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Remarks on 2-dimensional quasiperiodic tilings with rotational symmetries

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Abstract. We construct a sequentially compact space of patches. By using this construction, we analyze certain symmetries of tilings obtained by substitution rules.

1. Introduction

In 1982 a quasi-crystal with 5-fold rotational symmetry was discovered by Shechtman et al. (published in 1984 [13]). Before that, it had been believed that the structure of crystals was periodic, like a wallpaper pattern. Periodicty is another name for translational symmetry. 5-fold rotational symmetry is incompatible with translational symmetry and therefore quasi-crystals are not periodic. The most famous 2-dimensional mathematical model for a quasi-crystal would be a Penrose tiling with 5-fold rotational symmetry ([9], [10]). In addition, there are an Ammann-Beenker tiling with 8-fold rotational symmetry ([1], [2]) and a Danzer tiling with 7-fold rotational symmetry ([8]) in typical tilings.

We prepare several basic definitions (cf. [12]). A tiling $\mathscr{T}$ of the 2-dimensional Euclidean space $\mathbf{R}^2$ is a countable family of polygons T_i called tiles: $\mathscr{T} = \{T_i \mid i = 1, 2, \dots\}$ such that $\bigcup_{i=1}^{\infty} T_i = \mathbf{R}^2$ and $\mathrm{Int}\, T_i \cap \mathrm{Int}\, T_j = \emptyset$ if $i \neq j$. A nonperiodic tiling is one that admits no translation isomorphisms to itself. A quasiperiodic tiling is a nonperiodic tiling such that each local configuration of finitely many tiles in the tiling appears infinitely often.

Let $\mathscr{S} = \{S_1, S_2, \dots, S_l\}$ be a finite set of polygons S_j. When each tile T in a tiling $\mathscr{T}$ is congruent to some $S_i \in \mathscr{S}$, $\mathscr{S}$ is called a prototile set of $\mathscr{T}$.

The substitution rule is one of methods to construct quasiperiodic tilings with a given prototile set. Fix $\lambda (> 1)$. For $\mathscr{S} = \{S_1, S_2, \dots, S_l\}$, any S_k $(\in \mathscr{S})$ is decomposed into λ^{-1} scale-down copies $\lambda^{-1}\mathscr{S} =$

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$\{\lambda^{-1}S_1, \lambda^{-1}S_2, \dots, \lambda^{-1}S_l\}$ of $\mathscr{S}$. This decomposition is called a substitution rule of $\mathscr{S}$ if such a decomposition is possible. Let Φ denote a substitution rule of $\mathscr{S}$, and $\Phi[S_k]$ denote the decomposition of $S_k \in \mathscr{S}$. Similarly, let $\hat{\Phi}[S_k]$ denote the decomposition of λS_k into $\mathscr{S}$.

DEFINITION. A patch P is defined to be a set $P = \{T_\alpha\}_{\alpha \in A}$ of finite or countably infinitely many polygons which satisfies the following conditions (1)–(4):

(1) $\operatorname{Int} T_\alpha \cap \operatorname{Int} T_{\alpha'} = \varnothing \ (\alpha \neq \alpha')$,

(2) $\bigcup_{\alpha \in A} T_\alpha$ is connected,

(3) $\left(\bigcup_{\alpha \in A} T_\alpha\right) \cap (U_\varepsilon(v) - \{v\})$ is connected for any $\varepsilon > 0$ and any vertex v,

where $U_\varepsilon(v) = \{w \in \mathbf{R}^2 \mid d(w, v) < \varepsilon\}$, d is Euclidean metric of $\mathbf{R}^2$,

(4) (edge to edge) No vertex of tiles is in the interiors of edges of other tiles.

Note that a tiling is a patch of countably infinitely many tiles, and that any of possible patches does not necessarily appear in a tiling.

A patch $P = \{T_\alpha\}_{\alpha \in A}$ with a prototile set $\mathscr{S} = \{S_1, S_2, \dots, S_l\}$ is defined to be a patch such that there exists a polygon $S_j (\in \mathscr{S})$ congruent to T_α for any $T_\alpha (\in P)$.

