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Reaction-diffusion system approximation to the cross-diffusion competition system

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ABSTRACT. We study the stationary problem of a reaction-diffusion system with a small parameter ε, which approximates the cross-diffusion competition system proposed to study spatial segregation problem between two competing species. The convergence between two systems as $\varepsilon \downarrow 0$ is discussed from analytical and complementarily numerical point of views.

1. Introduction

Multiple species are directly or indirectly interacting with one another within ecological systems. As an example, it is well known that they compete to feed common resource. A macroscopic continuous model describing the competitive interaction of two ecological species is

$$\begin{aligned} u_t &= d_u \Delta u + (r_1 - a_1 u - b_1 v)u, \\ v_t &= d_v \Delta v + (r_2 - b_2 u - a_2 v)v, \end{aligned} \tag{1.1}$$

where $u(t,x)$ and $v(t,x)$ are the population densities of two competing species which move by diffusion, at time t and position x. d_u and d_v are the diffusion rates, r_i, a_i and b_i $(i = 1, 2)$ are the intrinsic growth rates, the intra-specific competition rates and the inter-specific competition rates of u and v. All of the parameters are positive constants. The system (1.1) has been intensively studied from analytical point of views ([1], [9], [10] for instance). Suppose that the parameters r_i, a_i and b_i $(i = 1, 2)$ satisfy the inequalities

$$\frac{a_1}{b_2} < \frac{r_1}{r_2} < \frac{b_1}{a_2}, \tag{1.2}$$

for which we ecologically say that two species are strongly competing. If (1.1) is considered in a convex domain with the zero-flux boundary conditions, we

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know that any non-negative solution (u, v) generically converges to either $\left(\frac{r_1}{a_1}, 0\right)$ or $\left(0, \frac{r_2}{a_2}\right)$, that is, the competitive exclusion principle occurs between the two species u and v ([1]). This result could be intuitively understood because strong competition holds between u and v. However, it is observed in natural fields that strongly competing species possibly coexist. Several explanations have been proposed for such coexistence. One of them is the repulsive effect between two competing species. In order to explain it theoretically, Shigesada et al. ([2]) proposed the following competition system with nonlinear diffusion:

$$\begin{aligned} u_t &= \Delta((d_u + \alpha_1 v)u) + (r_1 - a_1 u - b_1 v)u, \\ v_t &= \Delta((d_v + \alpha_2 u)v) + (r_2 - b_2 u - a_2 v)v, \end{aligned} \tag{1.3}$$

where α_1 and α_2 stand for the population pressure effects from one species to the other. It is obvious that (1.3) reduces to (1.1) when $\alpha_1 = \alpha_2 = 0$. (1.3) is called a cross-diffusion competition system. Since the first equation of (1.3) is rewritten as

$$u_t = \nabla((d_u + \alpha_1 v)\nabla u) + \alpha_1 \nabla(u \nabla v) + (r_1 - a_1 u - b_1 v)u, \tag{1.4}$$

one can see that the second term of the right hand side in (1.4) indicates the direct movement of u in the sense that u moves towards lower density of v when $\alpha_1 > 0$. This implies that v has the repulsive effect on u. For (1.3) with (1.2) it is numerically shown that even if the domain is convex, there exist stable non-constant equilibrium solutions exhibiting spatially segregating coexistence when either α_1 or α_2 is suitably large at least. This result indicates that the cross-diffusion mechanism (1.4) causes the possibility of coexistence of strongly competing species. For analytical studies on (1.3), we refer to [6], [8], [11], for instance.

