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Words, tilings and combinatorial spectra

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Abstract. We will introduce some combinatorics for given words. Such combinatorics can essentially determine the exact information of letters as well as the patterns of words. This method can induce a characterization of the so-called local indistinguishability for one dimensional tilings, which allows us to have a new development for tiling bialgebras. Using those combinatorics associated with words and one dimensional tilings, we can obtain their combinatorial spectra as certain sets of functions or positive real numbers. We will also discuss higher dimensional tilings. Furthermore, we will try to compute some genome examples.

0. Introduction

In this paper, we will give a totally new approach to tilings and words, and establish several characterizations for patterns. For subwords of a given word, we will introduce the associated matrices, graphs and multiplicities in Section 1, which is originally coming from a decomposition of tensor products (cf. [10]). We say that our approach here is combinatorial, since we use partially ordered sets, graphs, pilings, decompositions and multiplicities. We also use bialgebras and modules. In this sense, our approach might be algebraic. Note that upper case characters $A, B, \dots, X_1, X_2, \dots$ are used for our letters, and lower case characters $a, b, \dots, \lambda, \mu, \dots$ are used for our words and subwords. We will obtain a combinatorial characterization of words in Section 3, using partially ordered sets and multiplicities. That is, roughly saying, the multiplicities are equal $\Leftrightarrow$ the partially ordered sets are isomorphic $\Leftrightarrow$ the patterns are same. We will give several examples in Section 2 for convenience. In Section 4, we will review the notion of one dimensional tilings and the definition of local indistinguishability. Then we will use our method to characterize local indistinguishability for one dimensional tilings in Section 5. In Section 6, we will refine the result for tiling bialgebras to give a characterization of local indistinguishability. In Section 7, we will introduce $\mathrm{Spec}_f(a)$, called the func-

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tional spectrum of a word a, and $\mathrm{Spec}_f(\mathscr{T})$, called the functional simple spectrum of a one dimensional tiling $\mathscr{T}$. We also discuss higher dimensional cases in Section 8. We will try to compute several examples for some Genomes and for some tilings in Section 9. For a set $\{\dots\}$, we denote by $\#\{\dots\}$ its cardinality.

1. Words and combinatorics

Let a be a (finite) word using letters, and we denote by $\Omega(a)$ the set of letters appearing in a. If $a = X_1X_2\dots X_r$ with $X_i \in \Omega(a)$ for $1 \le i \le r$, then we say $l(a) = r$, which means that the length of a is r. We define for two words a and b to have the same pattern if there is a bijection ϕ from $\Omega(a)$ to $\Omega(b)$ such that $a = X_1X_2\dots X_r$ and $b = \phi(X_1)\phi(X_2)\dots\phi(X_r)$. Equivalently we sometimes say that the pattern of a is the same as the pattern of b.

A subword of $a = X_1X_2\dots X_r$ is defined to be

$$X_jX_{j+1}X_{j+2}\dots X_{j+p}$$

with $1 \le j \le r$ and $0 \le p \le r - j$, which is used here in a strong sense. Note that specialists sometimes call $X_{i_1}X_{i_2}\dots X_{i_s}$ $(1 \le i_1 < i_2 < \cdots < i_s \le r)$ a subword of a. But here, we always assume $i_{k+1} = i_k + 1$ when we say a subword. Of course, a subword can also be considered as another word. Let $S(a)$ be the set of all subwords of a. Adding one abstract independent symbol ε to $S(a)$ as a new letter, we define

$$W(a) = \{\varepsilon\} \cup S(a).$$

One may consider ε as an empty subword of a. For each $r = 0, 1, 2, 3\dots$, we put

$$W_r(a) = \{\lambda \in S(a) \mid l(\lambda) = r\} \qquad (r = 1, 2, 3, \dots),$$

and set $W_0(a) = \{\varepsilon\}$. Then, we see $W_1(a) = \Omega(a)$. If $\lambda, \mu \in S(a)$ with

$$\lambda = Y_1Y_2\dots Y_s, \qquad \mu = Z_1Z_2\dots Z_t,$$

then we make the following $s \times t$ matrix $M(\lambda, \mu) = (m_{ij})$ with entries in $\Omega(a) \cup \{\varepsilon\}$:

$$m_{ij} = \begin{cases} Y_i & \text{if } Y_i = Z_j, \\ \varepsilon & \text{if } Y_i \ne Z_j. \end{cases}$$

Using this $M(\lambda, \mu)$, we construct the associated graph, whose vertices are (i, j) for all $1 \le i \le s$ and $1 \le j \le t$, and whose edges (arrows) are defined by saying

$$(i, j) \text{ and } (k, \ell) \text{ are joined by a single arrow like:} \qquad (i, j) \to (k, \ell)$$

$$\text{if } k = i + 1,\ \ell = j + 1,\ m_{ij} \ne \varepsilon,\ m_{k\ell} \ne \varepsilon.$$

This graph is called $\Gamma(\lambda, \mu)$. Let $C(\lambda, \mu)$ be the set of all connected components of $\Gamma(\lambda, \mu)$. If $c \in C(\lambda, \mu)$ with

$$c = (i, j) \to (i+1, j+1) \to \cdots \to (i+p, j+p),$$

then we put $w(c) = m_{i,j} m_{i+1,j+1} \dots m_{i+p,j+p} \in S(a)$ in case of $m_{ij} \neq \varepsilon$ with $p \geq 0$, and we put $w(c) = \varepsilon \in W(a)$ in case of $m_{ij} = \varepsilon$ with $p = 0$. For $\lambda, \mu \in S(a)$ and $v \in W(a)$, we define

$$\mathscr{M}(a)_v(\lambda, \mu) = \#\{c \in C(\lambda, \mu) \mid w(c) = v\},$$

the multiplicity of (λ, μ) at v. We also set

$$\mathscr{M}(a)_v(\lambda, \varepsilon) = \delta_{v,\varepsilon} \cdot s,$$

$$\mathscr{M}(a)_v(\varepsilon, \mu) = \delta_{v,\varepsilon} \cdot t,$$

$$\mathscr{M}(a)_v(\varepsilon, \varepsilon) = \delta_{v,\varepsilon},$$

where δ means the usual Kronecker's delta ($\delta_{\varepsilon,\varepsilon} = 1$, $\delta_{v,\varepsilon} = 0$ if $v \neq \varepsilon$), and where $\lambda = Y_1 Y_2 \dots Y_s$ and $\mu = Z_1 Z_2 \dots Z_t$. Therefore, we obtain the following map

$$\mathscr{M}(a) : W(a) \times W(a) \times W(a) \to \mathbf{Z}_{\geq 0},$$

which is given by

$$(\lambda, \mu, v) \mapsto \mathscr{M}(a)_v(\lambda, \mu).$$

We call $\mathscr{M}(a)$ the combinatorics for a (which has a mathematical meaning in the sense of decomposition rule for tensor products). We note that $\mathscr{M}(a)_v(\lambda, \mu) = \mathscr{M}(a)_v(\mu, \lambda)$ for all $\lambda, \mu, v \in W(a)$.

If $a = AABAB$, $\lambda = ABA$ and $\mu = AABAB$, then we have

$$M(\lambda, \mu) = \begin{pmatrix} A & A & \varepsilon & A & \varepsilon \\ \varepsilon & \varepsilon & B & \varepsilon & B \\ A & A & \varepsilon & A & \varepsilon \end{pmatrix},$$

and

$$C(\lambda, \mu) = \left\{ \begin{array}{lllllll} (1,2) \to (2,3) \to (3,4), & & & & & & \\ (1,4) \to (2,5), & & & & & & \\ (1,1), & (3,1), & (3,2), & & & & \\ (1,3), & (1,5), & (2,1), & (2,2), & (2,4), & (3,3), & (3,5) \end{array} \right\}.$$

Therefore, we see $\mathscr{M}(a)_{ABA}(\lambda, \mu) = 1$, $\mathscr{M}(a)_{AB}(\lambda, \mu) = 1$, $\mathscr{M}(a)_A(\lambda, \mu) = 3$, and $\mathscr{M}(a)_\varepsilon(\lambda, \mu) = 7$. For other $v \in W(a)$, we have $\mathscr{M}(a)_v(\lambda, \mu) = 0$.

If two words a and b have the same combinatorics, that is, if there is a bijection

$$\theta : W(a) \to W(b)$$

such that

$$\mathscr{M}(a)_{\nu}(\lambda, \mu) = \mathscr{M}(b)_{\theta(\nu)}(\theta(\lambda), \theta(\mu))$$

for all $\lambda, \mu, \nu \in W(a)$, then we will say that a and b are combinatorially equivalent.

2. Words with length ≤ 3

We will compute several examples here, and make the tables of $\Gamma = \Gamma(a, a)$ for several words a, totalizing $M(a, a)$ and $C(a, a)$. Visually $\Gamma(a, a)$ tells us the whole information of $\mathscr{M}(a)$.