DEFINITION. We define the space $SP(\mathscr{S})$ of patches with a prototile set $\mathscr{S}$ by the following:

$$SP(\mathscr{S}) = \{P \mid O \in {}^{\exists}T \in P,\ P \text{ is a patch with } \mathscr{S}\},$$

where $O = (0, 0) \in \mathbf{R}^2$ is the origin. We give a metric h in $SP(\mathscr{S})$ by using Hausdorff distance H: For any $P_1, P_2 \in SP(\mathscr{S})$,

$$h(P_1, P_2) = \inf\{\varepsilon \mid H(\partial_\varepsilon(P_1), \partial_\varepsilon(P_2)) \leq \varepsilon\},$$

where $\partial_\varepsilon(P) = \bigcup_{T \in P}(\partial(T) \cap C_{1/\varepsilon})$. $\partial(T)$ is the boundary set of T and $C_{1/\varepsilon} = \{w \in \mathbf{R}^2 \mid d(w, O) \leq 1/\varepsilon\}$.

A continuous operator $\hat{\Phi} : SP(\mathscr{S}) \to SP(\mathscr{S})$ is induced by docomposition $\hat{\Phi}$ for each tile in a patch. A patch P is called a periodic point of $\hat{\Phi}$ when there exists a positive integer m such that $\hat{\Phi}^m(P) = P$.

Due to [11] the space $V(\mathscr{S})$ of tilings from a given prototile set $\mathscr{S}$ is a sequencial compact metric space. We see that $V(\mathscr{S})$ is a closed subspace in $SP(\mathscr{S})$.

THEOREM. *The space $SP(\mathscr{S})$ of patches with a prototile set $\mathscr{S}$ is a sequencial compact metric space.*

We remark that Theorem remains true in 3 and higher dimensional cases since the proof of Theorem in the section 2 works well by changing a little.

One of the purpose of this paper is to analyze certain symmetries of tilings obtained by using substitution rules:

Corollary. *Let $\mathscr{S}$ be a prototile set with a substitution rule Φ. If we have a prototile with a vertex of an angle π/n, then there exists a tiling $\mathscr{T}_n$ with n-fold rotational symmetry, which is a periodic point of $\hat{\Phi}$.*

Furthermore, for an integer n with $n = 5$ or $6 < n$, its tiling $\mathscr{T}_n$ is nonperiodic and has a non-trivial action of a non-crystallographic Coxeter group.

Note that a tiling $\mathscr{T}$ is an accumulation point in $SP(\mathscr{S})$, and that a patch P is an accumulation point in $SP(\mathscr{S})$ if P is a periodic point of $\hat{\Phi}$.

This note is arranged as follows. In the section 2 we prove Theorem and Corollary. In the section 3 we demonstrate some examples of tilings with rotational symmetries.

2. Proof of Theorem and Corollary

Proof of Theorem. We take any sequence $\{P_i\}$ with $P_i = \{T(i)_\alpha\}_{\alpha \in A(i)}$ $(i = 1, 2, \dots)$ in $SP(\mathscr{S})$. Because a prototile set $\mathscr{S}$ is a finite set, for some prototile S the sequence $\{P_i\}$ contains infinitely many patches P_k such that $O \in {}^{\exists}T_k \in P_k$ and that T_k and S are congruent by an orientation preserving isometry. By taking a subsequence and renumbering we have $\{P_i\}_{i=1,2,\dots}$ which satisfies that $O \in {}^{\exists}T_i \in P_i$ and that T_i and S are congruent by an orientation preserving isometry.

Due to a special case of the Selection Theorem [5, p. 154], if all T_j's have some point in common and are congruent to a tile T' for an infinite sequence $\{T_j\}$ of tiles, then some subsequence of $\{T_j\}$ converges to a tile congruent to a tile T' in terms of Hausdorff distance. By taking a subsequence we can assume that congruent transformations are orientation preserving isometries.

In addition, we can take a subsequence $\{P_{i_k}\}$ which satisfies one of the following (a)–(c);

(a) For any T_{i_k} with $O \in T_{i_k} \in P_{i_k}$, T_{i_k} and $T_\infty = \lim_{k \to \infty} T_{i_k}$ are congruent by some translation.

(b) For any T_{i_k} with $O \in T_{i_k} \in P_{i_k}$, T_{i_k} and T_∞ are congruent by some rotational transformation t_k and for the center p_k of t_k, $\{p_k\}$ converges to a point $p \in \mathbf{R}^2$.

(c) For any T_{i_k} with $O \in T_{i_k} \in P_{i_k}$, T_{i_k} and T_∞ are congruent by some rotational transformation t_k and for the center p_k of t_k, $\{p_k\}$ is divergent.

