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Bifurcation diagrams and critical subsystems of the Kowalevski gyrostat in two constant fields

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Abstract. The Kowalevski gyrostat in two constant fields is known as the unique example of an integrable rigid body problem described by the Hamiltonian system with three degrees of freedom not reducible to a family of systems in fewer dimensions. The practical explicit integration of this system can hardly be obtained by the existing techniques. Then the challenging problem becomes to fulfill the qualitative investigation based on the study of the Liouville foliation of the phase space. As the first approach to topological analysis of this system we find the stratified critical set of the momentum map; this set is represented as the union of manifolds with induced almost Hamiltonian systems having less than three degrees of freedom. We obtain the equations of the bifurcation diagram in three-dimensional space. These equations have the form convenient for the classification of the bifurcation sets arising on 5-dimensional iso-energetic levels.

1. Introduction

The famous integrable case of S. Kowalevski of the motion of a heavy rigid body about a fixed point [1] has received several generalizations. Some of them suppose restrictions to submanifolds in the phase space (partial cases), others are far from mechanics involving potential functions on the configuration space $SO(3)$ with singularities. The most essential generalization having the clear mechanical sense was found by A. G. Reyman and M. A. Semenov-Tian-Shansky in the work [2]. The authors introduce the dynamical system on the dual space of the Lie algebra $e(p,q)$ of the Lie group defined as the semi-direct product of $SO(p)$ and q copies of $\mathbf{R}^p$. Such systems are known as the Euler equations on Lie (co)algebras [3]. The case $p=3$, $q=2$ corresponds to the Euler–Poisson equations of the motion of a gyrostat in two constant fields.

For a rigid body without gyrostatic momentum, the model of two constant fields was introduced by O. I. Bogoyavlensky [3]. The physical object can be

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either a heavy electrically charged rigid body rotating in gravitational and constant electric fields, or a heavy magnet rotating in gravitational and constant magnetic fields. The corresponding equations are Hamiltonian on the orbit of the coadjoint action on $e(3,2)^*$. The typical orbit is diffeomorphic to $TSO(3) \cong \mathbf{R}^3 \times SO(3)$. Therefore, the gyrostat in two constant fields is the Hamiltonian system with three degrees of freedom. Bogoyavlensky [3] suggested the conditions of the Kowalevski type and found the analogue of the Kowalevski integral K for the top in two constant fields. H. Yehia [4] generalized this integral for the Kowalevski gyrostat in two constant fields. Almost simultaneously with Yehia, I. V. Komarov [5] and L. N. Gavrilov [6] proved the Liouville integrability of the Kowalevski gyrostat in the gravity field. But for two constant fields the Kowalevski gyrostat was not considered integrable due to the fact that the existence of the second field destroys the axial symmetry of the potential and, consequently, the corresponding cyclic integral. Finally, Reyman and Semenov-Tian-Shansky [2] found the Lax representation with a spectral parameter for the family of Euler equations on $e(p,q)^*$. For $p=3$, $q=2$ this representation immediately gave rise to the new integral for the Kowalevski gyrostat in two constant fields. When one of the fields vanishes this integral turns into the square of the cyclic integral.

The Kowalevski gyrostat in two constant fields does not have any explicit symmetry groups and, therefore, is not reducible, in a standard way, to a family of systems with two degrees of freedom. Phase topology of such systems has not been studied yet. The theory of n-dimensional integrable systems started in [7], [8] is not illustrated by an application to any real irreducible physical or geometrical problem with $n > 2$.

In the paper [9], the authors give a detailed exposition of the results of [2] as well as a study of the algebraic geometry of the Lax pair for the generalized Kowalevski system. They announce the possibility of its integration by the finite-band techniques and fulfill such integration for the classical top. For two constant fields the integration of the Kowalevski top is not given up-to-date. The problem of the Kowalevski *gyrostat* motion in two constant fields is not studied at all. The technical difficulties here are extremely high. It is not likely that, in the general regular case, the analytical solutions can be obtained having the form useful for the qualitative topological analysis or the computer simulation. However, there is a good experience of studying the critical subsystems, i.e., the systems with $n < 3$ degrees of freedom induced on $2n$-dimensional invariant submanifolds in the phase space consisting of the momentum map singularities. For the Kowalevski top in two constant fields we have now the complete description of all such singularities [10], [11], [12], [13], [14], [15] and the classification of the bifurcation diagrams for the restriction of this map to 5-dimensional iso-energetic surfaces [16], [17]. This result is a necessary part of the study of Liouville foliation of the integrable system and shows the actual need in the generalization of the Liouville invariants theory [8] for the dimensions greater than two.

The present paper contains similar results for the Kowalevski gyrostat in two constant fields. The 6-dimensional phase space is stratified by the rank of the momentum map. We find the equations of invariant submanifolds forming the set of its critical points (critical manifolds of rank 0, 1, or 2). The induced systems are Hamiltonian (almost everywhere) with less than three degrees of freedom. We straightforwardly prove that the image of these critical manifolds (the bifurcation diagram) lies in the discriminant set of the algebraic curve of the Lax representation given in [9]. Moreover, the spectral parameter on the Lax curve is explicitly expressed in terms of the constant s of the additional partial integral arising on the critical submanifolds. It then follows that the equations of the surfaces containing the bifurcation diagram are written in the parametric form such that the parameters are the energy constant h and the constant s. Fixing the value of h we come to explicit equations of the bifurcation diagrams induced on iso-energetic levels. The problem of classification of these diagrams seems quite complicated due to the existence of several physical parameters. Nevertheless, it is certainly solvable with the help of contemporary computer programs of analytical calculations.

First we show that the number of physical parameters for the gyrostat in two constant fields can be reduced by a simple procedure, which may be called the orthogonalization of the fields. More precisely, for the problems of gyrostat motion there exists a group of diffeomorphisms of the phase spaces (mentioned above orbits of the coadjoint action) that is an equivalence group for the corresponding dynamical systems. It appears that each equivalence class contains a problem with an orthonormal pair of radius vectors of the centers of forces application and with an orthogonal pair of the intensity vectors. Such force field is characterized by only one essential parameter—the ratio of the modules of the intensity vectors. For a dynamically symmetric gyrostat having the centers of forces application in the equatorial plane, the orthogonalization procedure along with the appropriate choice of the measure units leave, in addition to the forces ratio, only two physical parameters of the body itself, namely, the ratio of the equatorial and axial inertia moments and the non-zero axial component of the gyrostatic momentum. In the generalized Kowalevski case the first of them equals 2. Thus, the whole problem has, in fact, two essential parameters. In particular, each of the critical four-dimensional submanifolds found below provides a two-parametric family of completely integrable Hamiltonian systems with two degrees of freedom.

2. Gyrostat equations and parametrical reduction

Consider a rigid body $\mathscr{B}$ rotating around a fixed point O. Choose a trihedral at O moving along with the body and refer to it all vector and tensor objects. Denote by $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ the canonical unit basis in $\mathbf{R}^3$; then the moving trihedral itself is represented as $O\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. Let $\boldsymbol{\omega}$ be the vector of the angular velocity of $\mathscr{B}$. Suppose that $\mathscr{B}$ is bearing an axially symmetric rigid rotor $\mathscr{B}'$ rotating freely around its symmetry axis fixed in $\mathscr{B}$. Such system of two bodies is the simplest model of a gyrostat. The notion of a gyrostat was introduced by N. E. Zhoukovsky [18] for a body having cavities totally filled with homogeneous fluid. Both models have the common feature usually taken as the definition of a gyrostat: the total angular momentum of such system is $\mathbf{M} = \mathbf{I}\boldsymbol{\omega} + \boldsymbol{\lambda}$, where the inertia tensor $\mathbf{I}$ and the vector $\boldsymbol{\lambda}$ (called the gyrostatic momentum) are constant with respect to the moving trihedral. Using the term "gyrostat" we always suppose $\boldsymbol{\lambda} \neq 0$. In the case $\boldsymbol{\lambda} = 0$ we use the terms "rigid body" or "top" instead. The top is usually supposed to have a dynamical symmetry axis.

Let $\mathbf{M}_F$ denote the moment of external forces with respect to O (the rotating moment). Constant field is a force field inducing the rotating moment of the form $\mathbf{r} \times \boldsymbol{\alpha}$ with constant vector $\mathbf{r}$ and with $\boldsymbol{\alpha}$ corresponding to some physical vector fixed in inertial space; $\mathbf{r}$ points from O to the center of application of the field, $\boldsymbol{\alpha}$ is the field intensity. Along with the notation for the direct products of groups and vector spaces, we use the cross symbol for the standard vector product in $\mathbf{R}^3$ and for the defined below unusual binary operation involving 3×2-matrices and based on the vector product. It is needed in this section only and should not cause any ambiguity.