(1) $a = A$

$$\begin{array}{c|c} \Gamma & A \\ \hline A & A \end{array}$$

(2) $a = AA$

$$\begin{array}{c|ccc} \Gamma & A & & A \\ \hline A & A & & A \\ & & \searrow & \\ A & A & & A \end{array}$$

(3) $a = AB$

$$\begin{array}{c|ccc} \Gamma & A & & B \\ \hline A & A & & \varepsilon \\ & & \searrow & \\ B & \varepsilon & & B \end{array}$$

(4) $a = AAA$

$$\begin{array}{c|ccccc} \Gamma & A & & A & & A \\ \hline A & A & & A & & A \\ & & \searrow & & \searrow & \\ A & A & & A & & A \\ & & \searrow & & \searrow & \\ A & A & & A & & A \end{array}$$

(5) $a = AAB$

$$\begin{array}{c|ccccc} \Gamma & A & & A & & B \\ \hline A & A & & A & & \varepsilon \\ & & \searrow & & & \\ A & A & & A & & \varepsilon \\ & & & & \searrow & \\ B & \varepsilon & & \varepsilon & & B \end{array}$$

(6) $a = ABA$

$$\begin{array}{c|ccccc} \Gamma & A & & B & & A \\ \hline A & A & & \varepsilon & & A \\ & & \searrow & & & \\ B & \varepsilon & & B & & \varepsilon \\ & & & & \searrow & \\ A & A & & \varepsilon & & A \end{array}$$

(7) $a = ABC$

$$\begin{array}{c|ccccc} \Gamma & A & & B & & C \\ \hline A & A & & \varepsilon & & \varepsilon \\ & & \searrow & & & \\ B & \varepsilon & & B & & \varepsilon \\ & & & & \searrow & \\ C & \varepsilon & & \varepsilon & & C \end{array}$$

We confirm here the corresponding sets $W(a)$ in these examples:

$$W(A) = \{A, \varepsilon\}, \qquad W(AA) = \{AA, A, \varepsilon\}, \qquad W(AB) = \{AB, A, B, \varepsilon\},$$

$$W(AAA) = \{AAA, AA, A, \varepsilon\}, \qquad W(AAB) = \{AAB, AA, AB, A, B, \varepsilon\},$$

$$W(ABA) = \{ABA, AB, BA, A, B, \varepsilon\},$$

$$W(ABC) = \{ABC, AB, BC, A, B, C, \varepsilon\}.$$

3. Characterization of words

For a word $a = X_1X_2 \dots X_r$, we denote by ${}^t a$ the transpose word $X_r \dots X_2X_1$. We can regard $W(a)$ as a partially ordered set with its order $\leq$ given by saying that $b \leq c$ if b is a subword of c, where ε is a unique minimal element in $W(a)$. Then we obtain the following.

Theorem 1. *For two words a and a', the following three conditions are equivalent.*
(1) *The pattern of a is the same as the pattern of a' or ${}^t a'$.*
(2) *Two words a and a' are combinatorially equivalent.*
(3) *Two sets $W(a)$ and $W(a')$ are isomorphic as partially ordered sets.*

Proof. By the definition to be combinatorially equivalent, one sees that (1) implies (2). Since $M(a)_\mu(\lambda, \mu) \neq 0$ if and only if μ is a subword of λ, we can obtain that (2) implies (3). Here we note that ε is regarded as a subword of any word. Also we should note that $\varepsilon \in W(a)$ is uniquely characterized by the property that $\lambda = \varepsilon$ if $\mathscr{M}(a)_\nu(\lambda, \nu) = 0$ for all $\nu \in W(a)$ satisfying $\nu \neq \lambda$. Now we want to show: (3) $\Rightarrow$ (1). If $n = l(a) = l(a') \leq 3$ and the patterns of a and a' are different modulo transpose, then the structures of partially ordered sets $W(a)$ and $W(a')$ are not isomorphic as we could watch in the previous section. Therefore, we can suppose $n > 3$. Let $a = X_1X_2 \dots X_n$. We proceed by induction on n. For our purpose, it is enough to show that the structure of the partially ordered set $W(a)$ can uniquely characterize the pattern of a modulo transpose.

[The case of $W_{n-1}(a) = \{b\}$.]

In this case, we see

$$b = X_1X_2 \dots X_{n-1} = X_2X_3 \dots X_n.$$

This means $X_1 = X_2 = \cdots = X_n$, which uniquely gives the pattern of a. That is, $a = AA \dots A$. Hence, we are done.

[The case of $W_{n-1}(a) = \{b, c\}$ $(b \neq c)$, $W_{n-2}(a) = \{d, e\}$ $(d \neq e)$, $W_{n-2}(b) = \{d\}$, $W_{n-2}(c) = \{d, e\}$.]

In this case, we see that $b = AA \dots A$. Hence, we have $a = A \dots AB$ or $a = BA \dots A$. Then, the pattern of a is uniquely determined modulo transpose.

[The case of $W_{n-1}(a) = \{b, c\}$ $(b \neq c)$, $W_{n-2}(a) = \{d, e\}$ $(d \neq e)$, $W_{n-2}(b) = \{d, e\}$, $W_{n-2}(c) = \{d, e\}$.]

In this case, we see

$$a = ABAB \dots.$$

Hence, the pattern of a is uniquely determined.

[The case of $W_{n-1}(a) = \{b, c\}$ $(b \neq c)$, $W_{n-2}(a) = \{d, e, f\}$ (all distinct).]

In this case, we can assume that $d \leq b$ and $d \leq c$. Then, by induction, the pattern of d can be determined modulo transpose. Hence, we can suppose $d = d_1$ or $d = d_2$, where $d_1 = X_2 X_3 \dots X_{n-1}$ and $d_2 = X_{n-1} \dots X_2 = {}^t d_1$. Then, we have the following four cases:

$$\begin{cases} b = Y d_1 \\ c = d_1 Z \end{cases}, \quad \begin{cases} b = d_1 Y \\ c = Z d_1 \end{cases}, \quad \begin{cases} b = Y d_2 \\ c = d_2 Z \end{cases}, \quad \begin{cases} b = d_2 Y \\ c = Z d_2 \end{cases}.$$

By induction, b and c can uniquely be determined modulo transpose as patterns respectively. Hence the letters Y and Z are completely determined in terms of patterns using $W(a)$. This is exactly obtained by checking

$$\Omega(d) = \Omega(a) \text{ or not ?}; \qquad \Omega(d) = \Omega(b) \text{ or not ?}; \qquad \Omega(d) = \Omega(c) \text{ or not ?}$$

in $W(a)$. This means that we can almost decide what b is. Note that ${}^t(Y d_1) = d_2 Y$ and ${}^t(d_1 Y) = Y d_2$. Therefore, we can almost decide that one of the following two cases happens:

(case 1) $b = Y d_1$ or $b = d_2 Y$,

(case 2) $b = Y d_2$ or $b = d_1 Y$.

First, we suppose that we can completely decide which of them is exactly valid. If only (case 1) is valid, then

$$\begin{cases} b = Y d_1 \\ c = d_1 Z \end{cases}, \quad \text{or} \quad \begin{cases} b = d_2 Y \\ c = Z d_2 \end{cases}.$$

Hence,

$$a = Y d_1 Z \quad \text{or} \quad a = Z d_2 Y = {}^t(Y d_1 Z),$$

which means that the pattern of a can be determined modulo transpose. If only (case 2) is valid, then

$$\begin{cases} b = Y d_2 \\ c = d_2 Z \end{cases}, \quad \text{or} \quad \begin{cases} b = d_1 Y \\ c = Z d_1 \end{cases}.$$

Hence,

$$a = Y d_2 Z \quad \text{or} \quad a = Z d_1 Y = {}^t(Y d_2 Z),$$

which also means that the pattern of a can be determined modulo transpose. Next we suppose that we cannot decide whether (case 1) or (case 2) holds. Then, we reach

$$Yd = dY \qquad \text{or} \qquad {}^t(Yd) = {}^t dY = dY.$$

However, $Yd = dY$ means $d = YY\dots Y$ and $W_{n-2}(b) = \{d\}$, which is not our case here. Thus, in particular, we obtain $d = {}^t d$. Hence, in this case we have $a = YdZ$ or $a = ZdY = {}^t(YdZ)$. Therefore, in any case, the pattern of a is uniquely determined modulo transpose.

Thus, $W(a)$ with its partial order can completely give the pattern of a modulo transpose. Hence, (3) implies (1).

Theorem 2. *Let a, a' be words. Suppose that a bijection $\theta : W(a) \to W(a')$ gives a combinatorial equivalence between a and a'. If $a = X_1 X_2 \dots X_n$, then $a' = \theta(X_1)\theta(X_2)\dots\theta(X_n)$ or $a' = \theta(X_n)\dots\theta(X_2)\theta(X_1)$.*

Proof. If $n \le 2$, then we can easily confirm our statement here (cf. Section 2). Hence we suppose $n \ge 3$.

[The case of $W_{n-1}(a) = \{b\}$.]

In this case, we see

$$b = X_1 X_2 \dots X_{n-1} = X_2 X_3 \dots X_n.$$

This means $X_1 = X_2 = \cdots = X_n$, which uniquely gives the pattern of a. That is, $a = AA\dots A$. Hence, we also obtain $a' = \theta(a) = \theta(A)\theta(A)\dots\theta(A)$.

[The case of $W_{n-1}(a) = \{b, c\}$ $(b \ne c)$, $W_{n-2}(a) = \{d, e\}$ $(d \ne e)$, $W_{n-2}(b) = \{d\}$, $W_{n-2}(c) = \{d, e\}$.]

In this case, we see that $b = AA\dots A$. Hence, we have $a = AA\dots AB$ or $a = BAA\dots A$. Then, we also obtain

$$a' = \theta(a) = \theta(A)\theta(A)\dots\theta(A)\theta(B) \qquad \text{or} \qquad a' = \theta(a) = \theta(B)\theta(A)\theta(A)\dots\theta(A).$$

[The case of $W_{n-1}(a) = \{b, c\}$ $(b \ne c)$, $W_{n-2}(a) = \{d, e\}$ $(d \ne e)$, $W_{n-2}(b) = \{d, e\}$, $W_{n-2}(c) = \{d, e\}$.]

In this case, we see

$$a = ABAB\dots A \qquad (n = \text{odd}),$$

$$a = ABAB\dots B \qquad (n = \text{even}).$$

Hence, using the invariance $\mathscr{M}(a)_\nu(\lambda, \mu) = \mathscr{M}(a')_{\theta(\nu)}(\theta(\lambda), \theta(\mu))$, we obtain

$$a' = \theta(a) = \theta(A)\theta(B)\theta(A)\theta(B)\dots\theta(A)$$

if $n = \text{odd}$, and

$$a' = \theta(a) = \theta(A)\theta(B)\theta(A)\theta(B)\dots\theta(B) \qquad \text{or}$$
$$a' = \theta(a) = \theta(B)\theta(A)\theta(B)\theta(A)\dots\theta(A)$$

if $n =$ even.

[The case of $W_{n-1}(a) = \{b, c\}$ $(b \neq c)$, $W_{n-2}(a) = \{d, e, f\}$ (all distinct).]