Recently, Iida, Mimura and Ninomiya ([3]) have addressed the following question: Is there any reaction-diffusion system which approximates the cross-diffusion system (1.3)? In order to answer this question, they considered a simplified system of (1.3) with $\alpha_1 = \alpha > 0$ and $\alpha_2 = 0$ in a bounded domain $\Omega \in \mathbf{R}^N$ $(N \geq 1)$, that is

$$\begin{aligned} u_t &= \Delta((d_u + \alpha v)u) + (r_1 - a_1 u - b_1 v)u, \\ v_t &= d_v \Delta v + (r_2 - b_2 u - a_2 v)v, \end{aligned} \qquad t > 0,\ x \in \Omega \tag{1.5}$$

with the boundary and initial conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad t > 0,\ x \in \partial\Omega \tag{1.6}$$

and

$$\begin{aligned} u(0,x) &= u_{ini}(x) \geq 0 \\ v(0,x) &= v_{ini}(x) \geq 0 \end{aligned} \qquad x \in \Omega, \tag{1.7}$$

where ν is the outer normal vector on $\partial\Omega$ which is the smooth boundary of Ω. They proposed the following three component reaction-diffusion system with a sufficiently small parameter ε for (U_A, U_B, V):

$$\begin{aligned} U_{At} &= d_u \Delta U_A + (r_1 - a_1(U_A + U_B) - b_1 V) U_A \\ &\quad + \frac{1}{\varepsilon}(k(V) U_B - h(V) U_A), \\ U_{Bt} &= (d_u + M\alpha) \Delta U_B + (r_1 - a_1(U_A + U_B) - b_1 V) U_B \\ &\quad - \frac{1}{\varepsilon}(k(V) U_B - h(V) U_A), \\ V_t &= d_v \Delta V + (r_2 - b_2(U_A + U_B) - a_2 V) V, \end{aligned} \tag{1.8}$$

where M in (1.8) is a constant satisfying $M \geq \max\{\frac{r_2}{a_2}, \|v(0,\cdot)\|_{L^\infty(\Omega)}\}$, and thus $0 \leq v(t,x) \leq M$ for solution (u,v) of (1.5)–(1.6). The boundary and initial conditions are respectively given by

$$\frac{\partial U_A}{\partial \nu} = \frac{\partial U_B}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0, \qquad t > 0,\ x \in \partial\Omega. \tag{1.9}$$

and

$$\begin{aligned} U_A(0,x) &= \left(1 - \frac{v_{ini}(x)}{M}\right) u_{ini}(x), \\ U_B(0,x) &= \frac{v_{ini}(x)}{M} u_{ini}(x), \\ V(0,x) &= v_{ini}(x), \end{aligned} \qquad x \in \Omega, \tag{1.10}$$

For U_A and U_B in (1.8), we note that these convert each other with the rates $\frac{1}{\varepsilon}k(V)$ and $\frac{1}{\varepsilon}h(V)$ where $k(V)$ is a monotone decreasing function and $h(V)$ is a monotone increasing function satisfying

$$\frac{V}{M} = \frac{h(V)}{k(V) + h(V)}, \tag{1.11}$$

and that if ε is sufficiently small, U_A and U_B convert instantly. Then, the following theorem is known.

THEOREM 1 (M. Iida, M. Mimura and H. Ninomiya [3]). *Let* $(u, v) = (u(t, x), v(t, x))$ *be the solution of* (1.5)–(1.7). *Suppose that* (u, v) *is sufficiently smooth and uniformly bounded on* $[0, T] \times \bar{\Omega}$ *for some positive number* T. *Let* $(U_A, U_B, V) = (U_A(t, x; \varepsilon), U_B(t, x; \varepsilon), V(t, x; \varepsilon))$ *be the solution of* (1.8)–(1.10) *depending on a positive parameter* ε, *where smooth functions* h *and* k *satisfy* (1.11) *and*

$$k(s) \geq 0, \qquad h(s) \geq 0, \qquad k(s) + h(s) > 0$$

for $s \in [0, M]$. *Suppose that there exist positive numbers* M_0 *and* ε_0 *such that*

$$|U_A(t, x; \varepsilon)| + |U_B(t, x; \varepsilon)| + |V(t, x; \varepsilon)| \leq M_0$$

for $(t, x) \in [0, T] \times \bar{\Omega}$ *and* $\varepsilon \in (0, \varepsilon_0]$. *Then there is a positive constant* $C = C(u, v, \varepsilon_0, M_0, T)$ *independently of* ε *such that*

$$\sup_{t \in [0, T]} \|U_A(t, \cdot; \varepsilon) + U_B(t, \cdot; \varepsilon) - u(t, \cdot)\|_{L^2(\Omega)} \leq C\varepsilon,$$

$$\sup_{t \in [0, T]} \|V(t, \cdot; \varepsilon) - v(t, \cdot)\|_{L^2(\Omega)} \leq C\varepsilon$$

hold for $\varepsilon \in (0, \varepsilon_0]$.