For *two constant fields* the rotating moment is $\mathbf{M}_F = \mathbf{r}_1 \times \boldsymbol{\alpha} + \mathbf{r}_2 \times \boldsymbol{\beta}$ with $\mathbf{r}_1$, $\mathbf{r}_2$ constant in the body and $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ corresponding to the vectors fixed in inertial space. Obviously, $\mathbf{M}_F$ can be represented as the moment of *one constant field* if either $\mathbf{r}_1 \times \mathbf{r}_2 = 0$ or $\boldsymbol{\alpha} \times \boldsymbol{\beta} = 0$. Suppose that

$$\mathbf{r}_1 \times \mathbf{r}_2 \neq 0, \qquad \boldsymbol{\alpha} \times \boldsymbol{\beta} \neq 0. \tag{2.1}$$

Two constant fields satisfying (2.1) are said to be *independent*.

The equations defining the respective evolution of $\mathbf{M}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ in two constant fields are

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \boldsymbol{\omega} + \mathbf{r}_1 \times \boldsymbol{\alpha} + \mathbf{r}_2 \times \boldsymbol{\beta}, \qquad \frac{d\boldsymbol{\alpha}}{dt} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \qquad \frac{d\boldsymbol{\beta}}{dt} = \boldsymbol{\beta} \times \boldsymbol{\omega}. \tag{2.2}$$

These equations are Euler equations in the space $\mathbf{R}^9(\mathbf{M}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ considered as the dual space to $e(3,2)$. The Lie–Poisson bracket applied to the coordinate functions yields

$$\begin{gathered}\{M_i, M_j\} = \varepsilon_{ijk} M_k, \qquad \{M_i, \alpha_j\} = \varepsilon_{ijk}\alpha_k, \qquad \{M_i, \beta_j\} = \varepsilon_{ijk}\beta_k, \\ \{\alpha_i, \alpha_j\} = 0, \qquad \{\alpha_i, \beta_j\} = 0, \qquad \{\beta_i, \beta_j\} = 0. \end{gathered} \tag{2.3}$$

Such bracket is non-degenerate on each orbit of the coadjoint action. The orbits are defined by the geometric integrals (common level of the Casimir functions)

$$\boldsymbol{\alpha}\cdot\boldsymbol{\alpha} = c_{11}, \qquad \boldsymbol{\beta}\cdot\boldsymbol{\beta} = c_{22}, \qquad \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = c_{12}.$$

The dot stands for the standard scalar product in $\mathbf{R}^3$. If $c_{11} > 0$, $c_{22} > 0$, $c_{12}^2 < c_{11}c_{22}$, then the orbit in $\mathbf{R}^9$ is diffeomorphic to $\mathbf{R}^3 \times SO(3)$, and the induced Hamiltonian system has three degrees of freedom (see [3], [9] for the details). From physical point of view the constants c_{11}, c_{22}, c_{12} characterize the force fields intensities. Along with the coordinates of $\mathbf{r}_1$, $\mathbf{r}_2$ in the moving frame, we have 9 parameters of the interaction of the body with the external forces. We now show how to reduce this number.

Introduce some notation. Let $L(n,k)$ be the space of $n \times k$-matrices. Put $L(k) = L(k,k)$. Identify $\mathbf{R}^6 = \mathbf{R}^3 \times \mathbf{R}^3$ with $L(3,2)$ by the isomorphism j that joins two columns

$$A = j(\mathbf{a}_1, \mathbf{a}_2) = \|\mathbf{a}_1\ \mathbf{a}_2\| \in L(3,2), \qquad \mathbf{a}_1, \mathbf{a}_2 \in \mathbf{R}^3.$$

For the inverse map, write

$$j^{-1}(A) = (\mathbf{c}_1(A), \mathbf{c}_2(A)) \in \mathbf{R}^3 \times \mathbf{R}^3, \qquad A \in L(3,2).$$

If $A, B \in L(3,2)$, $\mathbf{a} \in \mathbf{R}^3$, by definition, put

$$\begin{gathered} A \times B = \sum_{i=1}^{2} \mathbf{c}_i(A) \times \mathbf{c}_i(B) \in \mathbf{R}^3; \\ \mathbf{a} \times A = j(\mathbf{a} \times \mathbf{c}_1(A), \mathbf{a} \times \mathbf{c}_2(A)) \in L(3,2). \end{gathered} \tag{2.4}$$

Lemma 1. *Let* $\Lambda \in SO(3)$, $D \in GL(2,\mathbf{R})$, $\mathbf{a} \in \mathbf{R}^3$, $A, B \in L(3,2)$. *Then*

$$\Lambda(A \times B) = (\Lambda A) \times (\Lambda B); \qquad (AD^{-1}) \times (BD^T) = A \times B;$$

$$\Lambda(\mathbf{a} \times A) = (\Lambda \mathbf{a}) \times (\Lambda A); \qquad \mathbf{a} \times (AD) = (\mathbf{a} \times A)D.$$

The proof is by direct calculation.

In notation (2.4) we write system (2.2) in the form

$$\mathbf{I}\frac{d\boldsymbol{\omega}}{dt} = (\mathbf{I}\boldsymbol{\omega} + \boldsymbol{\lambda}) \times \boldsymbol{\omega} + A \times U, \qquad \frac{dU}{dt} = -\boldsymbol{\omega} \times U. \tag{2.5}$$

Here $A = j(\mathbf{r}_1, \mathbf{r}_2)$ is a constant matrix, $U = j(\boldsymbol{\alpha}, \boldsymbol{\beta})$. The phase space of (2.5) is $\{(\boldsymbol{\omega}, U)\} = \mathbf{R}^3 \times L(3,2)$.

In fact, U in (2.5) is restricted by the geometric integrals; i.e., for some constant symmetric matrix $C \in L(2)$

$$U^T U = C. \tag{2.6}$$

Let $\mathcal{O}$ be the set defined by (2.6) in $L(3,2)$. In order to emphasize the dependence on C, we write $\mathcal{O} = \mathcal{O}(C)$.

Let $\mathfrak{P} = (\mathbf{I}, \boldsymbol{\lambda}, A, C)$ denote the complete set of constant parameters of the problem. Denote by $X_{\mathfrak{P}}$ the vector field on $\mathbf{R}^3 \times \mathcal{O}(C)$ induced by (2.5). Given the set $\mathfrak{P}$, the problem of motion of the gyrostat in two constant fields described by the dynamical system $X_{\mathfrak{P}}$ will be called, for short, the *DG-problem*.

Associate to $\Lambda \in SO(3)$, $D \in GL(2,\mathbf{R})$ the linear automorphisms $\Psi(\Lambda, D)$ and $\psi(\Lambda, D)$ of $\mathbf{R}^3 \times L(3,2)$ and $L(3) \times \mathbf{R}^3 \times L(3,2) \times L(2)$

$$\begin{aligned} &\Psi(\Lambda, D)(\boldsymbol{\omega}, U) = (\Lambda\boldsymbol{\omega}, \Lambda U D^T), \\ &\psi(\Lambda, D)(\mathbf{I}, \boldsymbol{\lambda}, A, C) = (\Lambda \mathbf{I} \Lambda^T, \Lambda\boldsymbol{\lambda}, \Lambda A D^{-1}, D C D^T). \end{aligned} \tag{2.7}$$

Equations (2.6) and (2.7) imply $\Psi(\Lambda, D)(\mathbf{R}^3 \times \mathcal{O}(C)) = \mathbf{R}^3 \times \mathcal{O}(DCD^T)$. Using Lemma 1 we obtain the following statement.

Lemma 2. *For each* $(\Lambda, D) \in SO(3) \times GL(2,\mathbf{R})$,

$$\Psi(\Lambda, D)_*(X_{\mathfrak{P}}(v)) = X_{\psi(\Lambda, D)(\mathfrak{P})}(\Psi(\Lambda, D)(v)), \qquad v \in \mathbf{R}^3 \times \mathcal{O}(C).$$

Thus, any two DG-problems determined by the sets of parameters $\mathfrak{P}$ and $\psi(\Lambda, D)(\mathfrak{P})$ are completely equivalent.

Let us call a DG-problem *canonical* if the centers of application of forces lie on the first two axes of the moving trihedral at unit distance from the fixed point and the intensities of the forces are orthogonal to each other.

Theorem 1. *For each DG-problem with independent forces there exists an equivalent canonical problem. Moreover, in both equivalent problems the centers of application of forces belong to the same plane in the body containing the fixed point.*

Proof. Let the DG-problem with the set of parameters $\mathfrak{P} = (\mathbf{I}, \boldsymbol{\lambda}, A, C)$ satisfy (2.1). This means that the symmetric matrices $A_* = (A^T A)^{-1}$ and C are positively definite. According to the well-known fact from linear algebra, A_* and C can be reduced, respectively, to the identity matrix and to a diagonal matrix via the same conjugation operator

$$DA_*D^T = E, \qquad DCD^T = \mathrm{diag}\{a^2, b^2\}, \qquad D \in GL(2,\mathbf{R}),\ a, b \in \mathbf{R}_+.$$

Then $\mathbf{c}_1(AD^{-1})$ and $\mathbf{c}_2(AD^{-1})$ form an orthonormal pair in $\mathbf{R}^3$. There exists

$\Lambda \in SO(3)$ such that $\Lambda \mathbf{c}_i(AD^{-1}) = \mathbf{e}_i$ $(i = 1, 2)$. The first statement is obtained by applying Lemma 2 with the previously chosen Λ, D to the initial vector field $X_{\mathfrak{P}}$.