In this case, we put

$$b = X_1X_2\dots X_{n-1}, \qquad c = X_2X_3\dots X_n,$$

and $d = X_2X_3\dots X_{n-1}$. By induction we see that

$$\theta(d) = \theta(X_2)\theta(X_3)\dots\theta(X_{n-1}) \qquad \text{or} \qquad \theta(d) = \theta(X_{n-1})\dots\theta(X_3)\theta(X_2).$$

Also, by induction, we obtain

$$\theta(b) = \theta(X_1)\theta(X_2)\dots\theta(X_{n-1}) \qquad \text{or} \qquad \theta(b) = \theta(X_{n-1})\dots\theta(X_2)\theta(X_1).$$

Furthermore, using the invariance of $\mathscr{M}$ again, we have

$$\theta(b) = \theta(X_1)\theta(d) \qquad \text{or} \qquad \theta(b) = \theta(d)\theta(X_1).$$

If $\theta(b) = \theta(X_1)\theta(X_2)\dots\theta(X_{n-1}) = \theta(X_1)\theta(d)$ with

$$\theta(d) = \theta(X_2)\theta(X_3)\dots\theta(X_{n-1}),$$

then

$$a' = \theta(a) = \theta(X_1)\theta(d)\theta(X_n) = \theta(X_1)\theta(X_2)\dots\theta(X_n).$$

If $\theta(b) = \theta(X_1)\theta(X_2)\dots\theta(X_{n-1}) = \theta(d)\theta(X_1)$ with

$$\theta(d) = \theta(X_2)\theta(X_3)\dots\theta(X_{n-1}),$$

then $\theta(b) = X'X'\dots X'$, which is not our case. If

$$\theta(b) = \theta(X_1)\theta(X_2)\dots\theta(X_{n-1}) = \theta(X_1)\theta(d)$$

with $\theta(d) = \theta(X_{n-1})\dots\theta(X_3)\theta(X_2)$, then ${}^t\theta(d) = \theta(d)$ and

$$a' = \theta(a) = \theta(X_1)\theta(d)\theta(X_n) = \theta(X_1)\theta(X_2)\dots\theta(X_n).$$

If $\theta(b) = \theta(X_1)\theta(X_2)\dots\theta(X_{n-1}) = \theta(d)\theta(X_1)$ with

$$\theta(d) = \theta(X_{n-1})\dots\theta(X_3)\theta(X_2),$$

then ${}^t\theta(b) = \theta(b)$ and

$$a' = \theta(a) = \theta(X_n)\theta(d)\theta(X_1) = \theta(X_n)\dots\theta(X_2)\theta(X_1).$$

If $\theta(b) = \theta(X_{n-1})\dots\theta(X_2)\theta(X_1) = \theta(X_1)\theta(d)$ with

$$\theta(d) = \theta(X_2)\theta(X_3)\dots\theta(X_{n-1}),$$

then ${}^t\theta(b) = \theta(b)$ and

$$a' = \theta(a) = \theta(X_1)\theta(d)\theta(X_n) = \theta(X_1)\theta(X_2)\dots\theta(X_n).$$

If $\theta(b) = \theta(X_{n-1})\dots\theta(X_2)\theta(X_1) = \theta(d)\theta(X_1)$ with

$$\theta(d) = \theta(X_2)\theta(X_3)\dots\theta(X_{n-1}),$$

then ${}^t\theta(d) = \theta(d)$ and

$$a' = \theta(a) = \theta(X_n)\theta(d)\theta(X_1) = \theta(X_n)\dots\theta(X_2)\theta(X_1).$$

If $\theta(b) = \theta(X_{n-1})\dots\theta(X_2)\theta(X_1) = \theta(X_1)\theta(d)$ with

$$\theta(d) = \theta(X_{n-1})\dots\theta(X_3)\theta(X_2),$$

then $\theta(b) = X'X'\dots X'$, which is not our case. If

$$\theta(b) = \theta(X_{n-1})\dots\theta(X_2)\theta(X_1) = \theta(d)\theta(X_1)$$

with $\theta(d) = \theta(X_{n-1})\dots\theta(X_3)\theta(X_2)$, then

$$a' = \theta(a) = \theta(X_n)\theta(d)\theta(X_1) = \theta(X_n)\dots\theta(X_2)\theta(X_1).$$

Hence, we proved our desired result.

The above results might partially be known, but there seems to be no good reference. Anyway, even for Human Genome in Bioinformatics, DNA or RNA strings could be controled by their combinatorics.

Example. Let $a = ABA$ and $a' = CDC$. Then

$$W(a) = \{ABA, AB, BA, A, B, \varepsilon\} \qquad \text{and} \qquad W(a') = \{CDC, CD, DC, C, D, \varepsilon\}.$$

Let $\theta_i : W(a) \to W(a')$ $(i = 1, 2)$ be maps defined by

$$\theta_1 : \begin{cases} ABA \mapsto CDC \\ AB \mapsto CD \\ BA \mapsto DC \\ A \mapsto C \\ B \mapsto D \\ \varepsilon \mapsto \varepsilon \end{cases} \qquad \theta_2 : \begin{cases} ABA \mapsto CDC \\ AB \mapsto CD \\ BA \mapsto DC \\ A \mapsto D \\ B \mapsto C \\ \varepsilon \mapsto \varepsilon \end{cases}.$$

Then, θ_1 gives a combinatorial equivalence between $W(a)$ and $W(a')$, and we see $\theta_1(a) = CDC = \theta_1(A)\theta_1(B)\theta_1(A)$. On the other hand, θ_2 induces an isomorphism between $W(a)$ and $W(a')$ as partially ordered sets. However, we find

$$\theta_2(a) = CDC \neq DCD = \theta_2(A)\theta_2(B)\theta_2(A).$$

We note that θ_2 does not preserve our $\mathscr{M}$.

4. One dimensional tilings

Let **R** be a real line. A tile in **R** is a connected closed bounded subset of **R**, namely a closed interval whose interior is nonempty. A tiling $\mathcal{T}$ of **R** is an infinite set of tiles which cover **R** overlapping, at most, at their boundaries. In this note, we identify a tiling of **R** with a bi-infinite sequence of letters, equivalently saying, a bi-infinite word of letters. Let $S(\mathcal{T})$ be the set of all finite subwords in $\mathcal{T}$. If $w = X_1 \dots X_r \in S(\mathcal{T})$, then $l(w) = r$ is called the length of w. Let $S_r(\mathcal{T})$ be the set of all finite subwords with length r. Put $\Omega(\mathcal{T}) = S_1(\mathcal{T})$, the set of all letters appearing in $\mathcal{T}$. For convenience, we assume that $\Omega(\mathcal{T})$ is finite.

Two tilings $\mathcal{T}$ and $\mathcal{T}'$ are called to be locally indistinguishable and we say $\mathcal{T} \sim_{l.i.} \mathcal{T}'$ if there is a bijection

$$\phi : S(\mathcal{T}) \to S(\mathcal{T}')$$

such that $\phi(S_r(\mathcal{T})) = S_r(\mathcal{T}')$ for all $r \geq 1$ and $\phi(w) = \phi(X_1)\phi(X_2)\dots\phi(X_r)$ for all $w = X_1X_2\dots X_r \in S(\mathcal{T})$.

For a tiling $\mathcal{T}$, we put $W(\mathcal{T}) = S(\mathcal{T}) \cup \{\varepsilon\}$, where ε is an abstract independent symbol as a new letter. Let $W_r(\mathcal{T}) = S_r(\mathcal{T})$ for $r \geq 1$, and set $W_0(\mathcal{T}) = \{\varepsilon\}$. For a tiling

$$\mathcal{T} = \dots X_{-3}X_{-2}X_{-1}X_0X_1X_2X_3\dots,$$

we denote by ${}^t\mathcal{T}$ the transpose of $\mathcal{T}$, that is,

$${}^t\mathcal{T} = \dots X_3X_2X_1X_0X_{-1}X_{-2}X_{-3}\dots.$$

5. Characterization of local indistinguishability

We will define a suitable map $\mathcal{M}(\mathcal{T})$ with

$$\mathcal{M}(\mathcal{T}) : W(\mathcal{T}) \times W(\mathcal{T}) \times W(\mathcal{T}) \to \mathbf{Z}_{\geq 0}$$

in the following way. Let $\lambda, \mu \in S(\mathcal{T})$ and $\nu \in W(\mathcal{T})$, then we can find an element $\tau \in S(\mathcal{T})$ such that λ, μ, ν are subwords of τ or such that λ, μ are subwords of τ with $\nu = \varepsilon$. Then, we put

$$\mathcal{M}(\mathcal{T})_\nu(\lambda, \mu) = \mathcal{M}(\mathcal{T})(\lambda, \mu, \nu) = \mathcal{M}(\tau)_\nu(\lambda, \mu),$$

the multiplicity of (λ, μ) at ν, which is well-defined. Furthermore, we also define

$$\mathcal{M}(\mathcal{T})_\nu(\lambda, \varepsilon) = \delta_{\nu,\varepsilon} \times s,$$

$$\mathcal{M}(\mathcal{T})_\nu(\varepsilon, \mu) = \delta_{\nu,\varepsilon} \times t,$$

$$\mathcal{M}(\mathcal{T})_\nu(\varepsilon, \varepsilon) = \delta_{\nu,\varepsilon},$$

where δ means the usual delta function, and where $\lambda = Y_1 Y_2 \dots Y_s$ and $\mu = Z_1 Z_2 \dots Z_t$. We call $\mathscr{M}(\mathscr{T})$ the combinatorics of $\mathscr{T}$.

Two tilings $\mathscr{T}$ and $\mathscr{T}'$ are called combinatorially equivalent if there is a bijection

$$\theta : W(\mathscr{T}) \to W(\mathscr{T}')$$

such that

$$\mathscr{M}(\mathscr{T})_\nu(\lambda, \mu) = \mathscr{M}(\mathscr{T}')_{\theta(\nu)}(\theta(\lambda), \theta(\mu))$$

for all $\lambda, \mu, \nu \in W(\mathscr{T})$. Then, we obtain the following.

