpha|/(n+1)}|u(y,s) - M_0 G(y,1+s)|.$$

Therefore applying Hölder's inequality, (3.3), (4.19), (4.24) and (4.25) yields that

$$\begin{aligned}
&\left|\int_{\mathbf{R}^n} y^\alpha\{(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)\}dy\right| \\
&\quad\le C\|u(s)\|_1^{1-|\alpha|/n}\|\nabla v(s) - M_0\nabla G(1+s)\|_{n/|\alpha|} \\
&\qquad + C\|\nabla G(1+s)\|_1^{1-|\alpha|/(n+1)}\|u(s) - M_0 G(1+s)\|_{(n+1)/|\alpha|} \\
&\quad\le C(1+s)^{-n/2-1+|\alpha|/2}.
\end{aligned}$$

This implies the desired estimate (5.5).

We now estimate $\|I_{2k}^n(t)\|_q$ $(k = 1, 2, 3)$. It follows from Minkowski's inequality, (3.3) and (5.5) that

$$\begin{aligned}
\|I_{21}^n(t)\|_q &\le \sum_{|\alpha|+2p\le n-1} C\|\nabla\partial_x^\alpha\partial_t^p G(1+t)\|_q \int_{t/2}^t (1+s)^p ds \\
&\quad \times \left|\int_{\mathbf{R}^n} y^\alpha\{(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)\}dy\right| \\
&\le \sum_{|\alpha|+2p\le n-1} Ct^{-n(1-1/q)/2-|\alpha|/2-p-1/2}\int_{t/2}^t (1+s)^{-n/2-1+|\alpha|/2+p}ds \\
&\le Ct^{-n(1-1/q)/2-(n+1)/2},
\end{aligned}$$

which implies that

$$\lim_{t\to\infty} t^{n(1-1/q)/2+n/2}\|I_{21}^n(t)\|_q = 0. \tag{5.6}$$

Similarly,

$$\begin{aligned}
\|I_{22}^n(t)\|_q &\le \sum_{\substack{|\alpha|+2p=n,\\ |\alpha|\ge 1}} Ct^{-n(1-1/q)/2-|\alpha|/2-p-1/2}\int_0^{t/2}(1+s)^p ds \\
&\quad \times \left|\int_{\mathbf{R}^n} y^\alpha\{(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)\}dy\right| \\
&\le Ct^{-n(1-1/q)/2-(n+1)/2}\int_0^{t/2}(1+s)^{-1}ds \\
&\le Ct^{-n(1-1/q)/2-(n+1)/2}\log t,
\end{aligned}$$

$$\begin{aligned}
\|I_{23}^n(t)\|_q &\le C\int_0^{t/2}(1+s)^{[(n-1)/2]+1}ds \\
&\quad \times \left|\int_{\mathbf{R}^n}\{(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)\}dy\right| \\
&\quad \times \int_0^1 \|\partial_t^{[(n-1)/2]+1}\nabla G(\cdot - y, 1+t-\tau(1+s))\|_q d\tau \\
&\le C\int_0^{t/2}(t-s)^{-n(1-1/q)/2-[(n-1)/2]-3/2}(1+s)^{-n/2+[(n-1)/2]}ds \\
&\le Ct^{-n(1-1/q)/2-[(n-1)/2]-3/2}\int_0^{t/2}(1+s)^{-n/2+[(n-1)/2]}ds \\
&\le C\times\begin{cases} t^{-n(1-1/q)/2-(n+1)/2} & \text{if } n \text{ is odd,} \\ t^{-n(1-1/q)/2-(n+1)/2}\log t & \text{if } n \text{ is even,}\end{cases}
\end{aligned}$$

which yield that

$$\lim_{t\to\infty} t^{n(1-1/q)/2+n/2}(\|I_{22}^n(t)\|_q + \|I_{23}^n(t)\|_q) = 0. \tag{5.7}$$

As a consequence, (5.2) follows from (5.4), (5.6) and (5.7). □

PROOF OF THEOREM 1. Let $t \ge 2$ and $1 \le q \le \infty$. Once the asymptotic behavior of u is shown, that of v is obtained by Lemma 9. Hence we prove only the asymptotic behavior of u.

First of all, by (5.5), we directly see that the convergence of integral

$$\int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p \{(u\nabla v)(y,s) - M_0^2 (G\nabla G)(y, 1+s)\} dyds \tag{5.8}$$

is ensured for $|\alpha| + 2p \le n-1$. The integrals

$$\int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u\nabla v\, dyds \qquad \text{and} \qquad \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p (G\nabla G)(y, 1+s) dyds$$

also converge for $|\alpha| + 2p \le n-2$ because using arguments similar to those in the proof of (5.5) gives

$$\sup_{s>0} (1+s)^{n/2+1/2-|\alpha|/2} \left| \int_{\mathbf{R}^n} y^\alpha u\nabla v\, dy \right| < \infty, \tag{5.9}$$

$$\sup_{s>0} (1+s)^{n/2+1/2-|\alpha|/2} \left| \int_{\mathbf{R}^n} y^\alpha (G\nabla G)(y, 1+s) dy \right| < \infty \tag{5.10}$$

for $|\alpha| \le n$. Hence we use these fact and (2.1) to get

$$\begin{aligned} u(t) &- \sum_{|\alpha|\le n} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^\alpha G(1+t) \int_{\mathbf{R}^n} y^\alpha u_0\, dy \\ &+ \sum_{|\alpha|+2p\le n-2} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u\nabla v\, dyds \\ &+ \sum_{|\alpha|+2p=n-1} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \int_0^t \int_{\mathbf{R}^n} y^\alpha (1+s)^p u\nabla v\, dyds \\ &+ \left\{ W(t) - M_0^2 \sum_{|\alpha|+2p\le n-2} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \right. \end{aligned}$$

$$
\begin{aligned}
&\cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p (G\nabla G)(y,1+s)dyds \\
&- M_0^2 \sum_{|\alpha|+2p=n-1} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \\
&\cdot \int_0^t \int_{\mathbf{R}^n} y^\alpha (1+s)^p (G\nabla G)(y,1+s)dyds \Bigg\} \\
= &\left\{ e^{t\Delta} u_0 - \sum_{|\alpha|\le n} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^\alpha G(1+t) \int_{\mathbf{R}^n} y^\alpha u_0 \, dy \right\} \\
&+ \sum_{|\alpha|+2p\le n-2} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \\
&\cdot \int_t^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p \{(u\nabla v)(y,s) - M_0^2 (G\nabla G)(y,1+s)\} dyds \\
&- \Bigg\{ \int_0^t e^{(t-s)\Delta} \nabla \cdot (u\nabla v)(s)ds - \sum_{|\alpha|+2p\le n-1} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \\
&\cdot \int_0^t \int_{\mathbf{R}^n} y^\alpha (1+s)^p \{(u\nabla v)(y,s) - M_0^2 (G\nabla G)(y,1+s)\} dyds - W(t) \Bigg\} \\
=: &\, M_1(t) + M_2(t) + M_3(t),
\end{aligned}
$$

where $M_0 = \int_{\mathbf{R}^n} u_0 \, dy$ and $W(t) = M_0^2 \int_0^t e^{(t-s)\Delta} \nabla \cdot (G\nabla G)(1+s)ds$ as before.