To finish the proof, notice that the transformation $A \mapsto AD^{-1}$ preserves the span of $\mathbf{c}_1(A)$, $\mathbf{c}_2(A)$. The matrix Λ in (2.7) stands for the change of the moving trihedral. Therefore, if $\mathbf{a} \in \mathbf{R}^3$ represents some physical vector in the initial problem, then $\Lambda\mathbf{a}$ is the same vector with respect to the body in the equivalent problem.

Remark 1. *The fact that any DG-problem can be reduced to the problem with one of the pairs* $\mathbf{r}_1$, $\mathbf{r}_2$ *or* $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ *orthonormal is obvious. Simultaneous orthogonalization of both pairs was first established in* [12] *for a rigid body and crucially simplifies all calculations.*

It follows from Theorem 1 that, without loss of generality, for independent forces we may suppose

$$\mathbf{r}_1 = \mathbf{e}_1, \qquad \mathbf{r}_2 = \mathbf{e}_2, \tag{2.8}$$

$$\boldsymbol{\alpha} \cdot \boldsymbol{\alpha} = a^2, \qquad \boldsymbol{\beta} \cdot \boldsymbol{\beta} = b^2, \qquad \boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0. \tag{2.9}$$

Change, if necessary, the order of $\mathbf{e}_1$, $\mathbf{e}_2$ (with simultaneous change of the direction of $\mathbf{e}_3$) to obtain $a \geqslant b > 0$.

Consider a dynamically symmetric top in two constant fields with the centers of application of forces in the equatorial plane of its inertia ellipsoid. Choose a moving trihedral such that $O\mathbf{e}_3$ is the symmetry axis. Then the inertia tensor $\mathbf{I}$ becomes diagonal. Let $a = b$. For any $\Theta \in SO(2)$ denote by $\hat{\Theta} \in SO(3)$ the corresponding rotation of $\mathbf{R}^3$ about $O\mathbf{e}_3$. Take in (2.7) $\Lambda = \hat{\Theta}$, $D = \Theta$. Under the conditions (2.8), (2.9), $\psi = \mathrm{Id}$ and Ψ becomes the symmetry group. The system (2.5) has the cyclic integral $\mathbf{I}\boldsymbol{\omega} \cdot (a^2\mathbf{e}_3 - \boldsymbol{\alpha} \times \boldsymbol{\beta})$. Therefore it is possible to reduce such a DG-problem to a family of systems with two degrees of freedom. For the analogue of the Kowalevski case this system becomes integrable [4].

Let us call a DG-problem *irreducible* if, in its canonical representation,

$$a > b > 0. \tag{2.10}$$

The following statements are needed in the future; they also reveal some features of a wide class of DG-problems.

Lemma 3. *In an irreducible DG-problem, the body has exactly four equilibria.*

Proof. The set of singular points of (2.5) is defined by $\boldsymbol{\omega} = 0$, $A \times U = 0$. For the equivalent canonical problem with (2.8) we have $\mathbf{e}_1 \times \boldsymbol{\alpha} + \mathbf{e}_2 \times \boldsymbol{\beta} = 0$.

Then the four vectors $\mathbf{e}_1$, $\boldsymbol{\alpha}$, $\mathbf{e}_2$, $\boldsymbol{\beta}$ are parallel to the same plane and $|\mathbf{e}_1 \times \boldsymbol{\alpha}| = |\mathbf{e}_2 \times \boldsymbol{\beta}|$. Given (2.10), this equality yields

$$\boldsymbol{\alpha} = \pm a\mathbf{e}_1, \qquad \boldsymbol{\beta} = \pm b\mathbf{e}_2. \tag{2.11}$$

Thus, in the canonical irreducible system, an equilibrium takes place only if the radius vectors of the centers of application are parallel to the corresponding fields intensities.

Note that the existence of the gyrostatic momentum does not change the equilibria. Therefore, the result here is the same as in the case of a rigid body in two constant fields [16].

LEMMA 4. *Let an irreducible DG-problem in its canonical form have the diagonal inertia tensor* $\mathbf{I} = \mathrm{diag}\{I_1, I_2, I_3\}$ *and* $\boldsymbol{\lambda} = 0$. *Then the body has the following families of pendulum type motions*

$$P_1 : \begin{cases} \boldsymbol{\omega} = \dot{\varphi}\mathbf{e}_1, \quad \boldsymbol{\alpha} \equiv \pm a\mathbf{e}_1, \quad \boldsymbol{\beta} = b(\mathbf{e}_2 \cos\varphi - \mathbf{e}_3 \sin\varphi), \\ I_1\ddot{\varphi} = -b\sin\varphi; \end{cases} \tag{2.12}$$

$$P_2 : \begin{cases} \boldsymbol{\omega} = \dot{\varphi}\mathbf{e}_2, \quad \boldsymbol{\beta} \equiv \pm b\mathbf{e}_2, \quad \boldsymbol{\alpha} = a(\mathbf{e}_1 \cos\varphi + \mathbf{e}_3 \sin\varphi), \\ I_2\ddot{\varphi} = -a\sin\varphi; \end{cases} \tag{2.13}$$

$$P_3 : \begin{cases} \boldsymbol{\omega} = \dot{\varphi}\mathbf{e}_3, \quad \boldsymbol{\alpha} \times \boldsymbol{\beta} \equiv \pm ab\mathbf{e}_3, \\ \boldsymbol{\alpha} = a(\mathbf{e}_1 \cos\varphi - \mathbf{e}_2 \sin\varphi), \quad \boldsymbol{\beta} = \pm b(\mathbf{e}_1 \sin\varphi + \mathbf{e}_2 \cos\varphi), \\ I_3\ddot{\varphi} = -(a \pm b)\sin\varphi. \end{cases} \tag{2.14}$$

If $\boldsymbol{\lambda} \neq 0$ *but* $\boldsymbol{\lambda} = \lambda\mathbf{e}_i$ *for some* $i = 1, 2, 3$, *then the only family remained is* P_i *with the corresponding index.*

The proof is obvious. The families (2.12)–(2.13) were first found in [12] (the case $\boldsymbol{\lambda} = 0$). The motions (2.14) for any axially symmetric gyrostat in two constant fields with the centers of forces application in the equatorial plane were found by Yehia [19]. Note that in the case (2.10) these families are the only motions with a fixed direction of the angular velocity. In particular, the body in two independent constant fields does not have any uniform rotations.

3. Critical set of the Kowalevski gyrostat

Suppose that the irreducible DG-problem has the diagonal inertia tensor with the principal moments of inertia satisfying the ratio 2 : 2 : 1, the gyrostatic momentum is directed along the dynamical symmetry axis $\boldsymbol{\lambda} = \lambda\mathbf{e}_3$ and the centers of the fields application lie in the equatorial plane $\mathbf{r}_1 \perp \mathbf{e}_3$, $\mathbf{r}_2 \perp \mathbf{e}_3$. These are the conditions of the integrable case [2] of the Kowalevski gyrostat in two constant fields. The orthogonalization procedure in this case does not change the $\mathbf{e}_3$-axis and we obtain (2.8), (2.9). Choosing the appropriate units of measurement, represent system (2.5) in the form

$$\begin{gathered}
2\dot{\omega}_1 = \omega_2(\omega_3 - \lambda) + \beta_3, \quad 2\dot{\omega}_2 = -\omega_1(\omega_3 - \lambda) - \alpha_3, \quad \dot{\omega}_3 = \alpha_2 - \beta_1, \\
\begin{aligned}
\dot{\alpha}_1 &= \alpha_2\omega_3 - \alpha_3\omega_2, & \dot{\beta}_1 &= \beta_2\omega_3 - \beta_3\omega_2, \\
\dot{\alpha}_2 &= \alpha_3\omega_1 - \alpha_1\omega_3, & \dot{\beta}_2 &= \beta_3\omega_1 - \beta_1\omega_3, \\
\dot{\alpha}_3 &= \alpha_1\omega_2 - \alpha_2\omega_1, & \dot{\beta}_3 &= \beta_1\omega_2 - \beta_2\omega_1.
\end{aligned}
\end{gathered} \tag{3.1}$$

The phase space is $P^6 = \mathbf{R}^3 \times \mathcal{O}$, where $\mathcal{O} \subset \mathbf{R}^3 \times \mathbf{R}^3$ is defined by (2.9); $\mathcal{O}$ is diffeomorphic to $SO(3)$.

The complete set of the first integrals in involution on P^6 includes the energy integral H, generalized Kowalevski integral K [3], [4], and the integral G found in [2]. After the parametrical reduction, these integrals are

$$\begin{aligned}
H &= \omega_1^2 + \omega_2^2 + \frac{1}{2}\omega_3^2 - \alpha_1 - \beta_2, \\
K &= (\omega_1^2 - \omega_2^2 + \alpha_1 - \beta_2)^2 + (2\omega_1\omega_2 + \alpha_2 + \beta_1)^2 \\
&\quad + 2\lambda[(\omega_3 - \lambda)(\omega_1^2 + \omega_2^2) + 2\omega_1\alpha_3 + 2\omega_2\beta_3], \\
G &= \frac{1}{4}(M_\alpha^2 + M_\beta^2) + \frac{1}{2}(\omega_3 - \lambda)M_\gamma - b^2\alpha_1 - a^2\beta_2.
\end{aligned}$$

Here $M_\alpha = (\mathbf{I}\boldsymbol{\omega} + \boldsymbol{\lambda}) \cdot \boldsymbol{\alpha}$, $M_\beta = (\mathbf{I}\boldsymbol{\omega} + \boldsymbol{\lambda}) \cdot \boldsymbol{\beta}$, $M_\gamma = (\mathbf{I}\boldsymbol{\omega} + \boldsymbol{\lambda}) \cdot (\boldsymbol{\alpha} \times \boldsymbol{\beta})$.