Theorem 3. *Let $\mathscr{T}$ and $\mathscr{T}'$ be a couple of one dimensional tilings. Then, the following two conditions are equivalent.*
(1) $\mathscr{T} \sim_{l.i.} \mathscr{T}'$, *or* $\mathscr{T} \sim_{l.i.} {}^t\mathscr{T}'$.
(2) $\mathscr{T}$ *and* $\mathscr{T}'$ *are combinatorially equivalent.*

Proof. Since (1) easily implies (2), we only need to show (2) $\Rightarrow$ (1). The special letter ε can be uniquely determined as in the proof of Theorem 1. Therefore, $\theta(\varepsilon) = \varepsilon$. Hence, we also see $\theta(W_r(\mathscr{T})) = W_r(\mathscr{T}')$. Again using Theorem 1 and Theorem 2, we can reach that if $\lambda = Y_1 \dots Y_s \in S(\mathscr{T})$, then $\theta(Y_1) \dots \theta(Y_s) \in S(\mathscr{T}')$ or $\theta(Y_s) \dots \theta(Y_1) \in S(\mathscr{T}')$. Now we put

$$\begin{aligned} S(\mathscr{T})^+ = \{\lambda \in S(\mathscr{T}) \mid \lambda = {} & Y_1 Y_2 \dots Y_s, \theta(Y_1)\theta(Y_2)\dots\theta(Y_s) \in S(\mathscr{T}'), \\ & \theta(Y_s)\dots\theta(Y_2)\theta(Y_1) \notin S(\mathscr{T}')\}, \\ S(\mathscr{T})^0 = \{\lambda \in S(\mathscr{T}) \mid \lambda = {} & Y_1 Y_2 \dots Y_s, \theta(Y_1)\theta(Y_2)\dots\theta(Y_s) \in S(\mathscr{T}'), \\ & \theta(Y_s)\dots\theta(Y_2)\theta(Y_1) \in S(\mathscr{T}')\}, \\ S(\mathscr{T})^- = \{\lambda \in S(\mathscr{T}) \mid \lambda = {} & Y_1 Y_2 \dots Y_s, \theta(Y_1)\theta(Y_2)\dots\theta(Y_s) \notin S(\mathscr{T}'), \\ & \theta(Y_s)\dots\theta(Y_2)\theta(Y_1) \in S(\mathscr{T}')\}. \end{aligned}$$

We suppose that both $S(\mathscr{T})^+$ and $S(\mathscr{T}')^-$ are non-empty. We choose $b \in S(\mathscr{T})^+$ and $c \in S(\mathscr{T})^-$. Then we can also find an element $a \in S(\mathscr{T})$ such that b and c are subwords of a. If a lies in $S(\mathscr{T})^+$, then we have $c \in S(\mathscr{T})^0$, which is a contradiction. Similarly we see that a cannot belong to $S(\mathscr{T})^0$. If $a \in S(\mathscr{T})^-$, then we have $b \in S(\mathscr{T})^0$, which is also a contradiction. Therefore, we obtain that either $S(\mathscr{T})^+$ or $S(\mathscr{T})^-$ is empty. This means $S(\mathscr{T}) = S(\mathscr{T})^+ \cup S(\mathscr{T})^0$ or $S(\mathscr{T}) = S(\mathscr{T})^0 \cup S(\mathscr{T})^-$. Hence, we see that $\mathscr{T}$ and $\mathscr{T}'$ are locally indistinguishable if $S(\mathscr{T}) = S(\mathscr{T})^+ \cup S(\mathscr{T})^0$, or $\mathscr{T}$ and ${}^t\mathscr{T}'$ are locally indistinguishable if $S(\mathscr{T}) = S(\mathscr{T})^0 \cup S(\mathscr{T})^-$. We should explain it more precisely. We suppose first that $S(\mathscr{T}) = S(\mathscr{T})^+ \cup S(\mathscr{T})^0$. Then, we can confirm that every pattern $a = Y_1 Y_2 \dots Y_s \in S(\mathscr{T})$ appears in $S(\mathscr{T}')$ as $\theta(Y_1)\theta(Y_2)\dots\theta(Y_s)$. Let $b = \theta(c) \in S(\mathscr{T}')$ with $c = Z_1 Z_2 \dots Z_s \in S(\mathscr{T})$. If

$b = \theta(Z_1)\theta(Z_2)\dots\theta(Z_s)$, then the pattern of b appears in $S(\mathcal{T})$ as $\theta^{-1}(b) = c = Z_1Z_2\dots Z_s$. If $b = \theta(Z_s)\dots\theta(Z_2)\theta(Z_1)$, then $c = Z_1Z_2\dots Z_s \in S(\mathcal{T})^0$ and ${}^t b \in S(\mathcal{T}')$, and the bijectivity of θ implies that we can find $c' \in S(\mathcal{T})$ such that $\theta(c') = {}^t b$. Such an element c' must be ${}^t c$ by Theorem 1 or Theorem 2. Therefore, the pattern $Z_s\dots Z_2Z_1$ corresponding to b appears in $S(\mathcal{T})$ as $c' = {}^t c \in S(\mathcal{T})$. Thus, $\mathcal{T}$ and $\mathcal{T}'$ are locally indistinguishable. In the case when $S(\mathcal{T}) = S(\mathcal{T})^0 \cup S(\mathcal{T})^-$, we can similarly establish that $\mathcal{T}$ and ${}^t\mathcal{T}'$ are locally indistinguishable.

6. Tiling bialgebras

For a one dimensional tiling $\mathcal{T}$, we can construct the associated bialgebra, denoted here by $\mathfrak{B}(\mathcal{T})$ and called the tiling bialgebra. We shall review it. We consider a triplet (i, a, j), where $a \in S(\mathcal{T})$ and $1 \le i, j \le l(a)$. Put $\mathfrak{M}(\mathcal{T}) = \{\boldsymbol{e}, \boldsymbol{z}, (i, a, j) \mid a \in S(\mathcal{T}), 1 \le i, j \le l(a)\}$, where $\boldsymbol{e}$ and $\boldsymbol{z}$ are new abstract independent symbols. For $(i, a, j), (k, b, \ell) \in \mathfrak{M}(\mathcal{T})$, we define the product of (i, a, j) and (k, b, ℓ) as follows (cf. [8]). Pile up the j-th position of a and the k-th position of b. If one gets $c \in S(\mathcal{T})$ by this piling, then we define $(i, a, j) \cdot (k, b, \ell) = (p, c, q)$, where p is the position of c corresponding to i and q is the position of c corresponding to ℓ satisfying $1 \le p, q \le l(c)$. Otherwise, we define $(i, a, j) \cdot (k, b, \ell) = \boldsymbol{z}$. We also define $\boldsymbol{m} \cdot \boldsymbol{e} = \boldsymbol{e} \cdot \boldsymbol{m} = \boldsymbol{m}$ and $\boldsymbol{m} \cdot \boldsymbol{z} = \boldsymbol{z} \cdot \boldsymbol{m} = \boldsymbol{z}$ for all $\boldsymbol{m} \in \mathfrak{M}(\mathcal{T})$. Then, $\mathfrak{M}(\mathcal{T})$ becomes a monoid. Let $\mathbf{C}[\mathfrak{M}(\mathcal{T})] = \bigoplus_{\boldsymbol{m} \in \mathfrak{M}(\mathcal{T})} \mathbf{C}\boldsymbol{m}$ be the monoid bialgebra of $\mathfrak{M}(\mathcal{T})$ over the field $\mathbf{C}$ of complex numbers (cf. [1]). To avoid redundancy, we set $\mathfrak{B}(\mathcal{T}) = \mathbf{C}[\mathfrak{M}(\mathcal{T})]/\mathbf{C}\boldsymbol{z}$, the quotient bialgebra of $\mathbf{C}[\mathfrak{M}(\mathcal{T})]$ by $\mathbf{C}\boldsymbol{z}$. We also use the same notation (i, a, j) for $(i, a, j) \bmod \mathbf{C}\boldsymbol{z}$. Such a bialgebra has a triangular decomposition:

$$\mathfrak{B}(\mathcal{T}) = \mathfrak{B}(\mathcal{T})_- \oplus \mathfrak{B}(\mathcal{T})_0 \oplus \mathfrak{B}(\mathcal{T})_+.$$

Then, the following two conditions are equivalent (cf. [1], [10]).

(1) Two one dimensional tilings $\mathcal{T}$ and $\mathcal{T}'$ are locally indistinguishable: $\mathcal{T} \sim_{l.i.} \mathcal{T}'$.

(2) $\mathfrak{B}(\mathcal{T})$ and $\mathfrak{B}(\mathcal{T}')$ are isomorphic as bialgebras with triangular decompositions.

Here, we will give some improvement of this result in Theorem 4 below, using our previous discussion. A $\mathfrak{B}(\mathcal{T})$-module V is called standard if V is finite dimensional and the number of group-like elements of $\mathfrak{B}(\mathcal{T})$ acting on V nontrivially is finite. For $a \in S(\mathcal{T})$, we set

$$V_a = \mathfrak{B}(\mathcal{T}).(1, a, 1) \Bigg/ \left(\sum_{(i,b,j) \in \mathfrak{B}(\mathcal{T}).(1,a,1),\, b \neq a} \mathbf{C}(i, b, j) \right) = \bigoplus_{k=1}^{l(a)} \mathbf{C}\overline{(k, a, 1)},$$

and $V_\varepsilon = \mathbf{C}$ (a trivial module), where $\dim V_a = l(a)$ and $\dim V_\varepsilon = 1$. Then, we easily see that $\{V_\lambda \mid \lambda \in W(\mathscr{T})\}$ is a complete set of representatives of irreducible standard modules, and that every standard module is a direct sum of irreducible ones, since $\mathfrak{B}(\mathscr{T})$ acts on a standard module as a semisimple matrix algebra $\bigoplus_{i=1}^s M_{n_i}(\mathbf{C})$ for some $n_1, \dots, n_s$ (by our definition). In particular, we obtain the complete reducibility for standard modules. Furthermore, we have

$$V_\lambda \otimes V_\mu = \bigoplus\nolimits_{\nu \in W(\mathscr{T})} V_\nu^{\oplus \mathscr{M}(\mathscr{T})_\nu(\lambda,\mu)}$$

for $\lambda, \mu \in W(\mathscr{T})$, since each group-like element $\boldsymbol{g}$ acts on $V_\lambda \otimes V_\mu$ as $\boldsymbol{g} \otimes \boldsymbol{g}$.