Next, we estimate $\|M_k(t)\|_q$ $(k = 1,2,3)$. Since $t^{n(1-1/q)/2+n/2}\|M_1(t)\|_q$ and $t^{n(1-1/q)/2+n/2}\|M_3(t)\|_q$ tend to 0 as $t \to \infty$ by making use of (3.6) with $m = n$ and Proposition 4, respectively, we consider only the estimate of $\|M_2(t)\|_q$. The estimation of $\|M_2(t)\|_q$ can be achieved as follows:

$$
\begin{aligned}
\|M_2(t)\|_q &\le \sum_{|\alpha|+2p\le n-2} C\|\nabla \partial_x^\alpha \partial_t^p G(1+t)\|_q \\
&\qquad \times \left| \int_t^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p \{(u\nabla v)(y,s) - M_0^2 (G\nabla G)(y,1+s)\} dyds \right| \\
&\le \sum_{|\alpha|+2p\le n-2} Ct^{-n(1-1/q)/2-|\alpha|/2-p-1/2} \int_t^\infty s^{-n/2-1+|\alpha|/2+p} \, ds \\
&\le Ct^{-n(1-1/q)/2-(n+1)/2},
\end{aligned}
$$

which yields that $\lim_{t\to\infty} t^{n(1-1/q)/2+n/2}\|M_2(t)\|_q = 0$. Here we have used (3.3) and (5.5). Therefore,

$$\begin{aligned}
t^{n(1-1/q)/2+n/2}\Bigg\| u(t) &- \sum_{|\alpha|\le n}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_x^\alpha G(1+t)\int_{\mathbf{R}^n} y^\alpha u_0\,dy \\
&+ \sum_{|\alpha|+2p\le n-2}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot\int_0^\infty\int_{\mathbf{R}^n} y^\alpha(1+s)^p u\nabla v\,dyds \\
&+ \sum_{|\alpha|+2p=n-1}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot\int_0^t\int_{\mathbf{R}^n} y^\alpha(1+s)^p u\nabla v\,dyds \\
&+ \Bigg\{ W(t) - M_0^2\sum_{|\alpha|+2p\le n-2}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot \\
&\qquad\cdot\int_0^\infty\int_{\mathbf{R}^n} y^\alpha(1+s)^p(G\nabla G)(y,1+s)dyds \\
&\quad - M_0^2\sum_{|\alpha|+2p=n-1}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot \\
&\qquad\cdot\int_0^t\int_{\mathbf{R}^n} y^\alpha(1+s)^p(G\nabla G)(y,1+s)dyds\Bigg\}\Bigg\|_q \to 0
\end{aligned}\tag{5.11}$$

as $t\to\infty$.

Now we shall prove the following claim in order to obtain the higher-order asymptotic expansion of u in more detail.

CLAIM 9. *For any integer l with $1\le l<n$, we have*

$$\begin{aligned}
&M_0^2\sum_{|\alpha|+2p=l-1}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot\int_0^\infty\int_{\mathbf{R}^n} y^\alpha(1+s)^p(G\nabla G)(y,1+s)dyds \\
&= \begin{cases} 0 & \text{if } l \text{ is odd}, \\ \dfrac{M_0^2}{2(8\pi)^{n/2}}\displaystyle\sum_{|\alpha|+p=(l-2)/2}\frac{(-1)^p\Delta\partial_x^{2\alpha}\partial_t^p G(1+t)}{2^{|\alpha|-1}\alpha!p!\{(n-2)-2(|\alpha|+p)\}} & \text{if } l \text{ is even}. \end{cases}
\end{aligned}\tag{5.12}$$

Furthermore, if n is odd,

$$\begin{aligned}
M_0^2\sum_{|\alpha|+2p=n-1}&\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot \\
&\cdot\int_0^t\int_{\mathbf{R}^n} y^\alpha(1+s)^p(G\nabla G)(y,1+s)dyds = 0
\end{aligned}\tag{5.13}$$

and

$$M_0^2 \sum_{|\alpha|+2p=n-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot$$

$$\cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p (G\nabla G)(y, 1+s) dyds = 0, \tag{5.14}$$

and if n is even,

$$M_0^2 \sum_{|\alpha|+2p=n-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot$$

$$\cdot \int_0^t \int_{\mathbf{R}^n} y^\alpha (1+s)^p (G\nabla G)(y, 1+s) dyds$$

$$= \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{|\alpha|+p=(n-2)/2} \frac{(-1)^p}{2^{|\alpha|}\alpha!p!} \Delta \partial_x^{2\alpha} \partial_t^p G(1+t) \log(1+t). \tag{5.15}$$

Proof. Since we can show the desired equalities (5.13), (5.14) and (5.15) by using arguments similar to those in the proof of (5.12), we show only the equality (5.12). A direct calculation gives

$$M_0^2 \sum_{|\alpha|+2p=l-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot$$

$$\cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p (G\nabla G)(y, 1+s) dyds$$

$$= -\frac{M_0^2}{2} \sum_{|\alpha|+2p=l-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot$$

$$\cdot \int_0^\infty \int_{\mathbf{R}^n} y y^\alpha (1+s)^{-1+p} G^2(y, 1+s) dyds$$

$$= -\frac{M_0^2}{2(4\pi)^n} \sum_{j=1}^n \sum_{|\alpha|+2p=l-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!} \partial_j \partial_x^\alpha \partial_t^p G(1+t)$$

$$\times \int_0^\infty \int_{\mathbf{R}^n} y_j y^\alpha (1+s)^{-n-1+p} \exp\left(-\frac{|y|^2}{2(1+s)}\right) dyds$$

$$= \frac{M_0^2}{2(4\pi)^n} \sum_{j=1}^{n} \sum_{\substack{|\alpha|+2p=l,\\ \alpha_j \neq 0}} \frac{\alpha_j(-1)^{|\alpha|+p}}{\alpha!p!} \partial_x^\alpha \partial_t^p G(1+t)$$

$$\times \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^{-n-1+p} \exp\left(-\frac{|y|^2}{2(1+s)}\right) dyds$$

$$= \frac{M_0^2}{2(4\pi)^n} \sum_{j=1}^{n} \sum_{\substack{|\alpha|+2p=l,\\ \alpha_j \neq 0}} \frac{\alpha_j(-1)^{|\alpha|+p}}{\alpha!p!} \partial_x^\alpha \partial_t^p G(1+t)$$

$$\times \int_0^\infty (1+s)^{-n/2+|\alpha|/2+p-1} ds \int_{\mathbf{R}^n} y^\alpha e^{-|y|^2/2}\, dy. \tag{5.16}$$

If l is even, then it follows from (5.16) that

$$M_0^2 \sum_{|\alpha|+2p=l-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot$$

$$\cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p (G\nabla G)(y, 1+s) dyds$$

$$= \frac{M_0^2}{2(4\pi)^n} \sum_{j=1}^{n} \sum_{\substack{|\alpha|+p=l/2,\\ \alpha_j \neq 0}} \frac{2\alpha_j(-1)^p}{(2\alpha)!p!} \partial_x^{2\alpha} \partial_t^p G(1+t)$$

$$\times \int_0^\infty (1+s)^{-n/2+|\alpha|+p-1} ds \int_{\mathbf{R}^n} y^{2\alpha} e^{-|y|^2/2}\, dy$$

$$= \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{j=1}^{n} \sum_{\substack{|\alpha|+p=l/2,\\ \alpha_j \neq 0}} \frac{\alpha_j(-1)^p}{2^{|\alpha|-1}\alpha!p!} \partial_x^{2\alpha} \partial_t^p G(1+t) \int_0^\infty (1+s)^{-n/2+|\alpha|+p-1} ds$$

$$= \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{j=1}^{n} \sum_{|\alpha|+p=(l-2)/2} \frac{(-1)^p}{2^{|\alpha|}\alpha!p!} \partial_j^2 \partial_x^{2\alpha} \partial_t^p G(1+t) \int_0^\infty (1+s)^{-n/2+|\alpha|+p} ds$$

$$= \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{|\alpha|+p=(l-2)/2} \frac{(-1)^p \Delta \partial_x^{2\alpha} \partial_t^p G(1+t)}{2^{|\alpha|-1}\alpha!p!\{(n-2)-2(|\alpha|+p)\}}.$$