Introduce the momentum map

$$J : P^6 \to \mathbf{R}^3, \qquad J(\zeta) = (G(\zeta), K(\zeta), H(\zeta)) \tag{3.2}$$

and denote by $\mathfrak{C} \subset P^6$ the set of critical points of J. By definition, the bifurcation diagram of J is the set $\Sigma \subset \mathbf{R}^3$ over which J fails to be locally trivial; Σ defines the cases when the integral manifolds

$$J_c = J^{-1}(c), \qquad c = (g, k, h) \in \mathbf{R}^3$$

change its topological (and smooth) type. To find $\mathfrak{C}$ and Σ is the necessary part of the global topological analysis of the problem.

It follows from the Liouville–Arnold theorem that for $c \notin \Sigma$ the manifold J_c, if not empty, is the union of three-dimensional tori. The considered Hamiltonian system on P^6 is non-degenerate at least for small enough values of b. Therefore the trajectories on such tori are almost everywhere quasi-periodic with three independent frequencies. The critical set $\mathfrak{C}$ is preserved by the phase flow and consists of the trajectories, which typically have less than three frequencies. We call the trajectories in $\mathfrak{C}$ *the critical motions*. The set $\mathfrak{C}$ is stratified by the rank of J. Let $\mathfrak{C}_i = \{\zeta \in \mathfrak{C} : \operatorname{rank} J(\zeta) = i\}$ $(i = 0, 1, 2)$. It is natural to expect that $\mathfrak{C}_i$ consists of the Liouville tori of dimension i and the image $J(\mathfrak{C}_i)$, as a subset of Σ, is a smooth surface Σ_i of dimension i. More precisely, for each $i \leqslant 2$ we have to take

$$\Sigma_i = J(\mathfrak{C}_i) \Big\backslash \bigcup_{j=0}^{i-1} J(\mathfrak{C}_j).$$

Then, as a whole, we may consider Σ as a two-dimensional cell complex, Σ_i as its i-skeleton. For $i = 1, 2$ we will have $\partial\Sigma_i \subset \Sigma_{i-1}$.

For $c \in \Sigma_2$ the set $J_c \cap \mathfrak{C}$ consists of two-dimensional tori. Take the union of such tori over the values c from some open subset in Σ_2. The dynamical system restricted to this union will be Hamiltonian with two degrees of freedom (except, maybe, a set of positive codimension on which the 2-form induced by the symplectic structure degenerates). This system inherits the property of complete integrability. Thus the critical motions from $\mathfrak{C}_2$ are basically organized in several integrable subsystems with two degrees of freedom (*the critical subsystems of rank* 2). Similarly, the critical motions from $\mathfrak{C}_1$ may form two-dimensional symplectic submanifolds bearing the induced integrable systems with one degree of freedom (*the critical subsystems of rank* 1). At the same time parts of $\mathfrak{C}_1$ may appear as the critical motions with respect to the critical subsystems of rank 2. And finally, the bifurcations inside $\mathfrak{C}_1$ correspond to the set $\mathfrak{C}_0$, which in non-degenerate case consists of isolated equilibria (*the critical subsystems of rank* 0). Such stratification is typical (see, for example, [8]) but is destroyed by symmetries existing in reducible problems of the rigid body dynamics. In the irreducible case of the top ($\lambda = 0$, $0 < b/a < 1$) the critical subsystems and the bifurcation diagram of the map (3.2) are known. The critical set is formed by four non-degenerate equilibria (see Lemma 3), one critical subsystem of rank 1 and three critical subsystems of rank 2. The complete presentation of these results and the list of publications are given in [13], [17]. Except for the partial integrable case of Bogoyavlensky [3] (case $K = 0$), all of the critical subsystems have been either explicitly integrated or reduced to separated systems of equations [14], [15], [20] leading to hyper-elliptic quadratures.

Introduce the change of variables [11] based on the change given by S. Kowalevski and on the Lax representation [2] ($\mathrm{i}^2 = -1$)

$$\begin{gathered}
x_1 = (\alpha_1 - \beta_2) + \mathrm{i}(\alpha_2 + \beta_1), \qquad x_2 = (\alpha_1 - \beta_2) - \mathrm{i}(\alpha_2 + \beta_1), \\
y_1 = (\alpha_1 + \beta_2) + \mathrm{i}(\alpha_2 - \beta_1), \qquad y_2 = (\alpha_1 + \beta_2) - \mathrm{i}(\alpha_2 - \beta_1), \\
z_1 = \alpha_3 + \mathrm{i}\beta_3, \qquad z_2 = \alpha_3 - \mathrm{i}\beta_3, \\
w_1 = \omega_1 + \mathrm{i}\omega_2, \qquad w_2 = \omega_1 - \mathrm{i}\omega_2, \qquad w_3 = \omega_3.
\end{gathered} \tag{3.3}$$

Then system (3.1) yields

$$
\begin{gathered}
2w_1' = -w_1(w_3 - \lambda) - z_1, \quad 2w_2' = w_2(w_3 - \lambda) + z_2, \quad 2w_3' = y_2 - y_1, \\
\begin{aligned}
x_1' &= -x_1 w_3 + z_1 w_1, & x_2' &= x_2 w_3 - z_2 w_2, \\
y_1' &= -y_1 w_3 + z_2 w_1, & y_2' &= y_2 w_3 - z_1 w_2, \\
2z_1' &= x_1 w_2 - y_2 w_1, & 2z_2' &= -x_2 w_1 + y_1 w_2.
\end{aligned}
\end{gathered}
\tag{3.4}
$$

Here prime stands for $d/d(\mathrm{i}t)$.

Consider (3.3) as the map $\mathbf{R}^9 \to \mathbf{C}^9$ and denote its image by V^9. Equations (2.9) of the phase space P^6 in V^9 take the form

$$
z_1^2 + x_1 y_2 = r^2, \qquad z_2^2 + x_2 y_1 = r^2, \qquad x_1 x_2 + y_1 y_2 + 2z_1 z_2 = 2p^2. \tag{3.5}
$$

Here we introduce the positive constants

$$
p = \sqrt{a^2 + b^2}, \qquad r = \sqrt{a^2 - b^2}.
$$

The first integrals in new coordinates are

$$
\begin{aligned}
H &= w_1 w_2 + \frac{1}{2} w_3^2 - \frac{1}{2}(y_1 + y_2), \\
K &= (w_1^2 + x_1)(w_2^2 + x_2) + 2\lambda(w_1 w_2 w_3 + z_2 w_1 + z_1 w_2) - 2\lambda^2 w_1 w_2, \\
G &= \frac{1}{4}(p^2 - x_1 x_2) w_3^2 + \frac{1}{2}(x_2 z_1 w_1 + x_1 z_2 w_2) w_3 \\
&\quad + \frac{1}{4}(x_2 w_1 + y_1 w_2)(y_2 w_1 + x_1 w_2) - \frac{1}{4} p^2 (y_1 + y_2) \\
&\quad + \frac{1}{4} r^2 (x_1 + x_2) + \frac{1}{2}\lambda(z_1 z_2 w_3 + y_2 z_2 w_1 + y_1 z_1 w_2) + \frac{1}{4}\lambda^2 (p^2 - y_1 y_2).
\end{aligned}
\tag{3.6}
$$

Let f be an arbitrary function on V^9. For brevity, the term "critical point of f" will always mean a critical point of the restriction of f to P^6. Similarly, df means the restriction of the differential of f to the set of vectors tangent to P^6. While calculating critical points of various functions, it is convenient to avoid introducing Lagrange's multipliers for the restrictions (3.5).

Lemma 5. *Critical points of a function f on V^9, in the above sense, are defined by the system of equations*

$$
X_i f = 0 \qquad (i = 1, \dots 6), \tag{3.7}
$$

where

$$
\begin{aligned}
&X_1 = \frac{\partial}{\partial w_1}, \qquad X_2 = \frac{\partial}{\partial w_2}, \qquad X_3 = \frac{\partial}{\partial w_3},\\
&X_4 = z_2 \frac{\partial}{\partial x_2} + z_1 \frac{\partial}{\partial y_2} - \frac{1}{2} x_1 \frac{\partial}{\partial z_1} - \frac{1}{2} y_1 \frac{\partial}{\partial z_2},\\
&X_5 = z_1 \frac{\partial}{\partial x_1} + z_2 \frac{\partial}{\partial y_1} - \frac{1}{2} y_2 \frac{\partial}{\partial z_1} - \frac{1}{2} x_2 \frac{\partial}{\partial z_2},\\
&X_6 = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial y_2}.
\end{aligned}
$$

Indeed, six vector fields X_i are tangent to P^6 and linearly independent at any point of P^6.