This theorem can be ecologically interpreted as follows: One of the species V moves randomly with the diffusion rate d_v, and U_A and U_B move with the diffusion rate d_u and $d_u + M\alpha$, respectively, where these exchange each other instantly, depending on the density of the species V. If the exchange rates $k(V)$ and $h(V)$ satisfy (1.11), then, the cross-diffusion effect of (1.4) appears on $u(= U_A + U_B)$. Thus (1.8) is called a reaction-diffusion approximation to the cross-diffusion system (1.5).

Here we note that this convergence theorem does not give any information on asymptotic behavior of solutions for large time because it holds for a finite time interval $[0, T]$. This motivates us to consider the convergence problem between the stationary problems of (1.5)–(1.6) and (1.8)–(1.9), respectively:

$$\begin{aligned} 0 &= \Delta((d_u + \alpha v)u) + (r_1 - a_1 u - b_1 v)u, \\ 0 &= d_v \Delta v + (r_2 - b_2 u - a_2 v)v, \end{aligned} \qquad x \in \Omega, \tag{1.12}$$

with the boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega, \tag{1.13}$$

and

$$0 = d_u \Delta U_A + (r_1 - a_1(U_A + U_B) - b_1 V)U_A$$
$$+ \frac{1}{\varepsilon}(k(V)U_B - h(V)U_A),$$
$$0 = (d_u + M\alpha)\Delta U_B + (r_1 - a_1(U_A + U_B) - b_1 V)U_B \qquad x \in \Omega, \tag{1.14}$$
$$- \frac{1}{\varepsilon}(k(V)U_B - h(V)U_A),$$
$$0 = d_v \Delta V + (r_2 - b_2(U_A + U_B) - a_2 V)V,$$

with the boundary conditions

$$\frac{\partial U_A}{\partial \nu} = \frac{\partial U_B}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0, \qquad x \in \partial\Omega. \tag{1.15}$$

We now address the question "Do (non-negative) solutions $(U_A(x;\varepsilon) + U_B(x;\varepsilon), V(x;\varepsilon))$ of (1.14) and (1.15) converge to the corresponding ones $(u(x), v(x))$ of (1.12) and (1.13) as ε tends to zero?"

In Section 2, we numerically consider this problem from the viewpoints of global structures of the equilibrium solutions and in Section 3, we show that it holds if solutions of (1.12) and (1.13) are non-degenerate.

2. Numerical results

Here we simply consider the 1-dimensional problem of (1.12) and (1.13) in the interval $(0, 1)$. The first case is where $r_1 = 5.0$, $r_2 = 2.0$, $a_1 = 3.0$, $a_2 = 3.0$, $b_1 = 1.0$ and $b_2 = 1.0$ (weak competition). We note that when $\alpha = 0$, a stable equilibrium solution is $\left(\frac{13}{8}, \frac{1}{8}\right)$ only for any values of d_u and d_v ([1]). Using α as a free parameter, we take the spatially constant equilibrium $\left(\frac{13}{8}, \frac{1}{8}\right)$ as the trivial branch solution. It is stable for small α, while as α increases, it is destabilized so that there appear spatially non-constant equilibrium solutions exhibiting spatial segregation between two species ([8] for instance). By using a bifurcation software which is called AUTO ([7]), the structure of equilibrium solutions can be drawn for globally varied α. When $d_u = d_v = 0.005$, it surprisingly exhibits rather complex diagram of bifurcation branches which connect each other, as in Fig. 1. On the other hand, fixing $\alpha = 3.0$, we take $d = d_u = d_v$ as a free parameter. The global structure of equilibrium solutions