Here we have used the fact that for $m \in \mathbf{Z}_+$,

$$\int_{\mathbf{R}} y^m e^{-y^2/2}\, dy = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ (m-1)(m-3)\dots 3\cdot 1\cdot (2\pi)^{1/2} & \text{if } m \text{ is even.} \end{cases} \tag{5.17}$$

On the other hand, if l is odd, then (5.16) is zero due to (5.17). As a consequence, we obtain the desired equality (5.12). □

Proof of Theorem 1, continued. First, by applying Claim 9, the following term appearing in (5.11) is calculated as

$$
\begin{aligned}
&W(t) - M_0^2 \sum_{|\alpha|+2p \le n-2} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \\
&\quad \cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p (G \nabla G)(y, 1+s) dy ds \\
&\quad - M_0^2 \sum_{|\alpha|+2p = n-1} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \\
&\quad \cdot \int_0^t \int_{\mathbf{R}^n} y^\alpha (1+s)^p (G \nabla G)(y, 1+s) dy ds \\
&= \begin{cases}
W(t) - \dfrac{M_0^2}{2(8\pi)} \Delta G(1+t) \log(1+t) \qquad \text{if } n = 2, \\[2ex]
W(t) - \dfrac{M_0^2}{2(8\pi)^{n/2}} \displaystyle\sum_{|\alpha|+p \le [(n-3)/2]} \frac{(-1)^p \Delta \partial_x^{2\alpha} \partial_t^p G(1+t)}{2^{|\alpha|-1} \alpha! p! \{(n-2) - 2(|\alpha|+p)\}} \\
\qquad \text{if } n \text{ is odd with } n \ge 3, \\[2ex]
W(t) - \dfrac{M_0^2}{2(8\pi)^{n/2}} \displaystyle\sum_{|\alpha|+p < [(n-3)/2]} \frac{(-1)^p \Delta \partial_x^{2\alpha} \partial_t^p G(1+t)}{2^{|\alpha|-1} \alpha! p! \{(n-2) - 2(|\alpha|+p)\}} \\
\qquad - \dfrac{M_0^2}{2(8\pi)^{n/2}} \displaystyle\sum_{|\alpha|+p = (n-2)/2} \frac{(-1)^p}{2^{|\alpha|} \alpha! p!} \Delta \partial_x^{2\alpha} \partial_t^p G(1+t) \log(1+t) \\
\qquad \text{if } n \text{ is even with } n \ge 3
\end{cases} \\
&= R(t), \qquad (5.18)
\end{aligned}
$$

where $R(t)$ is the one defined by (2.3).

Next, for the n-th order term of asymptotic expansion of u in (5.11), we show the following: If n is odd,

$$
\begin{aligned}
&t^{n(1-1/q)/2+n/2} \Bigg\| \sum_{|\alpha|+2p=n-1} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \int_0^t \int_{\mathbf{R}^n} y^\alpha (1+s)^p u \nabla v \, dy ds \\
&\quad - \sum_{|\alpha|+2p=n-1} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u \nabla v \, dy ds \Bigg\|_q \to 0 \qquad (5.19)
\end{aligned}
$$

as $t \to \infty$, and if n is even,

$$t^{n(1-1/q)/2+n/2}\Bigg\| \sum_{|\alpha|+2p=n-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!} \nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot \int_0^t \int_{\mathbf{R}^n} y^\alpha(1+s)^p u\nabla v\, dyds$$

$$- \sum_{|\alpha|+2p=n-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!} \nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot$$

$$\cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha(1+s)^p\{(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)\}dyds$$

$$- \frac{M_0^2}{2(8\pi)^{n/2}} \sum_{|\alpha|+p=(n-2)/2} \frac{(-1)^p}{2^{|\alpha|}\alpha!p!} \Delta\partial_x^{2\alpha}\partial_t^p G(1+t)\log(1+t)\Bigg\|_q \to 0 \tag{5.20}$$

as $t \to \infty$.

Indeed, since the integral (5.8) is well-defined for $|\alpha| + 2p = n-1$, we see from (3.3) and (5.5) that

$$t^{n(1-1/q)/2+n/2}\Bigg\| \sum_{|\alpha|+2p=n-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!} \nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot \int_0^t \int_{\mathbf{R}^n} y^\alpha(1+s)^p u\nabla v\, dyds$$

$$- \sum_{|\alpha|+2p=n-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!} \nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot$$

$$\cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha(1+s)^p\{(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)\}dyds$$

$$- M_0^2 \sum_{|\alpha|+2p=n-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!} \nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot$$

$$\cdot \int_0^t \int_{\mathbf{R}^n} y^\alpha(1+s)^p (G\nabla G)(1+s)dyds\Bigg\|_q$$

$$= t^{n(1-1/q)/2+n/2}\Bigg\| \sum_{|\alpha|+2p=n-1} \frac{(-1)^{|\alpha|+p}}{\alpha!p!} \nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot$$

$$\cdot \int_t^\infty \int_{\mathbf{R}^n} y^\alpha(1+s)^p\{(u\nabla v)(y,s) - M_0^2(G\nabla G)(y,1+s)\}dyds\Bigg\|_q$$

$$\le \sum_{|\alpha|+2p=n-1} Ct^{n/2-|\alpha|/2-p-1/2} \int_t^\infty s^{-n/2-1+|\alpha|/2+p}\, ds \le Ct^{-1/2} \to 0 \tag{5.21}$$

as $t \to \infty$. If n is odd, (5.21) gives (5.19) by (5.13) and (5.14). On the other hand, if n is even, (5.21) implies (5.20) because of (5.15). Hence (2.5) is obtained by combining (5.18) and (5.19) with (5.11). Also, (2.6) follows from (5.11), (5.18) and (5.20). As a consequence, the proof of Theorem 1 is complete.

6. Proof of Theorem 2

In this section, let (u, v) be the solution to (P) satisfying (D) with $\gamma = n+1$, and let $n \geq 1$ and $1 \leq q \leq \infty$. We repeat arguments similar to those in Section 5, and use the same notation as in the section. First of all, we give L^q-estimates for the solution in order to prove Theorem 2. Using Proposition 3 with $M_0 = 0$ gives

$$\sup_{t>0}(1+t)^{n(1-1/q)/2+1/2}\|u(t)\|_q < \infty, \tag{6.1}$$

$$\sup_{t>0}(1+t)^{n(1-1/q)/2+1}\|\nabla v(t)\|_q < \infty. \tag{6.2}$$

Next we show only the asymptotic behavior of u because that of v is obtained by Lemma 9. From (2.1) the following equality holds:

$$\begin{aligned} &u(t) - \sum_{1\leq|\alpha|\leq n+1} \frac{(-1)^{|\alpha|}}{\alpha!}\partial_x^\alpha G(1+t)\int_{\mathbf{R}^n} y^\alpha u_0\, dy \\ &\quad + \sum_{|\alpha|+2p\leq n} \frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot\int_0^\infty\int_{\mathbf{R}^n} y^\alpha(1+s)^p u\nabla v\, dyds \\ &= \left\{ e^{t\Delta}u_0 - \sum_{1\leq|\alpha|\leq n+1}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_x^\alpha G(1+t)\int_{\mathbf{R}^n} y^\alpha u_0\, dy \right\} \\ &\quad - \left\{ \int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)ds - \sum_{|\alpha|+2p\leq n}\frac{(-1)^{|\alpha|+p}}{\alpha!p!}\nabla\partial_x^\alpha\partial_t^p G(1+t)\cdot \right. \\ &\qquad \left. \cdot\int_0^\infty\int_{\mathbf{R}^n} y^\alpha(1+s)^p u\nabla v\, dyds \right\}. \end{aligned}$$

Therefore it is sufficient to prove the following proposition due to (3.6) with $m = n+1$ and $M_0 = 0$:

Proposition 5. *Let* $n \geq 1$ *and* $1 \leq q \leq \infty$. *Then the convergence of integral* $\int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u\nabla v \, dyds$ *is assured for* $|\alpha| + 2p \leq n$, *and*

$$t^{n(1-1/q)/2+(n+1)/2} \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot (u\nabla v)(s)ds - \sum_{|\alpha|+2p\leq n} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \right.$$

$$\left. \cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u\nabla v \, dyds \right\|_q \to 0 \tag{6.3}$$

as $t \to \infty$.