The following two propositions define the strata $\mathfrak{C}_0$ and $\mathfrak{C}_1$ of the critical set.

Proposition 1. *The set $\mathfrak{C}_0$ consists exactly of the four equilibria existing in this problem.*

Proof. The condition of zero rank of the momentum map at a point $\zeta \in P^6$ supposes, in particular, that $dH = 0$. Then ζ is the point of equilibrium and it follows from Lemma 3 that ζ is one of the points (2.11). Using the complex variables we have

$$
\begin{gathered}
w_1 = w_2 = w_3 = 0, \qquad z_1 = z_2 = 0,\\
x_1 = x_2 = \varepsilon_1 a - \varepsilon_2 b, \qquad y_1 = y_2 = \varepsilon_1 a + \varepsilon_2 b \qquad (\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1).
\end{gathered}
\tag{3.8}
$$

Use equations (3.7) with $f = K$ and $f = G$ to obtain that $dK(\zeta) = 0$ and $dG(\zeta) = 0$. Hence, $\operatorname{rank} J(\zeta) = 0$.

Note that in classical problems of the rigid body dynamics with an axially symmetric force field, the rank of the momentum map is everywhere not less than 1 due to the regularity of the cyclic integral. In our case, all equilibria are *non-degenerate* (in the Morse sense) critical points of the Hamilton function (see [16]). This is the reason why these points are critical for any first integral of the system. It is easily shown that the Morse indices of the potential energy function for these four equilibria equal exactly 0, 1, 2, 3. Therefore, only one of them (with minimal energy value) is stable.

It is essential that in the sequel $\lambda \neq 0$.

Proposition 2. *The set $\mathfrak{C}_1$ is completely defined by the condition*

$$\operatorname{rank}\{dK, dH\} = 1$$

and consists of the points of the following trajectories:

1) *pendulum motions* (2.14) *except for the equilibria;*
2) *motions defined by the equations*

$$w_1 = q(w)\sqrt{w}, \qquad w_2 = \frac{\sqrt{w}}{q(w)}, \qquad w_3 = \frac{\lambda}{\sigma} w \neq 0, \tag{3.9}$$

$$\begin{aligned}
x_1 &= \frac{1}{\sigma u}[r^2\lambda^2\sigma^2 - (\lambda^2+\sigma)uq^2(w)w], \\
x_2 &= \frac{1}{\sigma u}\left[r^2\lambda^2\sigma^2 - (\lambda^2+\sigma)u\frac{w}{q^2(w)}\right], \\
y_1 &= \sigma\left(1 + \frac{\sigma}{\lambda^2} - \frac{r^4\lambda^2\sigma}{u^2}\right) + \frac{r^2\lambda^2}{u}q^2(w)w, \\
y_2 &= \sigma\left(1 + \frac{\sigma}{\lambda^2} - \frac{r^4\lambda^2\sigma}{u^2}\right) + \frac{r^2\lambda^2}{u}\frac{w}{q^2(w)}, \\
z_1 &= -\frac{r^2\lambda\sigma}{u}\frac{\sqrt{w}}{q(w)} + \frac{\lambda^2+\sigma}{\lambda}q(w)\sqrt{w}, \\
z_2 &= -\frac{r^2\lambda\sigma}{u}q(w)\sqrt{w} + \frac{\lambda^2+\sigma}{\lambda}\frac{\sqrt{w}}{q(w)}.
\end{aligned} \tag{3.10}$$

Here $q(w)$ *is the root of the equation* $q^4 - 2Q(w)q^2 + 1 = 0$, *where*

$$Q(w) = \frac{\sigma u^3 + (\lambda^2+\sigma)[\lambda^2 w^2 + \sigma^2(2w-\sigma)]u^2 + r^4\lambda^4\sigma^4}{2r^2\lambda^2\sigma^2(\lambda^2+\sigma)uw}; \tag{3.11}$$

σ, u *are constants satisfying the equation*

$$\begin{aligned}
&\lambda^2(\lambda^2+\sigma)^2u^5 + (\lambda^2+\sigma)[2p^2\lambda^4 - (\lambda^2+\sigma)^3\sigma]\sigma u^4 \\
&\quad + r^4\lambda^6\sigma^2u^3 + 2r^4\lambda^4\sigma^4(\lambda^2+\sigma)^2u^2 - r^8\lambda^8\sigma^6 = 0.
\end{aligned} \tag{3.12}$$

The evolution $w(t)$ *is defined by the equation*

$$\left(\frac{dw}{dt}\right)^2 = -\frac{\lambda^2}{4\sigma^2}P_+(w)P_-(w), \tag{3.13}$$

where

$$P_\pm(w) = w^2 + 2\sigma^2\frac{u \pm r^2\lambda^2}{\lambda^2 u}w + \frac{\sigma[u^3 - (\lambda^2+\sigma)\sigma^2u^2 + r^4\lambda^4\sigma^3]}{(\lambda^2+\sigma)\lambda^2u^2}. \tag{3.14}$$

Proof. It follows from above that $dH \neq 0$ at the points of $\mathfrak{C}_1$. Then to investigate the dependence of the functions K and H it is sufficient to introduce the function with one Lagrange's multiplier σ. Write equations (3.7) for the function $f = K - 2\sigma H$,

$$(w_1^2 + x_1)w_2 + \lambda[z_1 + w_1(w_3 - \lambda)] - \sigma w_1 = 0,$$
$$(w_2^2 + x_2)w_1 + \lambda[z_2 + w_2(w_3 - \lambda)] - \sigma w_2 = 0, \tag{3.15}$$

$$\lambda w_1 w_2 - \sigma w_3 = 0, \tag{3.16}$$

$$(w_1^2 + x_1)z_2 - \lambda(w_2 x_1 + w_1 y_1) + \sigma z_1 = 0,$$
$$(w_2^2 + x_2)z_1 - \lambda(w_1 x_2 + w_2 y_2) + \sigma z_2 = 0, \tag{3.17}$$

$$x_1 w_2^2 - x_2 w_1^2 + \sigma(y_1 - y_2) = 0. \tag{3.18}$$

First consider the critical points of the function K. For this purpose we must put $\sigma = 0$. From (3.16) we get $w_1 = w_2 = 0$. Then (3.15) imply $z_1 = z_2 = 0$. Equations (3.17) and (3.18) become identities. The same values satisfy (3.7) with $f = 4G + (x_1x_2 - y_1y_2)H$. Hence, $dK = 0$ and $4\,dG + (x_1x_2 - y_1y_2)dH = 0$. Since $dH \neq 0$, it means that $\operatorname{rank} J = 1$. The initial variables on the corresponding trajectories are $\omega_1 = \omega_2 \equiv 0$, $\alpha_3 = \beta_3 \equiv 0$. Substitute these values in (3.1) to obtain solutions (2.14).

Let $\sigma \neq 0$. In this case $w_1 w_2 \neq 0$. Indeed, assuming the converse, from (3.15), (3.16), (3.5) we come to the points (3.8) of the set $\mathfrak{C}_0$. Therefore, satisfying (3.16) we can introduce new variables $w \neq 0$, q as shown in (3.9). Four equations (3.15), (3.17) form the linear system in y_1, y_2, z_1, z_2, from which we obtain these variables as the functions of x_1, x_2, w, q identically satisfying (3.18). Denote

$$u = (w - \sigma)^2(\lambda^2 + \sigma) - \sigma x_1 x_2. \tag{3.19}$$

Then the first two equations (3.5) are easily solved for x_1, x_2 as the functions of w, q, u. As a result we obtain the expressions (3.10). Let

$$Q = \frac{1}{2}\left(q^2 + \frac{1}{q^2}\right).$$

Then the substitution of x_1, x_2 from (3.10) back to (3.19) gives (3.11). The last unused equation (3.5) provides the relation (3.12) between u and the constants λ, σ. It shows that u defined as (3.19) appears to be a constant.

Thus, all phase variables are expressed via one variable w, for which from (3.4) we find the differential equation (3.13). Note that due to (3.14) the solutions are elliptic functions of time.

The expressions $Q(w)$, $P_\pm(w)$ formally have singularities in the case

$$\sigma = -\lambda^2. \tag{3.20}$$

Let $\sigma = -\lambda^2 + \varepsilon$, $\varepsilon \to 0$. The continuous solution $u(\varepsilon)$ of (3.12) is

$$u = r^{4/3}\lambda^{10/3} + \frac{2}{3}r^{-4/3}\lambda^{4/3}(p^2\lambda^{4/3} - 2r^{8/3})\varepsilon + O(\varepsilon^2)$$

and from (3.11), (3.14) we obtain the regular limit values

$$\lim_{\sigma\to-\lambda^2} Q(w) = \frac{r^{4/3}\lambda^{1/3}(w+\lambda^2)^2 + r^{8/3}\lambda^{5/3} - 2p^2\lambda^3}{r^2\lambda w},$$

$$\lim_{\sigma\to-\lambda^2} P_\pm(w) = w^2 + 2\lambda^{2/3}(\lambda^{4/3} \pm r^{2/3})w + \lambda^{4/3}(\lambda^{8/3} + r^{4/3}).$$

To finish the proof, we need to show that at the points of the trajectories found we really have $\operatorname{rank} J = 1$, i.e., the linear dependence of dK and dH implies the linear dependence of dG and dH. Indeed, equations (3.7) with

$$f = 2G - \left(p^2 + \frac{\lambda^2+\sigma}{\lambda^2\sigma}u\right)H$$

are satisfied both by (2.14) and by (3.9), (3.10). Therefore, $\operatorname{rank}\{dG, dH\} = 1$ and, consequently, $\operatorname{rank}\{dK, dG, dH\} = 1$. This completes the proof.