Theorem 4. *Notation is as above. Then, the following two conditions are equivalent.*

(1) $\mathscr{T} \sim_{l.i.} \mathscr{T}'$, *or* $\mathscr{T} \sim_{l.i.} {}^t\mathscr{T}'$.

(2) $\mathfrak{B}(\mathscr{T}) \simeq \mathfrak{B}(\mathscr{T}')$ *or* $\mathfrak{B}(\mathscr{T}) \simeq \mathfrak{B}({}^t\mathscr{T}')$ *as bialgebras.*

Proof. $(1) \Rightarrow (2)$ is trivial. We need to show $(2) \Rightarrow (1)$. Suppose $\mathfrak{B}(\mathscr{T}) \simeq \mathfrak{B}(\mathscr{T}')$. Then, both structures of standard modules are equivalent. Hence, both combinatorics $\mathscr{M}(\mathscr{T})$ and $\mathscr{M}(\mathscr{T}')$ are equivalent. By Theorem 3, we obtain $\mathscr{T} \sim_{l.i.} \mathscr{T}'$, or $\mathscr{T} \sim_{l.i.} {}^t\mathscr{T}'$. In the case when $\mathfrak{B}(\mathscr{T}) \simeq \mathfrak{B}({}^t\mathscr{T}')$, we can show $\mathscr{T} \sim_{l.i.} {}^t\mathscr{T}'$, or $\mathscr{T} \sim_{l.i.} \mathscr{T}'$ similarly.

7. Combinatorial spectra

We already established several characterizations for patterns of words and local indistinguishability of tilings. This seems to be theoretically satisfactory. However, we sometimes need good invariants. How can we define them? Here we will present one approach using our combinatorics (or multiplicities) developed before. Namely, in this section, we would like to define a spectral map, called

$$f : a \to f_a(t) \in \mathbf{R}[[t]],$$

which gives a formal power series $f_a(t)$ in t with real coefficients, where $\mathbf{R}$ is the field of real numbers, for each word a, satisfying $f_a(t) = l(a)$ for $a = AA\dots A$. Using induction on $l(a)$, we will define the map f. We set $f_\varepsilon(t) = t$. For a word a, we put $D(a) = \{w(c) \mid c \in C(a,a), w(c) \neq a\}$. Now we consider the following equation in $f_a(t) = \sum_{i=0}^\infty c_i t^i$:

$$f_a(t)^2 = f_a(t) + \sum_{d \in D(a)} \mathscr{M}(a)_d(a,a) f_d(t),$$

and solve it as a formal power series in t with a positive constant term. This is a reccursive definition of our map f here, and the functional equation above is corresponding to $V_\lambda \otimes V_\lambda = \bigoplus_{\nu \in W(\mathcal{T})} V_\nu^{\oplus \mathcal{M}(\mathcal{T})_\nu(\lambda,\lambda)}$. We call $f_a(t)$ the spectral function of a.

We note that our $f_a(t)$ is well defined. The equation above means

$$\sum_{k=0}^{\infty}(c_0c_k + c_1c_{k-1} + \cdots + c_kc_0)t^k = \left(\sum_{k=0}^{\infty} c_kt^k\right) + \left(\sum_{k=0}^{\infty} b_kt^k\right),$$

where the b_i are inductively given and b_0 is nonnegative at least by our definition. Then, c_0 must satisfy $c_0^2 = c_0 + b_0$. Hence,

$$c_0 = \frac{1 \pm \sqrt{1+4b_0}}{2},$$

one is positive and another nonpositive. Therefore, we choose the positive c_0 by our assumption. More precisely, we see $c_0 \geq 1$. If $k > 0$ and $c_0, c_1, \ldots, c_{k-1}$ are defined, then we should solve $2c_0c_k + \cdots = c_k + b_k$ and we can uniquely obtain c_k. In particular, if $a = X_1X_2\ldots X_r$ and all $X_1, \ldots, X_r$ are distinct, then we see $f_a(t) = 1 + \sum_{i=1}^{\infty} c_it^i$ with $c_i \in \mathbf{Z}$.

We show a few simple examples: If $a = A$ with $l(a) = 1$, then we obtain $f_A(t) = \sum_{i=0}^{\infty} c_it^i \in \mathbf{R}[[t]]$ with $c_0 > 0$ satisfying

$$f_A(t)^2 = f_A(t),$$

which implies $f_A(t) = 1$. Inductively we also obtain $f_a(t) = n$ for $a = AA\ldots A$ with $l(a) = n$. If $a = AB$, then we should solve

$$f_a(t)^2 = f_a(t) + 2f_\varepsilon(t) = f_a(t) + 2t,$$

and we have $f_{AB}(t) = 1 + 2t - 4t^2 + 16t^3 - 80t^4 + \cdots$. If $a = AAB$, then we reach the following equation:

$$f_a(t)^2 = f_a(t) + 2f_A(t) + 4f_\varepsilon(t) = f_a(t) + 2 + 4t,$$

and in fact we see

$$f_{AAB}(t) = 2 + \frac{4}{3}t - \frac{16}{27}t^2 + \frac{128}{243}t^3 - \frac{1280}{2187}t^4 + \cdots.$$

If $a = AABB$, then we have the following equation:

$$f_a(t)^2 = f_a(t) + 2f_A(t) + 2f_B(t) + 8f_\varepsilon(t) = f_a(t) + 4 + 8t,$$

which shows the possibility of irrational coefficients, namely in this case we find

$$c_0 = \frac{1+\sqrt{17}}{2}.$$

Using this definition, we can define the functional spectrum of a as follows. If $a = X_1 X_2 \dots X_n$ with $l(a) = n$, then we put $a(i,j) = X_i X_{i+1} \dots X_j$ for each $1 \le i \le j \le n$. Then, we set $F(a) = \{f_{a(i,j)}(t) \mid 1 \le i \le j \le n\}$, and, for each $g(t) \in F(a)$, we also set

$$\mathrm{mult}(g(t)) = \#\{(i,j) \mid 1 \le i \le j \le n, f_{a(i,j)}(t) = g(t)\}.$$

Then, we define the functional spectrum of a as the set of elements in $F(a)$ with multiplicities. Namely,

$$\mathrm{Spec}_f(a) = \{g(t)[\mathrm{mult}(g(t))] \mid g(t) \in F(a)\}.$$

If $F(a) = \{g_1(t), g_2(t), \dots, g_k(t)\}$, then we sometimes denote $\mathrm{Spec}_f(a)$ by

$$\mathrm{Spec}_f(a) = \{g_1(t)[m_1], g_2(t)[m_2], \dots, g_k(t)[m_k]\},$$

where $m_i = \mathrm{mult}(g_i(t))$. We make a list of spectral functions $f_a(t)$ for words a of short lengths. We should also note $\mathrm{Spec}_f(a) = \mathrm{Spec}_f({}^t a)$.

Spectral Functions of Words (Examples)

Word	Spectral Function
A	1
AA	2
AB	$1 + 2t - 4t^2 + 16t^3 - 80t^4 + 488t^5 - 2688t^6 + 16896t^7 - 109824t^8 + 732160t^9 - 4978688t^{10} + \cdots$
AAA	3
AAB	$2 + \frac{4}{3}t - \frac{16}{27}t^2 + \frac{128}{243}t^3 - \frac{1280}{2187}t^4 + \frac{14336}{19683}t^5 - \frac{57344}{59049}t^6 + \frac{720896}{531441}t^7 - \frac{9371648}{4782969}t^8 + \frac{374865920}{129140163}t^9 - \frac{5098176512}{1162261467}t^{10} + \cdots$
ABA	$2 + \frac{4}{3}t - \frac{16}{27}t^2 + \frac{128}{243}t^3 - \frac{1280}{2187}t^4 + \frac{14336}{19683}t^5 - \frac{57344}{59049}t^6 + \frac{720896}{531441}t^7 - \frac{9371648}{4782969}t^8 + \frac{374865920}{129140163}t^9 - \frac{5098176512}{1162261467}t^{10} + \cdots$
ABC	$1 + 6t - 36t^2 + 432t^3 - 6480t^4 + 108864t^5 - 1959552t^6 + 36951552t^7 - 720555264t^8 + 14411105280t^9 - 293986547712t^{10} + \cdots$
$AAAA$	4
$AAAB$	$3 + \frac{6}{5}t - \frac{36}{125}t^2 + \frac{432}{3125}t^3 - \frac{1296}{15625}t^4 + \frac{108864}{1953125}t^5 - \frac{1959552}{48828125}t^6 + \frac{36951552}{1220703125}t^7 - \frac{720555264}{30517578125}t^8 + \frac{2882221056}{152587890625}t^9 - \frac{293986547712}{19073486328125}t^{10} + \cdots$
$AABA$	$3 + \frac{6}{5}t - \frac{36}{125}t^2 + \frac{432}{3125}t^3 - \frac{1296}{15625}t^4 + \frac{108864}{1953125}t^5 - \frac{1959552}{48828125}t^6 + \frac{36951552}{1220703125}t^7 - \frac{720555264}{30517578125}t^8 + \frac{2882221056}{152587890625}t^9 - \frac{293986547712}{19073486328125}t^{10} + \cdots$
$AABB$	$\frac{1+\sqrt{17}}{2} + \frac{8}{\sqrt{17}}t - \frac{64}{17\sqrt{17}}t^2 + \frac{1024}{289\sqrt{17}}t^3 - \frac{20480}{4913\sqrt{17}}t^4 + \frac{458752}{83521\sqrt{17}}t^5 - \frac{11010048}{1419857\sqrt{17}}t^6 + \frac{276824064}{24137569}t^7 - \frac{7197425664}{410338673\sqrt{17}}t^8 + \frac{191931351040}{6975757441\sqrt{17}}t^9 - \frac{307090161664}{6975757441\sqrt{17}}t^{10} + \cdots$
$AABC$	$2 + \frac{10}{3}t - \frac{100}{27}t^2 + \frac{2000}{243}t^3 - \frac{50000}{2187}t^4 + \frac{1400000}{19683}t^5 - \frac{14000000}{59049}t^6 + \frac{440000000}{531441}t^7 - \frac{14300000000}{4782969}t^8 + \frac{1430000000000}{129140163}t^9 - \frac{48620000000000}{1162261467}t^{10} + \cdots$
$ABAB$	$2 + 4t - 8t^2 + 32t^3 - 160t^4 + 896t^5 - 5376t^6 + 33792t^7 - 219648t^8 + 1464320t^9 - 9957376t^{10} + \cdots$
$ABAC$	$2 + \frac{10}{3}t - \frac{100}{27}t^2 + \frac{2000}{243}t^3 - \frac{50000}{2187}t^4 + \frac{1400000}{19683}t^5 - \frac{14000000}{59049}t^6 + \frac{440000000}{531441}t^7 - \frac{14300000000}{4782969}t^8 + \frac{1430000000000}{129140163}t^9 - \frac{48620000000000}{1162261467}t^{10} + \cdots$