Proof. Fix $t \geq 2$. Firstly, we easily observe that the convergence of integral $\int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u\nabla v \, dyds$ is ensured for $|\alpha| + 2p \leq n$ since it follows from (6.1) and (6.2) that

$$\sup_{s>0} (1+s)^{n/2+3/2-|\alpha|/2} \left| \int_{\mathbf{R}^n} y^\alpha u\nabla v \, dy \right| < \infty$$

for $1 \leq j \leq n$ and $|\alpha| \leq n+1$ by using arguments similar to those in the proof of (5.5).

Next, as in the proof of Proposition 4, the following holds:

$$\int_0^t e^{(t-s)\Delta} \nabla \cdot (u\nabla v)(s)ds - \sum_{|\alpha|+2p\leq n} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot$$

$$\cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u\nabla v \, dyds$$

$$=: I_1(t) + \left\{ I_{21}^{n+1}(t) - \sum_{|\alpha|+2p\leq n} \frac{(-1)^{|\alpha|+p}}{\alpha! p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \right.$$

$$\left. \cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u\nabla v \, dyds \right\} + I_{22}^{n+1}(t) + I_{23}^{n+1}(t).$$

Hence calculations similar to those in the proof of Proposition 4 give the following estimates:

$$\|I_1(t)\|_q \le Ct^{-n(1-1/q)/2-(n+2)/2},$$

$$\begin{aligned}
&\left\| I_{21}^{n+1}(t) - \sum_{|\alpha|+2p\le n} \frac{(-1)^{|\alpha|+p}}{\alpha!p!} \nabla \partial_x^\alpha \partial_t^p G(1+t) \cdot \int_0^\infty \int_{\mathbf{R}^n} y^\alpha (1+s)^p u\nabla v \, dyds \right\|_q \\
&\quad \le \sum_{|\alpha|+2p\le n} Ct^{-n(1-1/q)/2-|\alpha|/2-p-1/2} \int_{t/2}^\infty (1+s)^p ds \left| \int_{\mathbf{R}^n} y^\alpha u\nabla v \, dy \right| \\
&\quad \le \sum_{|\alpha|+2p\le n} Ct^{-n(1-1/q)/2-|\alpha|/2-p-1/2} \int_{t/2}^\infty (1+s)^{-n/2-3/2+|\alpha|/2+p} ds \\
&\quad \le Ct^{-n(1-1/q)/2-(n+2)/2},
\end{aligned}$$

$$\begin{aligned}
\|I_{22}^{n+1}(t)\|_q &\le \sum_{\substack{|\alpha|+2p=n+1,\\ |\alpha|\ge 1}} Ct^{-n(1-1/q)/2-(n+2)/2} \int_0^{t/2} (1+s)^p ds \left| \int_{\mathbf{R}^n} y^\alpha u\nabla v \, dy \right| \\
&\le \sum_{\substack{|\alpha|+2p=n+1,\\ |\alpha|\ge 1}} Ct^{-n(1-1/q)/2-(n+2)/2} \int_0^{t/2} (1+s)^{-1} ds \\
&\le Ct^{-n(1-1/q)/2-(n+2)/2} \log t,
\end{aligned}$$

$$\begin{aligned}
\|I_{23}^{n+1}(t)\|_q &\le Ct^{-n(1-1/q)/2-3/2-[n/2]} \int_0^{t/2} (1+s)^{[n/2]+1} ds \left| \int_{\mathbf{R}^n} u\nabla v \, dy \right| \\
&\le Ct^{-n(1-1/q)/2-3/2-[n/2]} \int_0^{t/2} (1+s)^{-n/2-1/2+[n/2]} ds \\
&\le C \times \begin{cases} t^{-n(1-1/q)/2-(n+2)/2} \log t & \text{if } n \text{ is odd,} \\ t^{-n(1-1/q)/2-(n+2)/2} & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

As a consequence, (6.3) follows from these estimates.

7. Proof of Theorem 3

In this section, we shall show the existence of solutions to (P) satisfying (D) by applying the contraction mapping principle. The proof consists of several steps.

STEP 1. Let us define the space X_0 by

$$X_0 = C([0,\infty); L^1(\mathbf{R}^n)) \cap C([0,\infty); \mathscr{B}(\mathbf{R}^n))$$

and consider the Banach space

$$X = \{u \in X_0 \mid \|u\|_X < \infty\}$$

with the norm

$$\|u\|_X = \sup_{x \in \mathbf{R}^n, t>0} (1+|x|)^{\gamma}|u(x,t)| + \sup_{x \in \mathbf{R}^n, t>0} (1+t)^{\gamma/2}|u(x,t)| + \sup_{t>0} \|u(t)\|_1,$$

where γ is either n or $n+1$. We here note that $u \in X$ implies

$$|u(x,t)| \le (1+|x|)^{\alpha-\gamma}(1+t)^{-\alpha/2}\|u\|_X \qquad \text{for } 0 \le \alpha \le \gamma. \tag{7.1}$$

For $K > 0$, we define the closed subset B_K of X by

$$B_K = \{u \in X \mid \|u\|_X \le K\}.$$

Given $u \in B_K$, define v by

$$\begin{aligned} v(t) &= e^{-t}e^{t\Delta}v_0 + \int_0^t e^{-(t-s)}e^{(t-s)\Delta}u(s)ds \\ &=: [V(v_0)](t) + [J(u)](t) \end{aligned} \tag{7.2}$$

and then $\Phi(u)$ by

$$\begin{aligned} [\Phi(u)](t) &= e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}(u\nabla v)(s)ds \\ &=: [U(u_0)](t) - I[(u,\nabla v)](t). \end{aligned}$$

We now put $\|v_0\|_{L^1 \cap L^\infty_\gamma} := \|v_0\|_1 + \|v_0\|_{L^\infty_\gamma}$, $\|\nabla v_0\|_{L^1 \cap L^\infty_\gamma} := \|\nabla v_0\|_1 + \|\nabla v_0\|_{L^\infty_\gamma}$ and

$$l_\gamma(u_0) = \begin{cases} \|u_0\|_1 + \|u_0\|_{L^\infty_\gamma} & \text{if } \gamma = n, \\ \|u_0\|_{L^1_1} + \|u_0\|_{L^\infty_\gamma} & \text{if } \gamma = n+1. \end{cases} \tag{7.3}$$

Using Lemma 7 and $U(u_0), V(v_0), \nabla V(v_0) \in X_0$, we easily see that $U(u_0)$, $V(v_0)$ and $\nabla V(v_0)$ are in X, and satisfy

$$\|U(u_0)\|_X \le C_1^* l_\gamma(u_0), \tag{7.4}$$

$$\|V(v_0)\|_X \le C_2^*\|v_0\|_{L^1 \cap L^\infty_\gamma}, \qquad \|\nabla V(v_0)\|_X \le C_3^*\|\nabla v_0\|_{L^1 \cap L^\infty_\gamma}, \tag{7.5}$$

where C_k^* $(k = 1,2,3)$ are positive constants. Furthermore, Lemma 6 gives v, $\nabla v \in X_0$, which together with Lemma 2.2 of [19] implies $I(u,\nabla v) \in X_0$. Therefore we see that $\Phi(u)$ belongs to X_0.