Remark 2. *The motions described in Proposition* 2 *are periodic except for the cases when they become double asymptotic to the existing equilibria. For the second family it happens when $P_\pm(w)$ has a multiple root which can be only zero.*

The next lemma is needed for the future and follows immediately from the properties of analytical functions.

Lemma 6. *Let M be an analytical manifold, X an analytical vector field on M, and f, g, h analytical functions on M. Suppose $x(t)$ is a trajectory of X.*

(i) *If $f(x(t))g(x(t)) \equiv 0$, then either $f(x(t)) \equiv 0$ or $g(x(t)) \equiv 0$.*

(ii) *If the equation*

$$g(x)h(x) - f(x) = 0$$

holds along the solution $x(t)$ and $g(x(t))$ is not identically zero, then the function

$$\phi(t) = \frac{f(x(t))}{g(x(t))}$$

has no singularities.

The following theorem completes the description of the critical set of the momentum map for the gyrostat.

Theorem 2. *The set of critical points of the momentum map* (3.2) *consists of the following subsets in* P^6*:*

1) *the set* $\mathfrak{L}$ *defined by the system*

$$w_1 = 0, \qquad w_2 = 0, \qquad z_1 = 0, \qquad z_2 = 0; \tag{3.21}$$

2) *the set* $\mathfrak{N}$ *defined by the system*

$$F_1 = 0, \qquad F_2 = 0, \tag{3.22}$$

where

$$\begin{aligned}
F_1 = {} & (w_1w_2 + \lambda w_3)(x_1w_2 + \lambda z_1)\lambda y_1 \\
& - w_2(w_1^2 + x_1)(x_2z_1w_1 + x_1z_2w_2 - x_1x_2w_3 + 2z_1z_2\lambda) \\
& - x_2(w_1w_3 + z_1)(w_1z_1 - x_1w_3)\lambda + (x_1w_3^2 - 2z_1w_1w_3 - z_1^2)z_2\lambda^2, \\
F_2 = {} & (w_1w_2 + \lambda w_3)(x_2w_1 + \lambda z_2)\lambda y_2 \\
& - w_1(w_2^2 + x_2)(x_2z_1w_1 + x_1z_2w_2 - x_1x_2w_3 + 2z_1z_2\lambda) \\
& - x_1(w_2w_3 + z_2)(w_2z_2 - x_2w_3)\lambda + (x_2w_3^2 - 2z_2w_2w_3 - z_2^2)z_1\lambda^2;
\end{aligned}$$

3) *the set* $\mathfrak{O}$ *defined by the system*

$$R_1 = 0, \qquad R_2 = 0, \tag{3.23}$$

where

$$\begin{aligned}
R_1 = {} & [y_1w_2 + x_2w_1 + z_2(w_3 + \lambda)]w_1(w_3 - \lambda) \\
& + x_2z_1w_1 + x_1z_2w_2 + z_1z_2(w_3 + \lambda), \\
R_2 = {} & [y_2w_1 + x_1w_2 + z_1(w_3 + \lambda)]w_2(w_3 - \lambda) \\
& + x_2z_1w_1 + x_1z_2w_2 + z_1z_2(w_3 + \lambda).
\end{aligned} \tag{3.24}$$

Proof. The sets $\mathfrak{C}_0$, $\mathfrak{C}_1$ are described by Propositions 1, 2. To find the equations of $\mathfrak{C}_2$ note that after Proposition 2 we have $\operatorname{rank}\{dK, dH\} = 2$ on $\mathfrak{C}_2$. Therefore in any zero linear combination of dG, dK, dH on $\mathfrak{C}_2$ the multiplier of dG is non-zero and can be chosen equal to any non-zero constant. Introducing the undefined multipliers S, T, write the condition on $\mathfrak{C}_2$ in the form

$$2\,dG + S\,dK + (T - p^2)dH = 0. \tag{3.25}$$

According to Lemma 5 rewrite (3.25) as the system of the following six equations

$$
\begin{gathered}
x_2(y_2+2S)w_1+2S(w_1w_2+\lambda w_3)w_2 \\
+(T-z_1z_2-2S\lambda^2)w_2+x_2z_1w_3+(y_2+2S)z_2\lambda=0, \\
x_1(y_1+2S)w_2+2S(w_1w_2+\lambda w_3)w_1 \\
+(T-z_1z_2-2S\lambda^2)w_1+x_1z_2w_3+(y_1+2S)z_1\lambda=0,
\end{gathered}
\tag{3.26}
$$

$$
x_2z_1w_1+x_1z_2w_2+(T-x_1x_2)w_3+(2Sw_1w_2+z_1z_2)\lambda=0, \tag{3.27}
$$

$$
\begin{gathered}
Tz_1+x_1z_2w_3^2+[(x_1x_2-2z_1z_2)w_1+(y_1z_1+x_1z_2)\lambda+x_1y_1w_2]w_3 \\
-(y_1z_1+x_1z_2)w_1w_2+x_1(y_1+2S)w_2\lambda-[x_2z_1+(y_2+2S)z_2]w_1^2 \\
+[(y_2+2S)y_1-2z_1z_2]w_1\lambda+y_1z_1\lambda^2-[(y_2+2S)x_1+z_1^2]z_2=0, \\
Tz_2+x_2z_1w_3^2+[(x_1x_2-2z_1z_2)w_2+(y_2z_2+x_2z_1)\lambda+x_2y_2w_1]w_3 \\
-(y_2z_2+x_2z_1)w_1w_2+x_2(y_2+2S)w_1\lambda-[x_1z_2+(y_1+2S)z_1]w_2^2 \\
+[(y_1+2S)y_2-2z_1z_2]w_2\lambda+y_2z_2\lambda^2-[(y_1+2S)x_2+z_2^2]z_1=0, \\
(T-x_1x_2)(y_1-y_2)+2(y_2+S)x_2w_1^2-2(y_1+S)x_1w_2^2 \\
+2(x_2z_1w_1-x_1z_2w_2)w_3+x_2z_1^2-x_1z_2^2+2(y_2z_2w_1-y_1z_1w_2)\lambda=0.
\end{gathered}
\tag{3.28}
$$

Denote $\mathfrak{K}=\mathfrak{L}\cup\mathfrak{N}\cup\mathfrak{O}$. We need to prove that $\mathfrak{K}=\mathfrak{C}$.

First, show that $\mathfrak{K}\subset\mathfrak{C}$. Obviously, $\mathfrak{L}\subset\mathfrak{C}_0\cup\mathfrak{C}_1$. Indeed on $\mathfrak{L}$ we have $dK\equiv 0$, $dG\equiv \pm ab\, dH$; $\mathfrak{L}$ consists of the pendulum motions (2.14) including four existing equilibria. In fact $\mathfrak{L}\subset\mathfrak{N}\cap\mathfrak{O}$. Nevertheless, we prefer to consider this set apart from the others because it is the phase space of the critical subsystem of rank 1 playing a special role in the bifurcation diagram described below.

Consider the set $\mathfrak{N}$ defined by (3.22). According to Lemma 6 the solution of this system in y_1, y_2 has singularities if and only if along the trajectory we have either

$$
w_1w_2+\lambda w_3\equiv 0 \tag{3.29}
$$

or

$$
(w_2x_1+\lambda z_1)(w_1x_2+\lambda z_2)\equiv 0. \tag{3.30}
$$

Assuming (3.29), after several differentiations in virtue of (3.4), we come to equations (3.15)–(3.18) in the case (3.20). The corresponding points belong to $\mathfrak{C}_1$. For (3.30) the same procedure leads either to (3.29) or to the system of equations having the only solutions of the form (3.21), i.e., to the set $\mathfrak{L}$. Hence the trajectories satisfying (3.29) or (3.30) lie completely in $\mathfrak{C}_0\cup\mathfrak{C}_1$.