We consider a patch P such that $O \in T_\infty \in P$. Let $\hat{P}_{i_k}$, $\hat{P}$ be subpatches of P_{i_k}, P respectively, that is, $\hat{P}_{i_k}$ and $\hat{P}$ are patches such that $\hat{P}_{i_k} \subset P_{i_k}$ and $\hat{P} \subset P$. Now, suppose that for any $\varepsilon > 0$ there exists an integer K such that $\hat{P}_{i_k}$ and $\hat{P}$ have the same configuration of tiles and that $C_{1/\varepsilon} \cap (\bigcup_{T \in P_{i_k}} T) \subset \bigcup_{T \in \hat{P}_{i_k}} T$ and $C_{1/\varepsilon} \cap (\bigcup_{T \in P} T) \subset \bigcup_{T \in \hat{P}} T$ for all $k \geq K$, where $C_{1/\varepsilon} = \{w \in \mathbf{R}^2 \mid d(w, O) \leq 1/\varepsilon\}$. If T_{i_k} and T_∞ are congruent by an orientation preserving isometry, then $\bigcup_{T \in \hat{P}_{i_k}} \partial(T)$ and $\bigcup_{T \in \hat{P}} \partial(T)$ are isometric by the same isometry.

In the case of (a), for any $\varepsilon > 0$ we get that $H(\partial_\varepsilon(P_{i_k}), \partial_\varepsilon(P)) \leq \varepsilon$ for all $k \geq K$ if $H(T_{i_k}, T_\infty) \leq \varepsilon$ for Hausdorff distance H. Hence we obtain that $h(P_{i_k}, P) \leq \varepsilon$ by the definition of h.

In the case of (b), for any $\varepsilon > 0$ there exists an integer M_1 such that $d(p_k, p) < r/2$ for all $k \geq M_1$, where $r = \sup\{d(p, x) \mid x \in \partial(T_\infty)\}$. We put $r_k = \sup\{d(p_k, x) \mid x \in \partial(T_\infty)\}$, $R_k = \sup\{d(p_k, x) \mid x \in \partial_\varepsilon(P_{i_k}) \cup \partial_\varepsilon(P)\}$ and $R = \sup\{d(p, x) \mid x \in \partial(C_{1/\varepsilon})\}$. Then we see that $(r/2) < r_k$ and $R + (r/2) > R_k$. Since $T_\infty = \lim_{k \to \infty} T_{i_k}$, we can take an integer M_2 such that $H(T_{i_k}, T_\infty) \leq (r/(2R + r))\varepsilon$ for all $k \geq M_2$. For $M = \max\{K, M_1, M_2\}$ we get that $H(\partial_\varepsilon(P_{i_k}), \partial_\varepsilon(P)) \leq \varepsilon$ for all $k \geq M$. Hence we obtain that $h(P_{i_k}, P) \leq \varepsilon$ by the definition of h.

In the case of (c), for any $\varepsilon > 0$ there exists an integer N_1 such that $d(p_k, O) > 1/\varepsilon$ for all $k \geq N_1$. Without the loss of generality, we can assume that $d(p_i, O) < d(p_{i+1}, O)$. Then we see that $d(p_{N_1}, O) < r_k$ and $2d(p_{N_1}, O) > R_k$. Since $T_\infty = \lim_{k \to \infty} T_{i_k}$, we can take an integer N_2 such that $H(T_{i_k}, T_\infty) \leq \varepsilon/2$ for all $k \geq N_2$. For $N = \max\{K, N_1, N_2\}$ we get that $H(\partial_\varepsilon(P_{i_k}), \partial_\varepsilon(P)) \leq \varepsilon$ for all $k \geq N$. Hence we obtain that $h(P_{i_k}, P) \leq \varepsilon$ by the definition of h.