Step 2. We show the following claim.

Claim 10. *There exist positive constants C_1, C_2 such that for $u \in B_K$,*

$$\|v\|_X \le C_1(\|v_0\|_{L^1 \cap L^\infty_\gamma} + \|u\|_X), \tag{7.6}$$

$$\|\nabla v\|_X \le C_2(\|\nabla v_0\|_{L^1 \cap L^\infty_\gamma} + \|u\|_X). \tag{7.7}$$

Proof. First of all, we show (7.6). By (7.5), it suffices to prove that

$$\|J(u)\|_X \le C\|u\|_X. \tag{7.8}$$

Since $J(\cdot)$ is linear on X, we can assume $\|u\|_X = 1$. Thus (7.1) gives

$$|u(x,t)| \le (1+|x|)^{\alpha-\gamma}(1+t)^{-\alpha/2} \qquad \text{for } 0 \le \alpha \le \gamma. \tag{7.9}$$

We now prove that

$$\sup_{x\in\mathbf{R}^n,\, t>0} (1+|x|)^{\gamma}|J(u)| < \infty. \tag{7.10}$$

For this purpose, we split $J(u)$ as follows:

$$J(u) = J_1(u) + J_2(u),$$

where

$$J_1(u) = \int_0^t \int_{|y|\le|x|/2} e^{-(t-s)} G(x-y,t-s)u(y,s)dyds,$$

$$J_2(u) = \int_0^t \int_{|y|\ge|x|/2} e^{-(t-s)} G(x-y,t-s)u(y,s)dyds.$$

From (7.9) with $\alpha = 0$ we obtain

$$\begin{aligned}|J_2(u)| &\le \int_0^t e^{-(t-s)}\, ds \int_{|y|\ge|x|/2} G(x-y,t-s)(1+|y|)^{-\gamma}dy \\ &\le C(1+|x|)^{-\gamma}\int_0^t e^{-(t-s)}\, ds \int_{\mathbf{R}^n} G(x-y,t-s)dy \\ &\le C(1+|x|)^{-\gamma}\end{aligned}$$

for $(x,t)\in\mathbf{R}^n\times(0,\infty)$, which implies that

$$\sup_{x\in\mathbf{R}^n,\, t>0} (1+|x|)^{\gamma}|J_2(u)| < \infty. \tag{7.11}$$

Assume that $|x|\ge 1$. Then we estimate $|J_1(u)|$. For $\gamma = n$, (3.1) with $k=0$ yields that

$$\begin{aligned}|J_1(u)| &\le C\int_0^t e^{-(t-s)}\, ds \int_{|y|\le|x|/2} |x-y|^{-n}e^{-|x-y|^2/\{8(t-s)\}}|u(y,s)|dy \\ &\le C|x|^{-n}\int_0^t e^{-(t-s)}\, ds\int_{\mathbf{R}^n}|u(y,s)|dy \\ &\le C(1+|x|)^{-n}\end{aligned}$$

since $|x-y|\ge|x|/2$ for $|y|\le|x|/2$.

For $\gamma = n+1$, we need to rewrite $J_1(u)$ as follows:

$$J_1(u) =: J_{11}(u) + J_{12}(u),$$

where

$$J_{11}(u) = \int_0^{t/2} \int_{|y| \le |x|/2} e^{-(t-s)} G(x-y, t-s) u(y,s) dyds,$$

$$J_{12}(u) = \int_{t/2}^{t} \int_{|y| \le |x|/2} e^{-(t-s)} G(x-y, t-s) u(y,s) dyds.$$

By using (3.1) with $k=0$ and $|x-y| \ge |x|/2$ for $|y| \le |x|/2$, we have

$$\begin{aligned} |J_{11}(u)| &\le Ce^{-t/2} \int_0^{t/2} \int_{|y| \le |x|/2} |x-y|^{-n} e^{-|x-y|^2/\{8(t-s)\}} |u(y,s)| dyds \\ &\le C|x|^{-n} e^{-t/2} e^{-|x|^2/(32t)} \int_0^{t/2} ds \int_{\mathbf{R}^n} |u(y,s)| dy \\ &\le C(1+|x|)^{-n-1}. \end{aligned}$$

(7.9) with $\alpha = n+1$ and (3.2) with $k=0$ also give

$$\begin{aligned} |J_{12}(u)| &\le C \int_{t/2}^{t} e^{-(t-s)} (t-s)^{-n/2} (1+s)^{-n/2-1/2} ds \int_{|y| \le |x|/2} e^{-|x-y|^2/\{8(t-s)\}}\, dy \\ &\le Ct^{-n/2-1/2} e^{-|x|^2/(32t)} \int_{t/2}^{t} e^{-(t-s)} (t-s)^{-n/2} ds \int_{\mathbf{R}^n} e^{-|x-y|^2/\{16(t-s)\}}\, dy \\ &\le C|x|^{-n-1} \int_{t/2}^{t} e^{-(t-s)}\, ds \le C(1+|x|)^{-n-1}, \end{aligned}$$

where we have used $|x-y| \ge |x|/2$ for $|y| \le |x|/2$. Therefore from these estimates we see that

$$|J_1(u)| \le C(1+|x|)^{-\gamma} \qquad \text{for } |x| \ge 1. \tag{7.12}$$

On the other hand, (7.9) with $\alpha = 0$ implies that for $(x,t) \in \mathbf{R}^n \times (0,\infty)$,

$$\begin{aligned} |J_1(u)| &\le \int_0^t \int_{|y| \le |x|/2} e^{-(t-s)} G(x-y, t-s) |u(y,s)| dyds \\ &\le \int_0^t \int_{\mathbf{R}^n} e^{-(t-s)} G(x-y, t-s) (1+|y|)^{-\gamma} dyds \\ &\le \int_0^t e^{-(t-s)}\, ds \int_{\mathbf{R}^n} G(x-y, t-s) dy \le \int_0^t e^{-(t-s)}\, ds \le C. \end{aligned}$$

Hence using the boundedness of $|J_1(u)|$ on $\mathbf{R}^n \times (0, \infty)$ and (7.12) yields that

$$\sup_{x \in \mathbf{R}^n, t > 0} (1 + |x|)^{\gamma} |J_1(u)| < \infty. \tag{7.13}$$

Consequently, combining (7.11) and (7.13), we obtain (7.10).