Word	Spectral Function
$ABBA$	$\frac{1+\sqrt{17}}{2} + \frac{8}{\sqrt{17}}t - \frac{64}{17\sqrt{17}}t^2 + \frac{1024}{289\sqrt{17}}t^3 - \frac{20480}{4913\sqrt{17}}t^4 + \frac{458752}{83521\sqrt{17}}t^5 - \frac{11010048}{1419857\sqrt{17}}t^6 + \frac{276824064}{24137569}t^7 - \frac{7197425664}{410338673\sqrt{17}}t^8 + \frac{191931351040}{6975757441\sqrt{17}}t^9 - \frac{307090161664}{6975757441\sqrt{17}}t^{10} + \cdots$
$ABBC$	$2 + \frac{10}{3}t - \frac{100}{27}t^2 + \frac{2000}{243}t^3 - \frac{50000}{2187}t^4 + \frac{1400000}{19683}t^5 - \frac{14000000}{59049}t^6 + \frac{440000000}{531441}t^7 - \frac{14300000000}{4782969}t^8 + \frac{1430000000000}{129140163}t^9 - \frac{48620000000000}{1162261467}t^{10} + \cdots$
$ABCA$	$2 + \frac{10}{3}t - \frac{100}{27}t^2 + \frac{2000}{243}t^3 - \frac{50000}{2187}t^4 + \frac{1400000}{19683}t^5 - \frac{14000000}{59049}t^6 + \frac{440000000}{531441}t^7 - \frac{14300000000}{4782969}t^8 + \frac{1430000000000}{129140163}t^9 - \frac{48620000000000}{1162261467}t^{10} + \cdots$
$ABCD$	$1 + 12t - 144t^2 + 3456t^3 - 103680t^4 + 3483648t^5 - 125411328t^6 + 4729798656t^7 - 184462147584t^8 + 7378485903360t^9 - 301042224857088t^{10} + \cdots$

It seems to be good to have an invariant, like $\mathrm{Spec}_f(a)$. However, each formal power series usually contains infinitely many nonzero coefficients. This sounds rather large as a datum. Recall that we set $f_\varepsilon(t) = t$ as an initial condition to define our formal power series. Now we will try to solve our equation using real numbers. We fix a nonnegative real number $u \in \mathbf{R}_{\geq 0}$ and we would like to define a specialized spectral map (which is a kind of specialization $t \mapsto u$), called

$$\sigma_u : a \mapsto \sigma_u(a) \in \mathbf{R}_{\geq 0}.$$

First we define

$$\sigma_u(\varepsilon) = u.$$

For a word a, we will define $\sigma_u(a)$ by induction on $l(a)$. Let us consider the following quadratic equation:

$$x^2 = x + \sum_{d \in D(a)} \mathscr{M}_d(a, a)\sigma_u(d).$$

We choose its positive solution, called $\sigma_u(a)$. Then, the specialized spectrum of a is given by

$$\mathrm{Spec}|_u(a) = \{v[\mathrm{mult}(v)] \mid v \in V(a)\},$$

where $V(a) = \{\sigma_u(a(i,j)) \mid 1 \leq i < j \leq n\}$ for $a = X_1 X_2 \dots X_n$ with $l(a) = n$ and $\mathrm{mult}(v) = \#\{(i,j) \mid 1 \leq i \leq j \leq n, \sigma_u(a(i,j)) = v\}$ for $v \in V(a)$. If $V(a) = \{v_1, v_2, \dots, v_k\}$, then we sometimes denote $\mathrm{Spec}|_u(a)$ by

$$\mathrm{Spec}|_u(a) = \{v_1[m_1], v_2[m_2], \dots, v_k[m_k]\},$$

where $m_i = \mathrm{mult}(v_i)$. Here, for convenience, we take $u = \dfrac{\pi}{6}$, which is transcendental and near $\dfrac{1}{2}$, and we put $\sigma = \sigma_{\pi/6}$. Then, we can show some list of $\mathrm{Spec}|_{\pi/6}(a)$ of words a with short lengths as follows.

Specialized Spectra of Words (Example)

Word	Value [Multiplicity]
A	1 [1]
AA	2 [1], 1 [2]
AB	1.63895 [1], 1 [2]
AAA	3 [1], 2 [2], 1 [3]
AAB	2.58432 [1], 2 [1], 1.63895 [1], 1 [3]
ABA	2.58432 [1], 1.63895 [2], 1 [3]
ABC	2.34163 [1], 1.63895 [2], 1 [3]
$AAAA$	4 [1], 3 [2], 2 [3], 1 [4]
$AAAB$	3.56457 [1], 3 [1], 2.58432 [1], 2 [2], 1.63895 [1], 1 [4]
$AABA$	3.56457 [1], 2.58432 [2], 2 [1], 1.63895 [2], 1 [4]
$AABB$	3.40496 [1], 2.58432 [2], 2 [2], 1.63895 [1], 1 [4]
$AABC$	3.23605 [1], 2.58432 [1], 2.34163 [1], 2 [1], 1.63895 [2], 1 [4]
$ABAB$	3.27789 [1], 2.58432 [2], 1.63895 [3], 1 [4]
$ABAC$	3.23605 [1], 2.58432 [1], 2.34163 [1], 1.63895 [3], 1 [4]
$ABBA$	3.40496 [1], 2.58432 [2], 2 [1], 1.63895 [2], 1 [4]
$ABBC$	3.23605 [1], 2.58432 [2], 2 [1], 1.63895 [2], 1 [4]
$ABCA$	3.23605 [1], 2.34163 [2], 1.63895 [3], 1 [4]
$ABCD$	3.05601 [1], 2.34163 [2], 1.63895 [3], 1 [4]
$AAAAA$	5 [1], 4 [2], 3 [3], 2 [4], 1 [5]
$AAAAB$	4.55448 [1], 4 [1], 3.56457 [1], 3 [2], 2.58432 [1], 2 [3], 1.63895 [1], 1 [5]
$AAABA$	4.55448 [1], 3.56457 [2], 3 [1], 2.58432 [2], 2 [2], 1.63895 [2], 1 [5]
$AAABB$	4.31224 [1], 3.56457 [1], 3.40496 [1], 3 [1], 2.58432 [2], 2 [3], 1.63895 [1], 1 [5]
$AAABC$	4.18516 [1], 3.56457 [1], 3.23605 [1], 3 [1], 2.58432 [1], 2.34163 [1], 2 [2], 1.63895 [2], 1 [5]
$AABAA$	4.55448 [1], 3.56457 [2], 2.58432 [3], 2 [2], 1.63895 [2], 1 [5]
$AABAB$	4.21633 [1], 3.56457 [1], 3.27789 [1], 2.58432 [3], 2 [1], 1.63895 [3], 1 [5]
$AABAC$	4.18516 [1], 3.56457 [1], 3.23605 [1], 2.58432 [2], 2.34163 [1], 2 [1], 1.63895 [3], 1 [5]
$AABBA$	4.31224 [1], 3.40496 [2], 2.58432 [3], 2 [2], 1.63895 [2], 1 [5]
$AABBC$	4.05353 [1], 3.40496 [1], 3.23605 [1], 2.58432 [3], 2 [2], 1.63895 [2], 1 [5]
$AABCA$	4.18516 [1], 3.23605 [2], 2.58432 [1], 2.34163 [2], 2 [1], 1.63895 [3], 1 [5]
$AABCB$	4.05353 [1], 3.23605 [2], 2.58432 [2], 2.34163 [1], 2 [1], 1.63895 [3], 1 [5]
$AABCC$	4.05353 [1], 3.23605 [2], 2.58432 [2], 2.34163 [1], 2 [2], 1.63895 [2], 1 [5]
$AABCD$	3.91684 [1], 3.23605 [1], 3.05601 [1], 2.58432 [1], 2.34163 [2], 2 [1], 1.63895 [3], 1 [5]
$ABAAB$	4.21633 [1], 3.56457 [1], 3.40496 [1], 2.58432 [3], 2 [1], 1.63895 [3], 1 [5]
$ABAAC$	4.18516 [1], 3.56457 [1], 3.23605 [1], 2.58432 [3], 2 [1], 1.63895 [3], 1 [5]
$ABABA$	4.2016 [1], 3.27789 [2], 2.58432 [3], 1.63895 [4], 1 [5]
$ABABC$	3.95043 [1], 3.27789 [1], 3.23605 [1], 2.58432 [2], 2.34163 [1], 1.63895 [4], 1 [5]
$ABACA$	4.18516 [1], 3.23605 [2], 2.58432 [2], 2.34163 [1], 1.63895 [4], 1 [5]
$ABACB$	4.05353 [1], 3.23605 [2], 2.58432 [1], 2.34163 [2], 1.63895 [4], 1 [5]
$ABACD$	3.91684 [1], 3.23605 [1], 3.05601 [1], 2.58432 [1], 2.34163 [2], 1.63895 [4], 1 [5]
$ABBAC$	4.05353 [1], 3.40496 [1], 3.23605 [1], 2.58432 [2], 2.34163 [1], 2 [1], 1.63895 [3], 1 [5]
$ABBBA$	4.31224 [1], 3.56457 [2], 3 [1], 2.58432 [2], 2 [2], 1.63895 [2], 1 [5]
$ABBBC$	4.18516 [1], 3.56457 [2], 3 [1], 2.58432 [2], 2 [2], 1.63895 [2], 1 [5]
$ABBCA$	4.05353 [1], 3.23605 [2], 2.58432 [2], 2.34163 [1], 2 [1], 1.63895 [3], 1 [5]