Thus, the rest of the proof is only to choose a subsequence $\{P_{i_k}\} \subset \{P_i\}$ with the same configuration in larger and larger subpatches of finitely many tiles. By the definition of h, it suffices to choose a convergent subsequence of $\{P_i\}$ in the special case where P_i contains the common tile T_0 with $O \in T_0$. We choose a subsequence $\{P_{\ell_k}\}_{k=1,2,\ldots}$ of $\{P_i\}$ inductively. We put $P_i^{(1)} = \{T(i)_\alpha \in P_i \mid C_1 \cap T(i)_\alpha \neq \emptyset\}$, where $C_1 = \{w \in \mathbf{R}^2 \mid d(w, O) \leq 1\}$. For any $L > 0$ there are only finitely many patches $P = \{T_\alpha\}_{\alpha \in A}$ which satisfy that $T_0 \in P = \{T_\alpha\}_{\alpha \in A}$ and $C_L \cap T_\alpha \neq \emptyset$ for all $\alpha \in A$, where $C_L = \{w \in \mathbf{R}^2 \mid d(w, O) \leq L\}$. Hence $\{P_i^{(1)}\}_{i=1,2,\ldots}$ is a finite set and there exists $P_{i_0}^{(1)}$ such that $|\{j \mid P_{i_0}^{(1)} \subset P_j\}| = \infty$. We put $\Lambda_1 = \{j \mid P_{i_0}^{(1)} \subset P_j\}$ and $\ell_1 = \min \Lambda_1$. So, we choose P_{ℓ_1}.

Assume that we have Λ_{k-1} and ℓ_{k-1}. We put a subpatch $P_i^{(k)} = \{T(i)_\alpha \in P_i \mid C_k \cap T(i)_\alpha \neq \emptyset\}$ for each $i \in \Lambda_{k-1} - \{\ell_{k-1}\}$, where $C_k = \{w \in \mathbf{R}^2 \mid d(w, O) \leq k\}$. By the above argument, $\{P_i^{(k)}\}$ is a finite set and there exists $P_{i_0}^{(k)}$ such that $|\{j \mid P_{i_0}^{(k)} \subset P_j, j \in \Lambda_{k-1} - \{\ell_{k-1}\}\}| = \infty$. We put $\Lambda_k =$

$\{j \mid P_{i_0}^{(k)} \subset P_j, j \in \Lambda_{k-1} - \{\ell_{k-1}\}\}$ and $\ell_k = \min \Lambda_k$. So, we choose P_{ℓ_k}. Hence we can choose a subsequence $\{P_{\ell_k}\}$ of $\{P_i\}$ inductively. Then we put $P_\infty = \bigcup_k P_{i_0}^{(k)} \in SP(\mathscr{S})$. Since $h(P_{\ell_k}, P_\infty) < 1/k$, we obtain that $\{P_{\ell_k}\}_{k=1,2,\ldots}$ converges to P_∞.

Therefore, $SP(\mathscr{S})$ is sequencial compact and the proof of Theorem is completed.

Proof of Corollary. First, we prove the existence of a tiling with n-fold rotational symmetry. We consider a subset $\Delta(\pi/n) = \{(r\cos\theta, r\sin\theta) \mid -\pi/n \le \theta \le 0, r \ge 0\}$.

We show that $\Delta(\pi/n)$ is decomposed into prototiles. We take a prototile S_0 with a vertex of an angle π/n. We set a tile T_0 in $\Delta(\pi/n)$ which is a copy of S_0 and contains the origin $O = (0,0)$ as a vertex of an angle π/n. We take a sequence $\{\hat{\Phi}^k[T_0]\}$ in $SP(\mathscr{S})$. Since $SP(\mathscr{S})$ is sequencial compact by Theorem, we can choose a convergent subsequence $\{\hat{\Phi}^{k_i}[T_0]\}_{i=1,2,\ldots}$ of $\{\hat{\Phi}^k[T_0]\}$. Let P_∞ be a limit point of $\{\hat{\Phi}^{k_i}[T_0]\}_{i=1,2,\ldots}$. Because $\lambda > 1$, for any $R > 0$ there exists a sufficiently large j such that $U_R(O) \cap \Delta(\pi/n) \subset \bigcup_{T \in \hat{\Phi}^{k_j}[T_0]} T$, where $U_R(O) = \{w \in \mathbf{R}^2 \mid d(w, O) < R\}$. Hence P_∞ is a decomposition of $\Delta(\pi/n)$ and there exists a set $\{T_i\}$ of tiles such that $\bigcup_{i=1}^\infty T_i = \Delta(\pi/n)$ and $\mathrm{Int}\, T_i \cap \mathrm{Int}\, T_j = \varnothing$ if $i \ne j$.