Next we show that

$$\sup_{x \in \mathbf{R}^n, t > 0} (1 + t)^{\gamma/2} |J(u)| < \infty. \tag{7.14}$$

To prove this, we again divide $J(u)$ into two parts:

$$J(u) = J_3(u) + J_4(u),$$

where

$$J_3(u) = \int_0^{t/2} \int_{\mathbf{R}^n} e^{-(t-s)} G(x - y, t - s) u(y, s) dy ds,$$

$$J_4(u) = \int_{t/2}^{t} \int_{\mathbf{R}^n} e^{-(t-s)} G(x - y, t - s) u(y, s) dy ds.$$

Then the estimations of $J_3(u)$ and $J_4(u)$ can be achieved as follows: For $(x, t) \in \mathbf{R}^n \times (0, \infty)$,

$$\begin{aligned} |J_3(u)| &\leq \int_0^{t/2} \int_{\mathbf{R}^n} e^{-(t-s)} G(x - y, t - s)(1 + s)^{-\gamma/2} dy ds \\ &\leq e^{-t/2} \int_0^{t/2} (1 + s)^{-\gamma/2} ds \int_{\mathbf{R}^n} G(x - y, t - s) dy \\ &\leq e^{-t/2} \int_0^{t/2} (1 + s)^{-\gamma/2} ds \leq C(1 + t)^{-\gamma/2}, \\ |J_4(u)| &\leq \int_{t/2}^{t} e^{-(t-s)} (1 + s)^{-\gamma/2} ds \int_{\mathbf{R}^n} G(x - y, t - s) dy \\ &\leq C(1 + t)^{-\gamma/2} \int_{t/2}^{t} e^{-(t-s)}\, ds \leq C(1 + t)^{-\gamma/2}, \end{aligned}$$

where we have used (7.9) with $\alpha = \gamma$. Thus these estimates give (7.14).

Finally, from (3.4) we have

$$\|J(u)\|_1 \leq \int_0^t e^{-(t-s)} \|e^{(t-s)\Delta} u(s)\|_1 ds \leq \int_0^t e^{-(t-s)} \|u(s)\|_1 ds \leq C \qquad (t > 0),$$

which together with (7.10) and (7.14) implies (7.8). As a consequence, the desired estimate (7.6) is obtained.

We are going to show (7.7) by arguments similar to those in the proof of (7.6). By (7.5), it is sufficient to claim that

$$\|\nabla J(u)\|_X \le C\|u\|_X. \tag{7.15}$$

Here we can assume $\|u\|_X = 1$ since $\nabla J(u)$ is linear with respect to $u \in X$.

We now prove that

$$\sup_{x\in\mathbf{R}^n,\, t>0} (1+|x|)^{\gamma}|\nabla J(u)| < \infty. \tag{7.16}$$

First of all, we divide $\nabla J(u)$ into two parts:

$$\nabla J(u) =: \nabla J_1(u) + \nabla J_2(u),$$

where

$$\nabla J_1(u) = \int_0^t \int_{|y|\le|x|/2} e^{-(t-s)}\nabla G(x-y,t-s)u(y,s)dyds,$$

$$\nabla J_2(u) = \int_0^t \int_{|y|\ge|x|/2} e^{-(t-s)}\nabla G(x-y,t-s)u(y,s)dyds.$$

Fix $|x| \ge 1$. Since (3.1) and (3.2) with $k=1$ give

$$\begin{aligned}|\nabla G(x-y,t-s)| &\le C|x-y|^{\beta-(n+1)}(t-s)^{-\beta/2}e^{-|x-y|^2/\{8(t-s)\}}\\ &\le C|x-y|^{\beta-(n+1)}(t-s)^{-\beta/2} \end{aligned}\tag{7.17}$$

for β with $0\le\beta\le n+1$, using (7.17) with $\beta = n+1-\gamma$ implies that for $|x|\ge 1$,

$$\begin{aligned}|\nabla J_1(u)| &\le C\int_0^t e^{-(t-s)}(t-s)^{-n/2-1/2+\gamma/2}ds\int_{|y|\le|x|/2}|x-y|^{-\gamma}|u(y,s)|dy\\ &\le C|x|^{-\gamma}\int_0^t e^{-(t-s)}(t-s)^{-n/2-1/2+\gamma/2}ds\int_{\mathbf{R}^n}|u(y,s)|ds\\ &\le C(1+|x|)^{-\gamma}\int_0^t e^{-(t-s)}(t-s)^{-n/2-1/2+\gamma/2}ds\\ &\le C(1+|x|)^{-\gamma}. \end{aligned}\tag{7.18}$$

On the other hand, (7.9) with $\alpha = 0$ gives that for $(x,t)\in\mathbf{R}^n\times(0,\infty)$,

$$\begin{aligned}|\nabla J_1(u)| &\le \int_0^t \int_{|y|\le|x|/2} e^{-(t-s)}|\nabla G(x-y,t-s)|\,|u(y,s)|dyds\\ &\le \int_0^t\int_{\mathbf{R}^n} e^{-(t-s)}|\nabla G(x-y,t-s)|(1+|y|)^{-\gamma}dyds\end{aligned}$$

$$\leq \int_0^t e^{-(t-s)}\,ds \int_{\mathbf{R}^n} |\nabla G(x-y,t-s)|dy$$
$$\leq C\int_0^t e^{-(t-s)}(t-s)^{-1/2}ds \leq C.$$

Therefore, putting together the boundedness of $|\nabla J_1(u)|$ on $\mathbf{R}^n \times (0,\infty)$ and (7.18) yields that

$$\sup_{x\in\mathbf{R}^n,\, t>0} (1+|x|)^{\gamma}|\nabla J_1(u)| < \infty. \tag{7.19}$$

Furthermore, by (7.9) with $\alpha = 0$, we obtain

$$|\nabla J_2(u)| \leq \int_0^t e^{-(t-s)}\,ds \int_{|y|\geq|x|/2} |\nabla G(x-y,t-s)|(1+|y|)^{-\gamma}dy$$
$$\leq C(1+|x|)^{-\gamma}\int_0^t e^{-(t-s)}(t-s)^{-1/2}ds \leq C(1+|x|)^{-\gamma}$$

for $(x,t)\in\mathbf{R}^n\times(0,\infty)$. Hence the desired estimate (7.16) is obtained by combining this estimate and (7.19).

Next we claim that the following holds:

$$\sup_{x\in\mathbf{R}^n,\, t>0} (1+t)^{\gamma/2}|\nabla J(u)| < \infty. \tag{7.20}$$

In fact, $\nabla J(u)$ is again represented as follows:

$$\nabla J(u) =: \nabla J_3(u) + \nabla J_4(u),$$

where

$$\nabla J_3(u) = \int_0^{t/2}\int_{\mathbf{R}^n} e^{-(t-s)}\nabla G(x-y,t-s)u(y,s)dyds,$$
$$\nabla J_4(u) = \int_{t/2}^{t}\int_{\mathbf{R}^n} e^{-(t-s)}\nabla G(x-y,t-s)u(y,s)dyds.$$

Then the estimations of $\nabla J_3(u)$ and $\nabla J_4(u)$ can be achieved as follows: For $(x,t)\in\mathbf{R}^n\times(0,\infty)$,

$$|\nabla J_3(u)| \leq \int_0^{t/2}\int_{\mathbf{R}^n} e^{-(t-s)}|\nabla G(x-y,t-s)|(1+s)^{-\gamma/2}dyds$$
$$\leq e^{-t/2}\int_0^{t/2}(1+s)^{-\gamma/2}ds\int_{\mathbf{R}^n}|\nabla G(x-y,t-s)|dy$$
$$\leq Ce^{-t/2}\int_0^{t/2}(t-s)^{-1/2}(1+s)^{-\gamma/2}ds \leq C(1+t)^{-\gamma/2},$$

$$|\nabla J_4(u)| \le \int_{t/2}^{t} e^{-(t-s)}(1+s)^{-\gamma/2} ds \int_{\mathbf{R}^n} |\nabla G(x-y,t-s)| dy$$
$$\le C(1+t)^{-\gamma/2} \int_{t/2}^{t} e^{-(t-s)}(t-s)^{-1/2} ds \le C(1+t)^{-\gamma/2},$$

where we have used (7.9) with $\alpha = \gamma$. Hence from these estimates we get (7.20).