Denote $\mathfrak{N}_* = \mathfrak{N} \backslash (\mathfrak{C}_0 \cup \mathfrak{C}_1)$. On this set from (3.22) we obtain

$$\begin{aligned}
y_1 = {} & \frac{1}{(w_1w_2 + \lambda w_3)(w_2x_1 + \lambda z_1)\lambda}[w_2(w_1^2 + x_1)(x_2z_1w_1 + x_1z_2w_2 \\
& - x_1x_2w_3 + 2z_1z_2\lambda) + x_2(w_1w_3 + z_1)(w_1z_1 - x_1w_3)\lambda \\
& - (x_1w_3^2 - 2z_1w_1w_3 - z_1^2)z_2\lambda^2], \\
y_2 = {} & \frac{1}{(w_1w_2 + \lambda w_3)(w_1x_2 + \lambda z_2)\lambda}[w_1(w_2^2 + x_2)(x_2z_1w_1 + x_1z_2w_2 \\
& - x_1x_2w_3 + 2z_1z_2\lambda) + x_1(w_2w_3 + z_2)(w_2z_2 - x_2w_3)\lambda \\
& - (x_2w_3^2 - 2z_2w_2w_3 - z_2^2)z_1\lambda^2].
\end{aligned} \tag{3.31}$$

Let

$$S = \frac{x_1x_2w_3 - x_2z_1w_1 - x_1z_2w_2 - \lambda z_1z_2}{2\lambda(w_1w_2 + \lambda w_3)}, \qquad T = 2\lambda^2 S. \tag{3.32}$$

Substitute (3.31), (3.32) in (3.26)–(3.28) to obtain identities. Hence, $\mathfrak{N}_* \subset \mathfrak{C}_2$ and $\mathfrak{N} \subset \mathfrak{C}$.

Now consider the set $\mathfrak{O}$. It follows from (3.4), (3.5) that the identity $w_3 \equiv \lambda$ along any trajectory is impossible. The identity $w_1w_2 \equiv 0$ obviously leads to the points of $\mathfrak{L}$. Then by Lemma 6 on $\mathfrak{O}_* = \mathfrak{O} \backslash \mathfrak{L}$ we can use system (3.23) to express y_1, y_2:

$$\begin{aligned}
y_1 = {} & -\frac{1}{w_1w_2(w_3 - \lambda)}\{w_1(w_3 - \lambda)[x_2w_1 + z_2(w_3 + \lambda)] \\
& + x_2z_1w_1 + x_1z_2w_2 + z_1z_2(w_3 + \lambda)\}, \\
y_2 = {} & -\frac{1}{w_1w_2(w_3 - \lambda)}\{w_2(w_3 - \lambda)[x_1w_2 + z_1(w_3 + \lambda)] \\
& + x_2z_1w_1 + x_1z_2w_2 + z_1z_2(w_3 + \lambda)\}.
\end{aligned} \tag{3.33}$$

Put

$$S = \frac{x_2z_1w_1 + x_1z_2w_2 + z_1z_2(w_3 + \lambda)}{2w_1w_2(w_3 - \lambda)}, \qquad T = x_1x_2 + z_1z_2 - 2w_1w_2S. \tag{3.34}$$

These values together with (3.33) satisfy (3.26)–(3.28). Hence, $\mathfrak{O}_* \subset \mathfrak{C}_2$ and $\mathfrak{O} \subset \mathfrak{C}$.

To prove that, in turn, $\mathfrak{C} \subset \mathfrak{K}$ note first that $\mathfrak{C}_0 \subset \mathfrak{L} \cap \mathfrak{N} \cap \mathfrak{O}$, the solutions (3.9), (3.10) satisfy both systems (3.22), (3.23). Therefore, $\mathfrak{C}_0 \cup \mathfrak{C}_1 \subset \mathfrak{K}$. Take $\zeta \in \mathfrak{C}_2$ and suppose that $\zeta \notin \mathfrak{O}$. Then

$$R_1 R_2 \neq 0, \tag{3.35}$$

none of identities (3.29), (3.30) holds and from (3.24) we obtain

$$y_1 = \frac{R_1}{w_1 w_2 (w_3 - \lambda)} + y_1^0, \qquad y_2 = \frac{R_2}{w_1 w_2 (w_3 - \lambda)} + y_2^0, \tag{3.36}$$

where y_1^0, y_2^0 stand for the right-hand parts of (3.33). To finish the proof it is sufficient to show that the assumption (3.35) leads to $\zeta \in \mathfrak{N}$. The determinant of the system (3.26) with respect to T, $2S$ is equal to

$$\varDelta = x_1 w_2^2 - x_2 w_1^2 - (z_2 w_1 - z_1 w_2)\lambda.$$

If we suppose that $\varDelta \equiv 0$ on some time interval, then the sequence of the derivatives of this identity in virtue of (3.4) leads to (3.21), i.e., to the points of $\mathfrak{C}_0 \cup \mathfrak{C}_1$. Then for $\zeta \in \mathfrak{C}_2$ by Lemma 3 from (3.26) we find

$$S = \frac{1}{2\varDelta}(A_2 w_1 - A_1 w_2), \qquad T = \frac{1}{\varDelta}(A_1 B_1 - A_2 B_2). \tag{3.37}$$

Here

$$\begin{aligned}
A_1 &= (x_1 w_2 + \lambda z_1) y_1 + (x_1 w_3 - z_1 w_1) z_2, \\
B_1 &= (w_2^2 + x_2) w_1 + \lambda w_2 (w_3 - \lambda) + \lambda z_2, \\
A_2 &= (x_2 w_1 + \lambda z_2) y_2 + (x_2 w_3 - z_2 w_2) z_1, \\
B_2 &= (w_1^2 + x_1) w_2 + \lambda w_1 (w_3 - \lambda) + \lambda z_1.
\end{aligned}$$

Substitute (3.36) and (3.37) in (3.27). We get

$$(x_2 w_1 + \lambda z_2)\Theta_1 R_2 - (x_1 w_2 + \lambda z_1)\Theta_2 R_1 = 0, \tag{3.38}$$

where

$$\begin{aligned}
\Theta_1 &= w_1 (w_3 - \lambda)(w_1 w_2 + \lambda w_3) + w_3 (x_1 w_2 + \lambda z_1), \\
\Theta_2 &= w_2 (w_3 - \lambda)(w_1 w_2 + \lambda w_3) + w_3 (x_2 w_1 + \lambda z_2).
\end{aligned}$$

Satisfying (3.38) introduce the new function R such that

$$\begin{aligned}
\lambda (w_1 w_2 + \lambda w_3)(x_1 w_2 + \lambda z_1) R_1 - \Theta_1 R = 0, \\
\lambda (w_1 w_2 + \lambda w_3)(x_2 w_1 + \lambda z_2) R_2 - \Theta_2 R = 0.
\end{aligned} \tag{3.39}$$

It follows from (3.35) that $\Theta_1 \Theta_2 R \neq 0$. Then equations (3.28) reduce to one equation linear in R, from which we obtain

$$R = (w_1w_2 + \lambda^2)(x_2z_1w_1 + x_1z_2w_2 + \lambda z_1z_2) \\ - x_1x_2w_1w_2(w_3 - \lambda) + \lambda(w_1w_2 + \lambda w_3)z_1z_2.$$

Substitution of this value along with the expressions (3.24) in the system (3.39) gives the system (3.22). Therefore, $\zeta \in \mathfrak{N}$ and, consequently, $\mathfrak{C}_2 \subset \mathfrak{N} \cup \mathfrak{O}$. This inclusion yields $\mathfrak{C} \subset \mathfrak{K}$, and finally $\mathfrak{C} = \mathfrak{K}$. This proves the theorem.

Let us make some comments. The functions S, T used as Lagrange's multipliers and defined by (3.37) are in fact the partial integrals of the dynamical system induced on $\mathfrak{C}_2$. Indeed, let X denote the Hamiltonian vector field of the initial dynamical system (3.1) on P^6. Apply the Lie derivative along X to the condition (3.25) noting that for any first integral its differential is X-invariant. Then

$$\dot{S}\, dK + \dot{T}\, dH \equiv 0. \tag{3.40}$$

Since $\operatorname{rank}\{dG, dK, dH\} = 2$ on $\mathfrak{C}_2$, it follows from (3.25), (3.40) that $\dot{S} \equiv 0$ and $\dot{T} \equiv 0$.

On the set $\mathfrak{N}$ the substitution of (3.31) in (3.37) leads to expressions (3.32). Then S and T for this subsystem are dependent. On the contrary, on the set $\mathfrak{O}$ the functions (3.37) after (3.33) take the form (3.34). Such integrals in the case of the top ($\lambda = 0$) gave rise to the separation of variables in the equations of motion on $\mathfrak{O}$ [15].

Are the induced systems on $\mathfrak{N}$ and $\mathfrak{O}$ Hamiltonian? Consider the smooth 4-dimensional parts of these sets, which are obviously the invariant manifolds for X. To obtain the Hamiltonian subsystem with two degrees of freedom on an invariant 4-dimensional submanifold we must have the non-degenerate 2-form induced on it by the initial symplectic structure. It is known that for two independent functions φ_1, φ_2 on a symplectic manifold (P^{2n}, Ω) the restriction of Ω to the submanifold

$$M = \{\zeta : \varphi_1(\zeta) = 0, \varphi_2(\zeta) = 0\}$$

degenerates exactly at the points where the Poisson bracket $\{\varphi_1, \varphi_2\}$ vanishes. In our case using the rules (2.3) we obtain

$$\{F_1, F_2\} = \mathrm{i}\sqrt{2}\lambda(w_1w_2 + \lambda w_3)^{3/2}\sqrt{(w_2x_1 + \lambda z_1)(w_1x_2 + \lambda z_2)}\,C_{\mathfrak{N}},$$

$$\{R_1, R_2\} = -\mathrm{i}w_1w_2(w_3 - \lambda)C_{\mathfrak{O}},$$

where

$$\begin{aligned} C_{\mathfrak{N}} &= \frac{1}{S}(8\lambda^2S^3 - r^4)\sqrt{2S^2 - (2H + \lambda^2)S + p^2}, \\ C_{\mathfrak{O}} &= \frac{1}{S}[12S^4 - 4(2H - \lambda^2)S^3 + p^4 - r^4] \end{aligned} \tag{3.41}$$

are the first integrals of motion. Obviously, the obtained expressions for the Poisson brackets are not identically zero. Therefore, the induced systems are Hamiltonian almost everywhere on $\mathfrak{N}$ and $\mathfrak{O}$.