Word	Value [Multiplicity]
$ABBCD$	3.91684 [1], 3.23605 [2], 2.58432 [2], 2.34163 [1], 2 [1], 1.63895 [3], 1 [5]
$ABCAB$	3.95043 [1], 3.23605 [2], 2.34163 [3], 1.63895 [4], 1 [5]
$ABCAD$	3.91684 [1], 3.23605 [1], 3.05601 [1], 2.34163 [3], 1.63895 [4], 1 [5]
$ABCBA$	4.05353 [1], 3.23605 [2], 2.58432 [1], 2.34163 [2], 1.63895 [4], 1 [5]
$ABCBD$	3.91684 [1], 3.23605 [2], 2.58432 [1], 2.34163 [2], 1.63895 [4], 1 [5]
$ABCDA$	3.91684 [1], 3.05601 [2], 2.34163 [3], 1.63895 [4], 1 [5]

On the other hand, if we consider the spectra of tilings, then in general the multiplicities do not make any sense, that is, the multiplicity might be infinite. Therefore, for a one dimensional tiling $\mathscr{T}$, we define

$$\mathrm{Spec}_f(\mathscr{T}) = \{f_a(t) \mid a \in S(\mathscr{T})\} \qquad \text{and} \qquad \mathrm{Spec}|_u(\mathscr{T}) = \{\sigma_u(a) \mid a \in S(\mathscr{T})\},$$

which are called the functional simple spectrum of $\mathscr{T}$ and the specialized simple spectrum of $\mathscr{T}$ respectively. Both are infinite sets without multiplicities. Our definitions say that Spec_f and $\mathrm{Spec}|_u$ give invariants of locally indistinguishable classes of tilings. We will give two trivial examples here. If $a = AA\dots A$ with $l(a) = n$, then

$$\mathrm{Spec}_f(a) = \mathrm{Spec}|_u(a) = \{n[1], n-1[2], \dots, 2[n-1], 1[n]\}.$$

If $\mathscr{T} = \dots AAA \dots$, then

$$\mathrm{Spec}_f(\mathscr{T}) = \mathrm{Spec}|_u(\mathscr{T}) = \{\dots, n, n-1, \dots, 2, 1\} = \mathbf{N},$$

where $\mathbf{N}$ is the set of all natural numbers.

8. Higher dimensional tilings

One can easily imagine that our definition of $\mathrm{Spec}_f(\mathscr{T})$ for a one dimensional tiling $\mathscr{T}$ can be generalized to higher dimensional cases. In fact, even for higher dimensional tilings, we can also define their simple spectra as infinite sets of formal power series or positive real numbers, which are again invariants of locally indistinguishable classes as well as invariants modulo affine transformations $AT(\mathbf{R}^n)$.

Let $\mathscr{T}$ be a tiling of $\mathbf{R}^n$. That is, a tiling $\mathscr{T}$ of $\mathbf{R}^n$ is an infinite set of tiles, $T_\xi (\xi \in \Xi)$ with an index set Ξ, which cover $\mathbf{R}^n$ overlapping, at most, at their boundaries, where a tile T in $\mathbf{R}^n$ is a connected closed bounded subset of $\mathbf{R}^n$ satisfying

(T1) its interior T° is connected,

(T2) the closure of T° coincides with T.

A finite subset $a = \{T_1, \dots, T_k\}$ of a tiling $\mathscr{T}$ is called a patch if the interior of $\bigcup_{i=1}^k T_i$ is connected. We denote by $P(\mathscr{T})$ the set of all patches obtained from $\mathscr{T}$. We say that two tiles T and T' are equivalent if there is a vector

$\boldsymbol{x} \in \mathbf{R}^n$ such that $T + \boldsymbol{x} = T'$, where $T + \boldsymbol{x} = \{\boldsymbol{t} + \boldsymbol{x} \mid \boldsymbol{t} \in T\}$. Also we say that two patches a and a' are equivalent if there is a vector $\boldsymbol{x} \in \mathbf{R}^n$ such that $a + \boldsymbol{x} = a'$, where $a + \boldsymbol{x} = \{T + \boldsymbol{x} \mid T \in a\}$. Let $[T]$ (resp. $[a]$) be the equivalence class of tiles (resp. patches) containing T (resp. a), and let $[P(\mathscr{T})]$ be the set of all equivalence classes of patches. Let $a = \{T_1, \ldots, T_k\}$ be a patch. Then, a subset $\alpha = \{(T_{i_1}, T_{j_1}), \ldots, (T_{i_r}, T_{j_r})\}$ of $a \times a$ is called a diagonal patch in $a \times a$ if there is a vector $\boldsymbol{x} \in \mathbf{R}^n$ such that $T_{i_s} + \boldsymbol{x} = T_{j_s}$ for all $s = 1, \ldots, r$ and such that $\{T_{i_1}, \ldots, T_{i_r}\}$ is a patch. We put $[\alpha] = [\{T_{i_1}, \ldots, T_{i_r}\}]$ as a patch class, and we say that $[\alpha]$ is the patch type of α. Let $\mathscr{G}(a)$ be the set of all diagonal patches in $a \times a$, and let $\mathscr{C}(a)$ be the set of all maximal diagonal patches in $\mathscr{G}(a)$. Then, we put $\mathscr{D}(a) = \{\alpha \in \mathscr{C}(a) \mid [\alpha] \neq [a]\}$, and $q(a) = k^2 - \sum_{\alpha \in \mathscr{C}(a)} \#\alpha$, where $k = \#a$. Now we want to define the associated spectral map

$$f : [P(\mathscr{T})] \to \mathbf{R}[[t]] \qquad \text{with } f([a]) = f_a(t),$$

and the corresponding functional simple spectrum

$$\begin{aligned}\mathrm{Spec}_f(\mathscr{T}) &= \{f([a]) \mid [a] \in [P(\mathscr{T})]\} \\ &= \{f_a(t) \mid [a] \in [P(\mathscr{T})]\}\end{aligned}$$

of $\mathscr{T}$. As in Section 7, we should solve the equation

$$f_a(t)^2 = f_a(t) + \sum_{\beta \in \mathscr{D}(a)} f([\beta]) + q(a)t$$

in terms of $f_a(t) = \sum_{i=0}^{\infty} c_i t^i \in \mathbf{R}[[t]]$ with $c_0 > 0$. This reccursive definition gives our desired spectral map f, which can also imply a combinatorial way to define $\mathrm{Spec}_f(\mathscr{T})$, the functional simple spectrum of $\mathscr{T}$. Also, we can recursively define the specialized simple spectrum of $\mathscr{T}$ by

$$\mathrm{Spec}|_u(\mathscr{T}) = \{\sigma_u(a) \mid a \in S(\mathscr{T})\}.$$

9. Genome and tiling Examples

In this section, we will try to compute

$$f_a(t), \qquad \sigma(a) \qquad \text{or} \qquad \mathrm{Spec}|_{\pi/6}(a)$$

for some words a arising from Genomes and tilings. First we take a Rice Yellow Mottle Virus Satellite ssRNA as follows. The data is from GenBank in NCBI USA: $a =$

```
 1  CCAGCUGCGC  AGGGGGCGGA  GAUUUUGUUU  CGAGCCUUAC  CGACACUGAU
51  GAGCCAAGAG  GAACUUGGAG  GCACCCAGGA  AUUUCACCCG  GGUCGACCUG
```

```
101  GGCGGCUAGG  AGCCGUGCAC  AGGGCGUCGC  UGUGGAGCGA  GCCUGGCCUC
151  CAAGGGGCCU  GGAGGCGAAA  CCGGUCUGUU  GGGACCACUC  GGACCAUCAG
201  UCAUCGUGCU  CCGGCAGCUU  .
```

Then, we have $f_a(t) = \dfrac{1+\sqrt{\xi}}{2} + \left(\dfrac{1764112}{45\sqrt{\xi}} + \dfrac{432}{\sqrt{17\xi}} + \dfrac{136}{\sqrt{33\xi}}\right)t + \cdots$, where $\xi = 43057 + 68\sqrt{17} + 12\sqrt{33}$, and we obtain $\sigma(a) = 175.46608222745476$ and

$$\mathrm{Spec}|_{\pi/6}(a)$$

looks like

$$
\begin{aligned}
\{\quad & 175.46608222745476\ [1], 174.7062159380592\ [1], 174.64180445742593\ 1],\\
& 173.9738718957113\ [1], 173.88134559021285\ [1], 173.84391800343542\ [1],\\
& \cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
& 6.558261854738853\ [4], 6.544268798389065\ [4], 6.539192217329748\ [9],\\
& 6.530140424396209\ [1], 6.5200621736929385\ [42], 6.498368804475771\ [1],\\
& 6.472106634919677\ [1], 6.459785246890427\ [25], 6.450612665023148\ [1],\\
& \cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
& 3.0560096453612196\ [14], \quad 3\ [16], \quad 2.5843212570026712\ [130],\\
& 2.3416277185114787\ [72], 2\ [64], 1.6389458069621212\ [155], 1\ [220]\quad \}.
\end{aligned}
$$

Next, we take the following important Human Gene called SRY, which determines SEX and appears on the chromosome Y of XX (female) and XY (male). The data is from IEBI Ensembl Transcript Report: $a =$

```
  1  ATGCAATCAT  ATGCTTCTGC  TATGTTAAGC  GTATTCAACA  GCGATGATTA
 51  CAGTCCAGCT  GTGCAAGAGA  ATATTCCCGC  TCTCCGGAGA  AGCTCTTCCT
101  TCCTTTGCAC  TGAAAGCTGT  AACTCTAAGT  ATCAGTGTGA  AACGGGAGAA
151  AACAGTAAAG  GCAACGTCCA  GGATAGAGTG  AAGCGACCCA  TGAACGCATT
201  CATCGTGTGG  TCTCGCGATC  AGAGGCGCAA  GATGGCTCTA  GAGAATCCCA
251  GAATGCGAAA  CTCAGAGATC  AGCAAGCAGC  TGGGATACCA  GTGGAAAATG
301  CTTACTGAAG  CCGAAAAATG  GCCATTCTTC  CAGGAGGCAC  AGAAATTACA
351  GGCCATGCAC  AGAGAGAAAT  ACCCGAATTA  TAAGTATCGA  CCTCGTCGGA
401  AGGCGAAGAT  GCTGCCGAAG  AATTGCAGTT  TGCTTCCCGC  AGATCCCGCT
451  TCGGTACTCT  GCAGCGAAGT  GCAACTGGAC  AACAGGTTGT  ACAGGGATGA
501  CTGTACGAAA  GCCACACACT  CAAGAATGGA  GCACCAGCTA  GGCCACTTAC
551  CGCCCATCAA  CGCAGCCAGC  TCACCGCAGC  AACGGGACCG  CTACAGCCAC
601  TGGACAAAGC  TGTAG.
```

Then, we have $\sigma(a) = 487.18739815010457$.