Finally, from (3.4) we have

$$\|\nabla J(u)\|_1 \le \int_0^t e^{-(t-s)} \|\nabla e^{(t-s)\Delta} u(s)\|_1 ds$$
$$\le C \int_0^t e^{-(t-s)}(t-s)^{-1/2} \|u(s)\|_1 ds \le C \qquad (t > 0),$$

which together with (7.16) and (7.20) yields (7.15). Consequently, the desired estimate (7.7) is obtained. □

By Claim 10, we see that (u,v) is a solution to (P) on $\mathbf{R}^n \times [0,\infty)$ satisfying (D) if $u \in B_K$ is a fixed-point of Φ.

STEP 3. The following claim is a key one to prove Theorem 3.

CLAIM 11. *Let* $\varphi, \nabla\psi \in B_K$. *Then*

$$\|I(\varphi,\nabla\psi)\|_X \le C_3 \|\varphi\|_X \|\nabla\psi\|_X, \tag{7.21}$$

where C_3 *is a positive constant independent of* φ *and* ψ.

PROOF. Assume that $\|\varphi\|_X = \|\nabla\psi\|_X = 1$ because $I(\cdot,\cdot)$ is bilinear on $B_K \times B_K$, and note that for every $0 \le \lambda \le 2\gamma$ and every $0 \le \mu \le \gamma$, $\varphi, \nabla\psi \in B_K$ satisfy the following estimates:

$$|(\varphi\nabla\psi)(y,s)| \le (1+|y|)^{\lambda-2\gamma}(1+s)^{-\lambda/2}, \tag{7.22}$$

$$|\nabla\psi(y,s)| \le (1+|y|)^{\mu-\gamma}(1+s)^{-\mu/2}. \tag{7.23}$$

We now divide $I(\varphi,\nabla\psi)$ into two parts:

$$I(\varphi,\nabla\psi) = I_1(\varphi,\nabla\psi) + I_2(\varphi,\nabla\psi),$$

where

$$I_1(\varphi,\nabla\psi) = \int_0^t \int_{|y|\le|x|/2} \nabla G(x-y,t-s) \cdot (\varphi\nabla\psi)(y,s) dy ds,$$

$$I_2(\varphi,\nabla\psi) = \int_0^t \int_{|y|\ge|x|/2} \nabla G(x-y,t-s) \cdot (\varphi\nabla\psi)(y,s) dy ds.$$

Firstly, we estimate $|I_1(\varphi, \nabla\psi)|$. Fix $|x| \geq 1$. In the case $\gamma = n$, by applying (7.17) with $\beta = 1$ and (7.23) with $\mu = 1$, we have

$$\begin{aligned}
|I_1(\varphi, \nabla\psi)| &\leq \int_0^t \int_{|y|\leq|x|/2} |\nabla G(x-y, t-s)|\,|\varphi(y,s)|\,|\nabla\psi(y,s)|dyds \\
&\leq C|x|^{-n} \int_0^t (t-s)^{-1/2}(1+s)^{-1/2} ds \int_{\mathbf{R}^n} (1+|y|)^{-n+1}|\varphi(y,s)|dy \\
&\leq C|x|^{-n} \int_0^t (t-s)^{-1/2}s^{-1/2}\, ds \int_{\mathbf{R}^n} |\varphi(y,s)|dy \leq C(1+|x|)^{-n},
\end{aligned}$$

where we have used $|x-y| \geq |x|/2$ for $|y| \leq |x|/2$.

In the case $\gamma = n+1$, from (7.17) with $\beta = 0$ and (7.22) with $\lambda = n + 3/2$ we obtain

$$\begin{aligned}
|I_1(\varphi, \nabla\psi)| &\leq \int_0^t \int_{|y|\leq|x|/2} |\nabla G(x-y, t-s)|\,|(\varphi\nabla\psi)(y,s)|dyds \\
&\leq C|x|^{-n-1} \int_0^t (1+s)^{-n/2-3/4} ds \int_{\mathbf{R}^n} (1+|y|)^{-n-1/2} dy \\
&\leq C(1+|x|)^{-n-1}.
\end{aligned}$$

Therefore it follows from these estimates that for $|x| \geq 1$,

$$|I_1(\varphi, \nabla\psi)| \leq C(1+|x|)^{-\gamma}. \tag{7.24}$$

On the other hand, (7.22) with $\lambda = 1$ implies that for $(x,t) \in \mathbf{R}^n \times (0,\infty)$,

$$\begin{aligned}
|I_1(\varphi, \nabla\psi)| &\leq \int_0^t \int_{|y|\leq|x|/2} |\nabla G(x-y, t-s)|\,|(\varphi\nabla\psi)(y,s)|dyds \\
&\leq \int_0^t (1+s)^{-1/2} ds \int_{\mathbf{R}^n} |\nabla G(x-y, t-s)|(1+|y|)^{-2\gamma+1} dy \\
&\leq C \int_0^t (t-s)^{-1/2} s^{-1/2}\, ds \leq C.
\end{aligned} \tag{7.25}$$

Hence, it follows from the boundedness of $|I_1(\varphi, \nabla\psi)|$ on $\mathbf{R}^n \times (0,\infty)$ and (7.24) that

$$\sup_{x\in\mathbf{R}^n,\, t>0} (1+|x|)^{\gamma}|I_1(\varphi, \nabla\psi)| < \infty. \tag{7.26}$$

Next we estimate $|I_2(\varphi, \nabla\psi)|$. By using (7.22) with $\lambda = 1$, the estimation of $|I_2(\varphi, \nabla\psi)|$ can be achieved as follows: For $(x,t) \in \mathbf{R}^n \times (0,\infty)$,

$$|I_2(\varphi,\nabla\psi)| \le \int_0^t \int_{|y|\ge|x|/2} |\nabla G(x-y,t-s)|\,|(\varphi\nabla\psi)(y,s)|dyds$$
$$\le \int_0^t (1+s)^{-1/2}ds \int_{|y|\ge|x|/2} (1+|y|)^{-2\gamma+1}|\nabla G(x-y,t-s)|dy$$
$$\le C(1+|x|)^{-2\gamma+1}\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds \le C(1+|x|)^{-\gamma}.$$

This together with (7.26) implies that

$$\sup_{x\in\mathbf{R}^n,\,t>0}(1+|x|)^{\gamma}|I(\varphi,\nabla\psi)| < \infty. \tag{7.27}$$

Here we again split the integral $I(\varphi,\nabla\psi)$ as follows:

$$I(\varphi,\nabla\psi) =: I_3(\varphi,\nabla\psi) + I_4(\varphi,\nabla\psi),$$

where

$$I_3(\varphi,\nabla\psi) = \int_0^{t/2}\int_{\mathbf{R}^n} \nabla G(x-y,t-s)\cdot(\varphi\nabla\psi)(y,s)dyds,$$
$$I_4(\varphi,\nabla\psi) = \int_{t/2}^{t}\int_{\mathbf{R}^n} \nabla G(x-y,t-s)\cdot(\varphi\nabla\psi)(y,s)dyds.$$

First of all, we estimate $|I_3(\varphi,\nabla\psi)|$. Assume that $t\ge 1$. In the case $\gamma = n$, by using (7.17) with $\beta = n+1$ and (7.23) with $\mu = 1$, we have

$$|I_3(\varphi,\nabla\psi)| \le \int_0^{t/2}\int_{\mathbf{R}^n} |\nabla G(x-y,t-s)|\,|\varphi(y,s)|\,|\nabla\psi(y,s)|dyds$$
$$\le C\int_0^{t/2}(t-s)^{-n/2-1/2}(1+s)^{-1/2}ds\int_{\mathbf{R}^n}(1+|y|)^{-n+1}|\varphi(y,s)|dy$$
$$\le Ct^{-n/2-1/2}\int_0^{t/2} s^{-1/2}\,ds\int_{\mathbf{R}^n}|\varphi(y,s)|dy \le C(1+t)^{-n/2}.$$