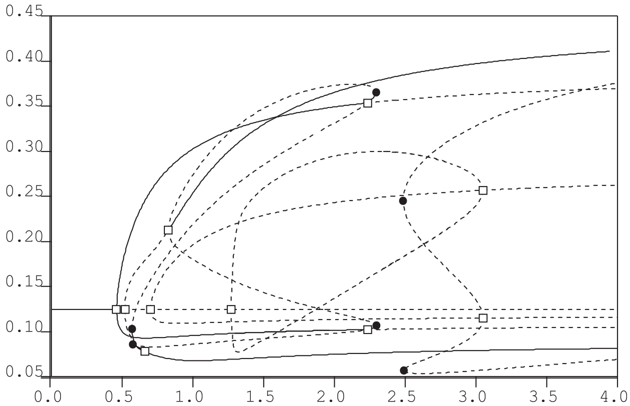


Fig. 1. Global structure of equilibrium solutions with a free parameter α, the vertical and horizontal axis imply the value of $v(0)$ and the free parameter α respectively. Solid (resp. dot) curves indicate stable (resp. unstable) branches where $d_u = d_v = 0.005$. ● implies a limitting point.

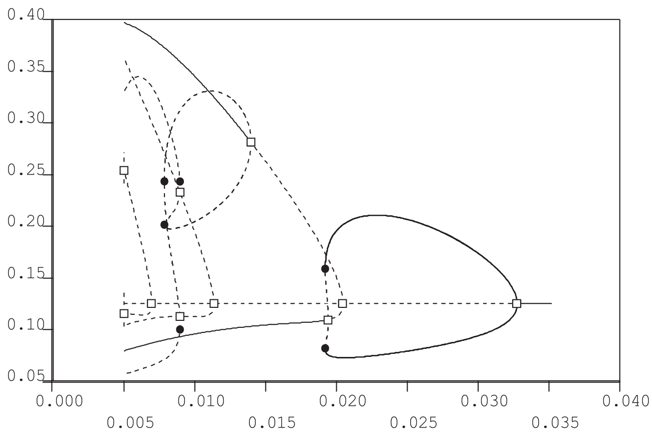


Fig. 2. Global structure of equilibrium solutions with a free parameter $d = d_u = d_v$, the vertical and horizontal axis imply the value of $v(0)$ and the free parameter d respectively. ● implies a limitting point.

with a parameter d is drawn in Fig. 2. For large d, $\left(\frac{13}{8}, \frac{1}{8}\right)$ is stable but as d decreases, it is destablized and there appear primarily stable spatially non-constant equilibrium solutions. The second case is where $r_1 = 2.0$, $r_2 = 5.0$, $a_1 = 1.0$, $a_2 = 1.0$, $b_1 = 0.5$, $b_2 = 3.0$ (strong competition). When $\alpha = 0$, the constant equilibrium solutions are $(2, 0)$, $(0, 5)$ and $(1, 2)$, where the first two solutions are stable and the last one is unstable. In this case we know that any

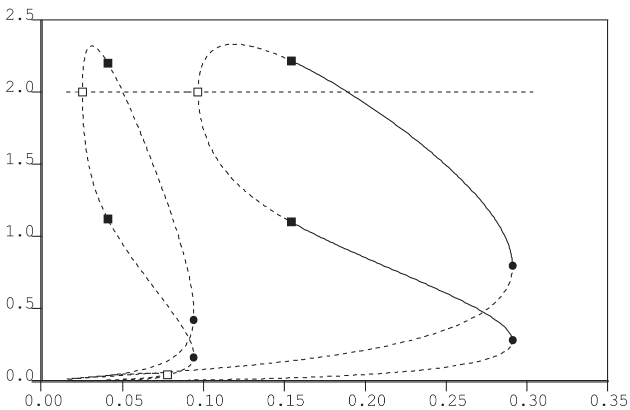
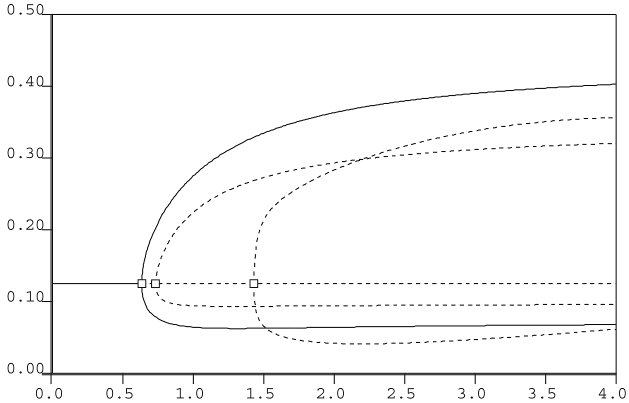
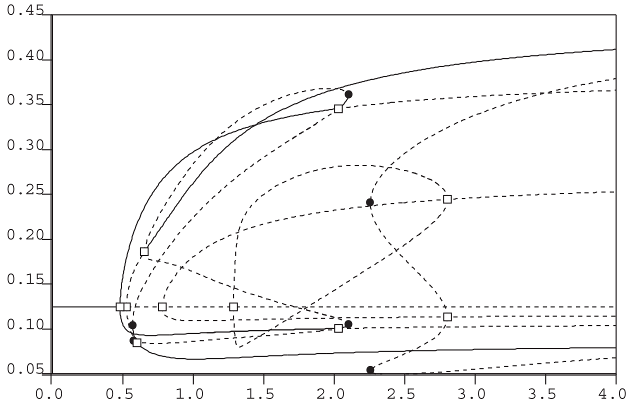