The decomposition of $\Delta(\pi/n)$ can be extended to the whole plane $\mathbf{R}^2$. In fact, we put $T_i(k,j) = r_{2\pi/n}^k R_1^j T_i$ $(k = 0, 1, 2, \ldots, n-1, j = 0, 1)$, where R_1 is the reflection in the x-axis and $r_{2\pi/n}$ is the $2\pi/n$ rotation around the origin. Then we see that $\bigcup_{i,k,j} T_i(k,j) = \mathbf{R}^2$ and $\mathrm{Int}\, T_i(k,j) \cap \mathrm{Int}\, T_{i'}(k',j') = \varnothing$ if $(i,j,k) \ne (i',j',k')$, and get that $\{T_i(k,j)\}$ is a tiling in $\mathbf{R}^2$. By the way of the construction we obtain that $\{T_i(k,j)\}$ has n-fold rotational symmetry.

Secondly, we show that the tiling just constructed above is a periodic point of $\hat{\Phi}$. We have only to prove that the decomposition P_∞ of $\Delta(\pi/n)$ is a periodic point of $\hat{\Phi}$ by the way of the construction. We take a patch $P \subset P_\infty$ which consists of all tiles T such that $O \in T \in P_\infty$. Since $P_\infty = \lim_{i\to\infty} \hat{\Phi}^{k_i}[T_0]$, there exists an integer N such that $P \subset \hat{\Phi}^{k_i}[T_0]$ for any $i \ge N$. We put $p = \min\{k \mid P \subset \hat{\Phi}^k[T_0]\}$ and $m = \min\{k - k' \mid P \subset \hat{\Phi}^k[T_0] \cap \hat{\Phi}^{k'}[T_0], k' < k\}$. The sequence $\{\hat{\Phi}^{km}[P]\}_{k=0,1,2,\ldots}$ converges to $\bigcup_k \hat{\Phi}^{km}[P]$ because $\hat{\Phi}^{im}[P] \subset \hat{\Phi}^{jm}[P]$ if $i < j$. We consider a sequence $\{\hat{\Phi}^{p+km}[T_0]\}_{k=0,1,2,\ldots}$. Since $P \subset \hat{\Phi}^p[T_0]$, we see that $\hat{\Phi}^{km}[P] \subset \hat{\Phi}^{p+km}[T_0]$. Hence we get that $\{\hat{\Phi}^{p+km}[T_0]\}$ converges to $\lim_{k\to\infty} \hat{\Phi}^{km}[P]$. By the minimality of p and m, $\{\hat{\Phi}^{k_i}[T_0]\}_{i\ge N}$ is a convergent subsequence of $\{\hat{\Phi}^{p+km}[T_0]\}_{k=0,1,2,\ldots}$. Then we get that $\{\hat{\Phi}^{p+km}[T_0]\}$ converges to $P_\infty = \lim_{i\to\infty} \hat{\Phi}^{k_i}[T_0]$. Therefore, we obtain that $P_\infty = \lim_{k\to\infty} \hat{\Phi}^{km}[P]$ and $\hat{\Phi}^m[P_\infty] = P_\infty$.

In addition, we see that the tiling obtained above is nonperiodic by the following proposition:

PROPOSITION ([4], [14]). *Let n be a positive integer with $n = 5$ or $6 < n$. Then the tilings with n-fold rotational symmetry are nonperiodic.*

The following lemma implies Proposition immediately (see [4], [14] for details).

LEMMA ([4], [14]). *Let n be a positive integer with $n = 5$ or $6 < n$. No tiling has more than one center of n-fold rotational symmetry.*

Due to [4] J. H. Conway introduced the proof of Lemma for $n = 5$ which is given by P. Barlow. By the similar argument Lemma is true for $6 < n$ (cf. [14]).

Next, we see that a Coxeter group $H_2^{(n)}$ acts on the tiling obtained above by the way of the construction because $\Delta(\pi/n)$ is the fundamental region of $H_2^{(n)} = \langle R_1, R_2 \mid (R_1 R_2)^n = R_1^2 = R_2^2 = e \rangle$, where R_1 is the reflection in the x-axis and R_2 is the reflection in the line $x \tan(-\pi/n) - y = 0$. For any integer n with $n = 5$ or $6 < n$, $H_2^{(n)}$ is a non-crystallographic Coxeter group (cf. [6]).

The proof of Corollary is completed.

REMARK. When the decomposition of $\Delta(\pi/n)$ is line symmetric in the line $x \tan(-\pi/2n) - y = 0$, the decomposition of $\Delta(\pi/n)$ can be extended to the tiling in the whole plane $\mathbf{R}^2$ by using only rotation $r_{\pi/n}$. This tiling has $2n$-fold rotational symmetry (for example, see an Ammann-Beenker tiling in 3.3).