In the case $\gamma = n+1$, (7.17) with $\beta = n+1$ and (7.22) with $\lambda = n+3/2$ imply that

$$|I_3(\varphi,\nabla\psi)| \le \int_0^{t/2}\int_{\mathbf{R}^n} |\nabla G(x-y,t-s)|\,|(\varphi\nabla\psi)(y,s)|dyds$$
$$\le C\int_0^{t/2}(t-s)^{-n/2-1/2}(1+s)^{-n/2-3/4}ds\int_{\mathbf{R}^n}(1+|y|)^{-n-1/2}dy$$
$$\le Ct^{-n/2-1/2}\int_0^{\infty}(1+s)^{-n/2-3/4}ds \le C(1+t)^{-n/2-1/2}.$$

Thus,

$$|I_3(\varphi, \nabla\psi)| \le C(1+t)^{-\gamma/2} \qquad \text{for } t \ge 1. \tag{7.28}$$

Since $|I_3(\varphi, \nabla\psi)|$ is bounded on $\mathbf{R}^n \times (0, \infty)$ by making use of the argument similar to that in the proof of (7.25), from this boundedness and (7.28) we see that

$$\sup_{x \in \mathbf{R}^n, t>0} (1+t)^{\gamma/2} |I_3(\varphi, \nabla\psi)| < \infty. \tag{7.29}$$

Next, we estimate $|I_4(\varphi, \nabla\psi)|$. It follows from (7.22) with $\lambda = 2\gamma$ that for $(x, t) \in \mathbf{R}^n \times (0, \infty)$,

$$\begin{aligned} |I_4(\varphi, \nabla\psi)| &\le \int_{t/2}^t \int_{\mathbf{R}^n} |\nabla G(x-y, t-s)| \, |(\varphi\nabla\psi)(y, s)| dy ds \\ &\le \int_{t/2}^t (1+s)^{-\gamma} ds \int_{\mathbf{R}^n} |\nabla G(x-y, t-s)| dy \\ &\le C \int_{t/2}^t (t-s)^{-1/2} (1+s)^{-\gamma} ds \le C(1+t)^{-\gamma/2}. \end{aligned}$$

Therefore this estimate and (7.29) yield that

$$\sup_{x \in \mathbf{R}^n, t>0} (1+t)^{\gamma/2} |I(\varphi, \nabla\psi)| < \infty. \tag{7.30}$$

To finish the proof of Claim 11, we estimate $\|I(\varphi, \nabla\psi)\|_1$. It follows from Minkowski's inequality and (7.23) with $\mu = \gamma$ that

$$\begin{aligned} \|I(\varphi, \nabla\psi)\|_1 &\le \int_0^t ds \int_{\mathbf{R}^n} |\varphi(y, s)| \, |\nabla\psi(y, s)| dy \int_{\mathbf{R}^n} |\nabla G(x-y, t-s)| dx \\ &\le \int_0^t (t-s)^{-1/2} (1+s)^{-\gamma/2} ds \int_{\mathbf{R}^n} |\varphi(y, s)| dy \\ &\le \int_0^t (t-s)^{-1/2} (1+s)^{-1/2} ds \le C \qquad \text{for } t > 0, \end{aligned}$$

which together with (7.27) and (7.30) implies (7.21). □

Step 4. We begin with the following claim.

Claim 12. *For* $u \in B_K$,

$$\|\Phi(u)\|_X \le C_0 l_\gamma(u_0) + C_0 \|\nabla v_0\|_{L^1 \cap L^\infty_\gamma} \|u\|_X + C_0 \|u\|_X^2, \tag{7.31}$$

and for $u_1, u_2 \in B_K$,

$$\|\Phi(u_1) - \Phi(u_2)\|_X \le C_0(\|\nabla v_0\|_{L^1\cap L^\infty_\gamma} + \|u_1\|_X + \|u_2\|_X)\|u_1 - u_2\|_X, \quad (7.32)$$

where $C_0 = \max\{C_1^*, C_2C_3\}$ *and* $l_\gamma(u_0)$ *is the one defined by* (7.3).

PROOF. First, we shall prove (7.32). A direct calculation gives

$$\begin{aligned}\Phi(u_1) - \Phi(u_2) &= -I(u_1 - u_2, \nabla v_1) - I(u_2, \nabla v_1 - \nabla v_2)\\ &=: I_1 + I_2.\end{aligned}$$

For I_1, making use of (7.7) with $v = v_1$, $u = u_1$ and (7.21) with $\varphi = u_1 - u_2$, $\nabla\psi = \nabla v_1$, we have

$$\begin{aligned}\|I_1\|_X &\le C_3\|u_1 - u_2\|_X\|\nabla v_1\|_X\\ &\le C_2C_3(\|\nabla v_0\|_{L^1\cap L^\infty_\gamma} + \|u_1\|_X)\|u_1 - u_2\|_X\\ &\le C_0(\|\nabla v_0\|_{L^1\cap L^\infty_\gamma} + \|u_1\|_X)\|u_1 - u_2\|_X.\end{aligned}$$

For I_2, applying (7.7) with $v = v_1 - v_2$, $\nabla v_0 = 0$ and $u = u_1 - u_2$ gives

$$\|\nabla v_1 - \nabla v_2\|_X \le C_2\|u_1 - u_2\|_X. \quad (7.33)$$

Therefore (7.21) with $\varphi = u_2$, $\nabla\psi = \nabla v_1 - \nabla v_2$ and (7.33) imply that

$$\begin{aligned}\|I_2\|_X &\le C_3\|u_2\|_X\|\nabla v_1 - \nabla v_2\|_X\\ &\le C_2C_3\|u_2\|_X\|u_1 - u_2\|_X \le C_0\|u_2\|_X\|u_1 - u_2\|_X.\end{aligned}$$

Consequently, (7.32) follows from these estimates.

The estimate (7.31) is also obtained by (7.4) and (7.32) with $u_1 = u$, $u_2 = 0$. □

STEP 5. We shall prove Theorem 3. Assuming

$$C_0^2 l_\gamma(u_0) < 1/16 \qquad \text{and} \qquad C_0\|\nabla v_0\|_{L^1\cap L^\infty_\gamma} < 1/2,$$

we define K_0 by

$$K_0 = \left(\frac{1}{2} - \sqrt{\frac{1}{4} - 4C_0^2 l_\gamma(u_0)}\right)\Big/(2C_0). \quad (7.34)$$

Then, by Claim 12, $u \in B_{K_0}$ implies that

$$\|\Phi[u]\|_X \le C_0 l_\gamma(u_0) + \frac{1}{2}K_0 + C_0K_0^2 = K_0.$$

Therefore $\Phi[u] \in B_{K_0}$ if $u \in B_{K_0}$. Since $2C_0K_0 < 1/2$ by (7.34), there exists $d \in (0,1)$ such that

$$\|\Phi[u_1] - \Phi[u_2]\|_X \le d\|u_1 - u_2\|_X \qquad \text{for every } u_1, u_2 \in B_{K_0}.$$

Hence, Φ is a contraction mapping in the closed subset B_{K_0} of X, which implies that Φ admits a unique fixed-point u in B_{K_0}. As a result, there exists a unique global solution to (P) satisfying (D). The proof of Theorem 3 is complete.

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