4. The bifurcation diagram

The Lax representation for the considered problem found in [2] can be written in the form

$$L' = LM - ML, \tag{4.1}$$

where

$$L = \left\| \begin{array}{cccc} 2\lambda & \dfrac{x_2}{\varkappa} & -2w_1 & \dfrac{z_2}{\varkappa} \\ -\dfrac{x_1}{\varkappa} & -2\lambda & -\dfrac{z_1}{\varkappa} & 2w_1 \\ -2w_1 & \dfrac{z_2}{\varkappa} & -2w_3 & -\dfrac{y_1}{\varkappa} - 4\varkappa \\ -\dfrac{z_1}{\varkappa} & 2w_2 & \dfrac{y_2}{\varkappa} + 4\varkappa & 2w_3 \end{array} \right\|, \qquad M = \left\| \begin{array}{cccc} -\dfrac{w_3}{2} & 0 & \dfrac{w_2}{2} & 0 \\ 0 & \dfrac{w_3}{2} & 0 & -\dfrac{w_1}{2} \\ \dfrac{w_1}{2} & 0 & \dfrac{w_2}{2} & \varkappa \\ 0 & -\dfrac{w_2}{2} & -\varkappa & -\dfrac{w_3}{2} \end{array} \right\|.$$

Here $\varkappa$ stands for the spectral parameter, the derivative in (4.1) is calculated in virtue of the system (3.4). The equation for the eigenvalues μ of the matrix L defines the algebraic curve associated with this representation [9]. Let $s = 2\varkappa^2$ and let h, k, g be the arbitrary constants of the integrals (3.6). The equation of the algebraic curve takes the form

$$\begin{aligned} &\mu^4 - 4\mu^2 \left[\frac{p^2}{s} - (2h + \lambda^2) + 2s \right] \\ &\qquad + 4\left[\frac{r^4}{s^2} + \frac{2}{s}(4g - 2p^2 h - p^2\lambda^2) + 4(k + 2\lambda^2 h) - 8\lambda^2 s \right] = 0. \end{aligned} \tag{4.2}$$

It is natural to suppose that the bifurcation diagram of the momentum map (3.2) is included in the set of values (g, k, h) such that this curve either have singular points or is reducible, i.e., the left-hand part of (4.2) splits into the product of some rational non-trivial expressions. In this way we can guess the result of the following statement. Nevertheless, to obtain the complete proof of it, we must fulfill the calculations on the above found critical manifolds.

THEOREM 3. *The bifurcation diagram of the momentum map $G \times K \times H$ is included in the union of the following (intersecting) subsets of $\mathbf{R}^3(g, k, h)$:*

1) *the pair of straight lines*

$$\Gamma_+ : \begin{cases} k = (a+b)^2, \\ g = -ab\left(h - \dfrac{\lambda^2}{2}\right); \end{cases} \qquad \Gamma_- : \begin{cases} k = (a-b)^2, \\ g = ab\left(h - \dfrac{\lambda^2}{2}\right); \end{cases} \tag{4.3}$$

2) *the surface*

$$\Gamma_1 : \begin{cases} k = 4\lambda^2 s - 2\lambda^2 h + \dfrac{r^4}{4s^2}, \\ g = -\lambda^2 s^2 + \dfrac{1}{2}p^2\left(h + \dfrac{\lambda^2}{2}\right) - \dfrac{r^4}{4s}, \quad s \in \mathbf{R}\backslash\{0\}; \end{cases} \tag{4.4}$$

3) *the surface*

$$\Gamma_2 : \begin{cases} k = 3s^2 - 4\left(h - \dfrac{\lambda^2}{2}\right)s + p^2 + \left(h - \dfrac{\lambda^2}{2}\right)^2 - \dfrac{p^4 - r^4}{4s^2}, \\ g = -s^3 + \left(h - \dfrac{\lambda^2}{2}\right)s^2 + \dfrac{p^4 - r^4}{4s}, \quad s \in \mathbf{R}\backslash\{0\}. \end{cases} \tag{4.5}$$

PROOF. Let $\zeta \in \mathfrak{L}$. Substitution of the values $z_1 = z_2 = 0$ in (3.5) yields $x_1 x_2 = (a \pm b)^2$, $y_1 y_2 = (a \mp b)^2$. Then from (3.6), (3.21) we obtain the equations defining the lines (4.3).

Let $\zeta \in \mathfrak{N}\backslash\mathfrak{L}$. Take the constant of the partial integral S defined by (3.32) for the parameter s in (4.4), substitute the expressions (3.6) for the corresponding constants, and fulfill the change (3.31). Then both equations (4.4) become the identities. Hence, $J(\mathfrak{N}\backslash\mathfrak{L}) \subset \Gamma_1$.

The inclusion $J(\mathfrak{O}\backslash\mathfrak{L}) \subset \Gamma_2$ is proved in a similar way. We take the constant of the partial integral S from (3.34) for the parameter s in (4.5) and fulfill the substitution (3.33).

REMARK 3. *Note that the shift of the energy level $\tilde{h} = h - \lambda^2/2$ makes the equations of the lines $\Gamma_\pm$ and the surface Γ_2 independent of λ. Thereby obtained equations are identical with the corresponding equations of the case $\lambda = 0$* [13]. *The surface Γ_1 is obtained as a perturbation (with respect to λ) of two tangent to each other sheets of the bifurcation diagram of the case $\lambda = 0$, i.e., the plane $k = 0$ and the slanted parabolic cylinder $(p^2 h - 2g)^2 - r^4 k = 0$. Thus, it is easy to view the evolution of the Appelrot classes* [21] *of the S. Kowalevski case in the process of two-way generalizations—adding the second force field and, afterwards, the non-zero gyrostatic momentum.*

Denote $\Gamma_\pm = \Gamma_+ \cup \Gamma_-$. It is easily seen that $\Gamma_\pm \cap \Gamma_1$ consists of the finite number of points and there exists the whole segment of $\Gamma_\pm\backslash\Gamma_2$. In terms of the critical subsystems described by Theorem 2 it means that the critical set $\mathfrak{L}$ does not lie completely in the interior of either set $\mathfrak{N}$ or $\mathfrak{O}$. This is an additional reason to consider the system on $\mathfrak{L}$ as a special case.

Now we can describe those singularities of the bifurcation diagram which correspond to the degenerations of the induced symplectic structure in the critical subsystems. For this purpose, use the expressions (3.41). The equation $C_{\mathfrak{N}} = 0$ define two curves on Γ_1: $8\lambda^2 s^3 - r^4 = 0$ and $2s^2 - (2h + \lambda^2)s + p^2 = 0$. The first one is the cuspidal edge of Γ_1 and the second is the tangency curve of Γ_1 and Γ_2. The equation $C_{\mathfrak{O}} = 0$ gives $12s^4 - 4s^3(2h - \lambda^2) + p^4 - r^4 = 0$ and corresponds to the cuspidal edge of Γ_2. Note also that the points of the common part of $\Gamma_\pm$ and Γ_2 form the line of self-intersection of Γ_2.

The equations established by Theorem 3 are in the following sense convenient. Let us fix the energy constant h. Then we obtain the parametric equations of a one-dimensional set in the plane (g, k) (with the finite number of singular points). This set is the bifurcation diagram Σ_h of the restriction of the pair of integrals G, K onto the iso-energetic surface $\{H = h\} \subset P^6$, which is always compact. In particular, all diagrams Σ_h lie in the bounded area of the (g, k)-plane and are easily drawn numerically. The analytical investigation of the types of the diagrams Σ_h with respect to the essential parameters (b/a, $\lambda/\sqrt{a}, h/a$) is a necessary but technically complicated problem. Nevertheless, it must be solvable. Indeed, the set of double points and cusps of the curves $\Gamma_{1,2}$ in the (g, k)-plane is already defined above and its evolution with respect to the parameters is easily investigated analytically. Moreover, the values of the first integrals on the motions (3.9)–(3.12) define the points of transversal intersections $\Gamma_1 \cap \Gamma_2$. This fact, at least, guarantees that the numerical algorithm can be built for effective calculation of knots of one-dimensional cell complex Σ_h for any h. In turn, it should be possible to find all cases of bifurcations of the set of these knots with respect to the parameters defining the above set of rank 1 critical motions.

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