Here we will draw the graph of $\mathrm{Spec}|_{\pi/6}$ for the Rice Yellow Mottle Virus Satellite ssRNA as follows.

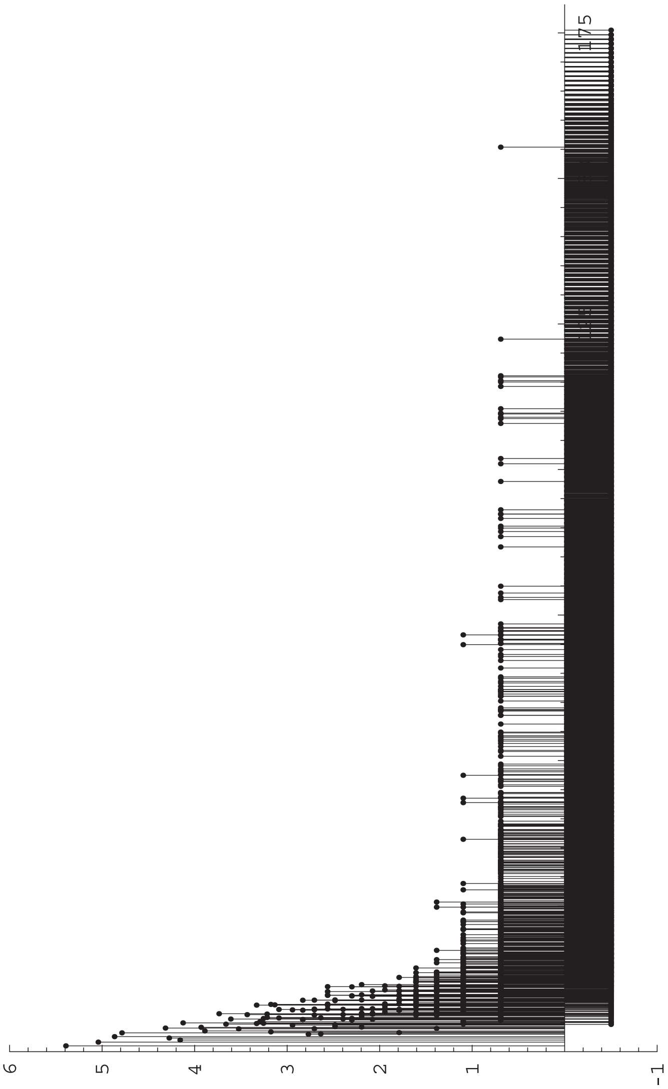

Fig. 1. A plot of $\mathrm{Spec}|_{\pi/6}(a)$ of the Rice Yellow Mottle Virus Satellite ssRNA shown as in Section 9. Note that the y-axis is plotted in the log scale with special arrangement, see the text for detail.

Fig. 1 is a plot of $\mathrm{Spec}|_{\pi/6}(a)$ of the Rice Yellow Mottle Virus Satellite ssRNA shown as in the above, in which we have plotted $\sigma(s)$ in the x-axis and

$$\begin{cases} \log(\mathrm{mult}(f_s(t))) & \text{for } \mathrm{mult}(f_s(t)) > 1, \\ -0.5 & \text{for } \mathrm{mult}(f_s(t)) = 1, \end{cases}$$

in the y-axis, for $s \in S(a)$.

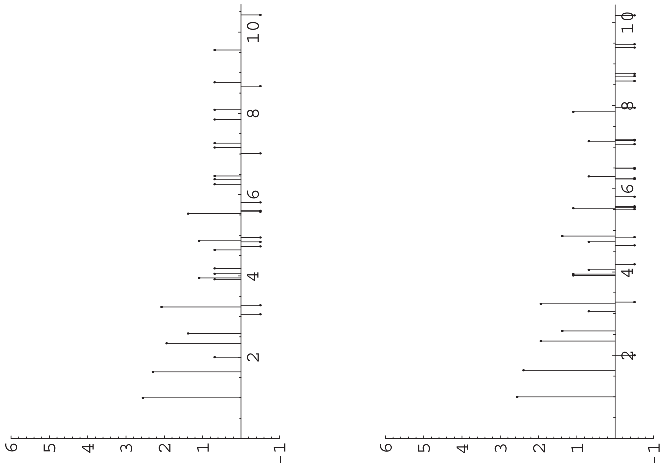

Fig. 2. Virus b (left) and SRY c (right)

Also we take four words with length 13 and compare them:

$b = CCAGCUGCGCAGG$
the first 13 letters in Rice Yellow Mottle Virus

$c = ATGCAATCATATG$
the first 13 letters in Human Gene SRY

$d = ABAABABAABAAB$
the first 13 letters in Fibonacci Tiling

$e = ABBABAABBAABA$
the first 13 letters in Thue-Morse Tiling

Then, we will compute $f_z(t)$ and $\mathrm{Spec}|_{\pi/6}(z)$ for $z \in \{b, c, d, e\}$ as follows.

$$f_b(t) = \frac{1+\sqrt{129}}{2} + \frac{46\sqrt{129}}{43}t - \frac{3492\sqrt{129}}{1849}t^2 + \cdots$$

$$f_c(t) = 5 + \frac{52}{3}t - \frac{3784}{81}t^2 + \frac{650144}{2187}t^3 - \frac{173262752}{59049}t^4 + \cdots$$

$$f_d(t) = 8 + \frac{1672}{225}t - \frac{4584304}{759375}t^2 + \frac{34550182976}{2562890625}t^3 + \cdots$$

$$f_e(t) = \frac{1+\sqrt{177+8\sqrt{17}}}{2} + \frac{96+332\sqrt{17}}{3\sqrt{177+8\sqrt{17}}}t + \cdots$$

We will draw $\mathrm{Spec}|_{\pi/6}$ of the above four sequences in Fig. 2 and Fig. 3.

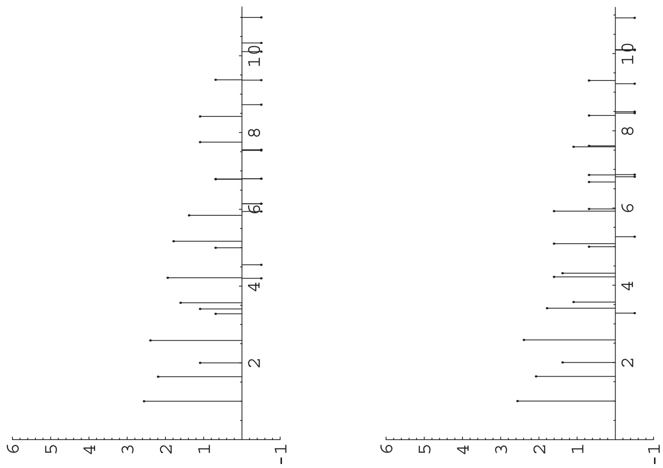

Fig. 3. Fibonacci d (left) and Thue-Morse e (right)

Observation. Let a be a word. Then: (1) $f_a(t) = l(a)$ if $\#\Omega(a) = 1$. (2) $f_a(t) = 1 + \sum_{i=1}^{\infty} a_i t^i \in \mathbf{Z}[[t]]$ if $\#\Omega(a) = l(a)$.

(3) The coefficients of the above $f_b(t)$ and $f_e(t)$ are likely to be irrational and the coefficients of the above $f_c(t)$ and $f_d(t)$ seem to be rational. It is very interesting to study how the irrationality appears in the coefficients of $f_a(t)$.

10. Remarks

There are many mathematical approaches to quasicrystals and aperiodic orders including interesting tilings (cf. [2], [3], [6], [7], [8], [9]). Especially in [8], some K-theoretical approach is given. In [10], we already found that a couple of one-dimensional tilings $\mathscr{T}$ and $\mathscr{T}'$ are locally indistinguishable (or locally nondistinguishable) if and only if the corresponding bialgebras with triangular decompositions are isomorphic in the sense that the corresponding isomorphism preserves their triangular decompositions. This is refined in this paper. Also we obtained groups and Lie algebras associated with one dimensional tilings, and we have seen that tiling groups have Gauss decompositions, and that tiling Lie algebras have additive Gauss decompositions (cf. [5]). We hope that our method here could have some good application to Bioinformatics as well as Material Science. Accidentally the first several coefficients of the Rice Yellow Mottle Virus are irrational, and the first several coefficients of Human Gene SRY are rational in our examples. We can also define a certain irrationality of a word a. For example, if $f_a(t) = \sum_{i=0}^{\infty} c_i t^i$, then we set $K_a = \mathbf{Q}(c_0, c_1, c_2, \ldots)$, a field extension of $\mathbf{Q}$. Then, we could reach a new application to Bioinformatics using pure mathematics, which is our hope near future (cf. [4], [11]). We obtained our data using Mathematica Computing System (cf. [12]).

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