3. Some examples

In this section we demonstrate three examples of typical tilings with rotational symmetries.

3.1. A Danzer tiling with 7-fold rotational symmetry. The prototiles of Danzer tilings are six types of triangles with arrows on the edges (three triangles a, b, c in Figure 1 and their mirror images).

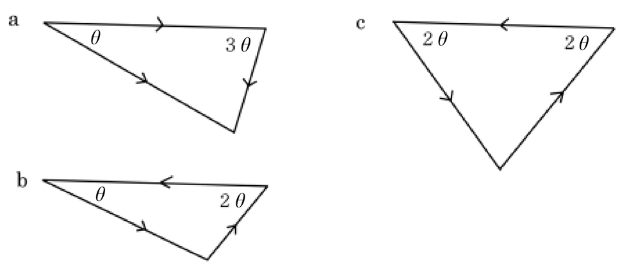

Fig. 1. Three prototiles with arrows of Danzer tilings ($\theta = \pi/7$)

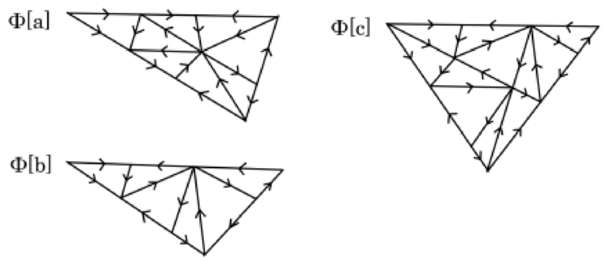

Fig. 2. The substitution rule of Danzer tilings

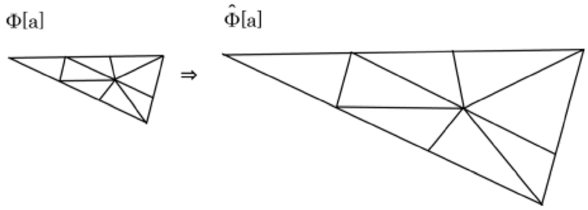

Fig. 3. Rescaling $\Phi[a]$ to $\hat{\Phi}[a]$

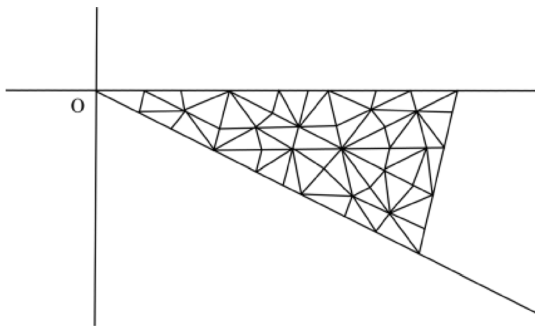

Fig. 4. The decomposition of $\Delta(\pi/7)$

We give the substitution rule of prototiles as in Figure 2 and their mirror images (cf. [8]). As an example we decompose a tile a into smaller copies of the prototiles and get $\Phi[a]$. Then we rescale these small tiles to the original size and get $\hat{\Phi}[a]$ as in Figure 3. Next, we docompose each tile in the patch which we have just constructed, rescale, decompose again, and so forth ad infinitum. The sequence $\{a, \hat{\Phi}[a], \hat{\Phi}^2[a], \ldots\}$ converges to a decomposition of $\Delta(\pi/7)$ as in Figure 4. We reflect the decomposition of $\Delta(\pi/7)$ in the x-axis and get Figure 5. In addition, we rotate Figure 5 by $2k\pi/7$ radian around the

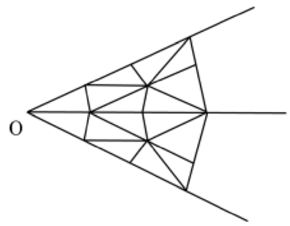

Fig. 5. The reflection of Fig. 4

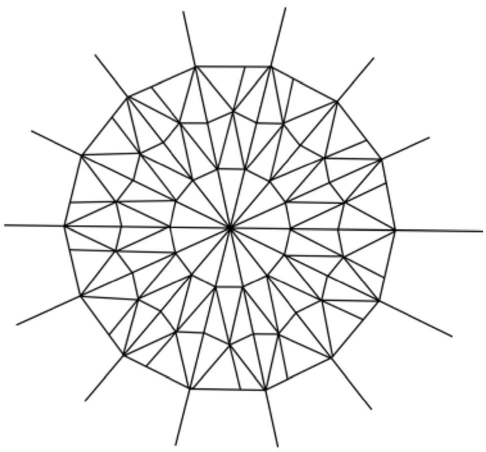

Fig. 6. A Danzer tiling with 7-fold rotational symmetry

origin ($k = 1, 2, \dots, 6$). Then a Danzer tiling with 7-fold rotational symmetry appears as in Figure 6.

When we take a sequence $\{b, \hat{\Phi}[b], \hat{\Phi}^2[b], \dots\}$, we see that a subsequence $\{\hat{\Phi}[b], \hat{\Phi}^2[b], \hat{\Phi}^3[b], \dots\}$ converges to a decomposition of $\Delta(\pi/7)$.

3.2. A Penrose tiling with 5-fold rotational symmetry. The prototiles of Penrose tilings are four types of triangles with arrows on the edges (two triangles a, b in Figure 7 and their mirror images). Note that the original

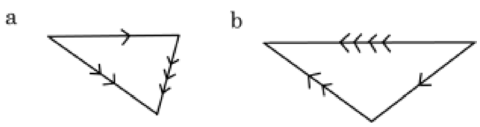

Fig. 7. Two prototiles with arrows of Penrose tilings

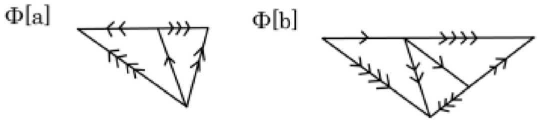

Fig. 8. The substitution rule of Penrose tilings

Penrose tilings are restorable from tilings that use four kinds of tiles in Figure 7 and their mirror images by erasing edges with threefold or fourfold arrows.

We give the substitution rule of prototiles as in Figure 8 and their mirror images (cf. [3]).

When we take a sequence $\{a, \hat{\Phi}[a], \hat{\Phi}^2[a], \ldots\}$, we see that a subsequence $\{\hat{\Phi}[a], \hat{\Phi}^5[a], \hat{\Phi}^9[a], \ldots\}$ converges to a decomposition of $\Delta(\pi/5)$.

When we take a sequence $\{b, \hat{\Phi}[b], \hat{\Phi}^2[b], \ldots\}$, we see that a subsequence $\{\hat{\Phi}[b], \hat{\Phi}^5[b], \hat{\Phi}^9[b], \ldots\}$ converges to a decomposition of $\Delta(\pi/5)$.

3.3. An Ammann-Beenker tiling with 8-fold rotational symmetry. The prototiles of Ammann-Beenker tilings are two decorated rhombs with acute angles $\pi/4$ and two decorated halves of a square (two tiles a, b in Figure 9 and their mirror images). Note that the original Ammann-Beenker tilings are restorable from tilings that use four kinds of decorated tiles in Figure 9 and their mirror images by identifying squares which appear in tilings as tiles.

We give the substitution rule of prototiles as in Figure 10 and their mirror images (cf. [7]).

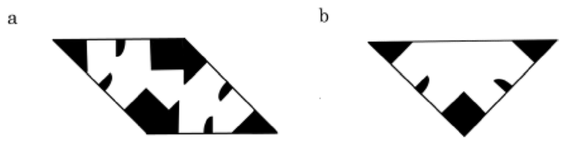

Fig. 9. Two prototiles of Ammann-Beenker tilings

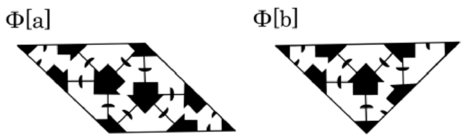

Fig. 10. The substitution rule of Ammann-Beenker tilings

When we take a sequence $\{a, \hat{\Phi}[a], \hat{\Phi}^2[a], \hat{\Phi}^3[a], \dots\}$, we see that it is convergent sequence and get a decomposition of $\Delta(\pi/4)$.

When we take a sequence $\{b, \hat{\Phi}[b], \hat{\Phi}^2[b], \dots\}$, we see that a subsequence $\{\hat{\Phi}[b], \hat{\Phi}^2[b], \hat{\Phi}^3[b], \dots\}$ converges to a decomposition of $\Delta(\pi/4)$. These two decompositions of $\Delta(\pi/4)$ is line symmetric in the line $x\tan(-\pi/8) - y = 0$. Hence we obtain a tiling with 8-fold rotational symmetry in each case.

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