Fig. 3. Global structure of equilibrium solutions with a free parameter $d = d_u = d_v$, the vertical and horizontal axis imply the value of $v(0)$ and the free parameter d respectively. ■ implies a Hopf bifurcation point. ● implies a limitting point.

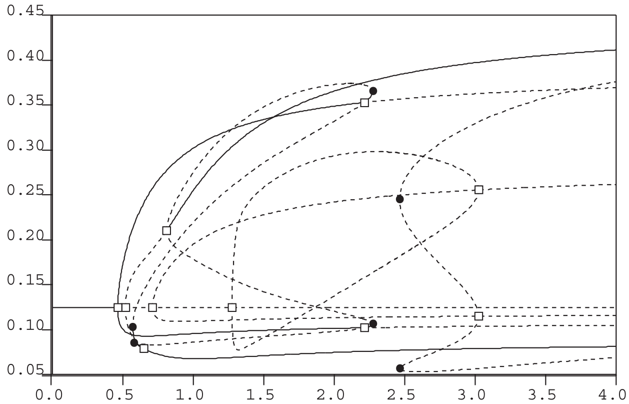
positive solution generically tends to either $(2, 0)$ or $(0, 5)$ for any values of d_u and d_v ([1]). This implies the occurrence of competitive exclusion principle. Here we take $\alpha = 3.0$ and $d = d_u = d_v$ as a free parameter. When d increases, a sub-critical bifurcation primarily occurs so that unstable spatially non-constant equilibrium solutions appear and there occurs a Hopf bifurcation on these branches so that these solutions become stable, as in Fig. 3. This result indicates that the cross-diffusion enhances the possibility of coexistence of two competing species even if the competitive interaction is strong. Next, we consider the following problem: How are the structures of equilibrium solutions $(U_A(x; \varepsilon), U_B(x; \varepsilon), V(x; \varepsilon))$ of the approximating problem (1.14) and (1.15)? and do these global structures converge to the ones of $(u(x), v(x))$ of (1.12) and (1.13) as ε tends to zero? Figs. 4(a), 4(b) and 4(c) show global structures of equilibrium solutions of (1.14) and (1.15) for different values of ε where we put $k(V) = 1 - \frac{V}{M}$ and $h(V) = \frac{V}{M}$ which satisfy (1.11) and the parameters are the same ones in Fig. 1 thus we choose $M = 1$. When $\varepsilon = 0.01$, the global structure is rather simple, as in Fig. 4(a). As ε decreases, it becames complex and when $\varepsilon = 0.0001$, the equilibrium solution structure of Fig. 4(c) surprisingly resembles the one in Fig. 1. Figs. 5 and 6 show the global structures with a free parameter d for different values of ε. These results clearly indicate that for a sufficiently small positive ε, the three component reaction-diffusion system (1.8) seems a nice approximation to the cross-diffusion system (1.5) from not only the evolutional problem but also the stationary problem viewpoints.



(a) $\varepsilon = 0.01$

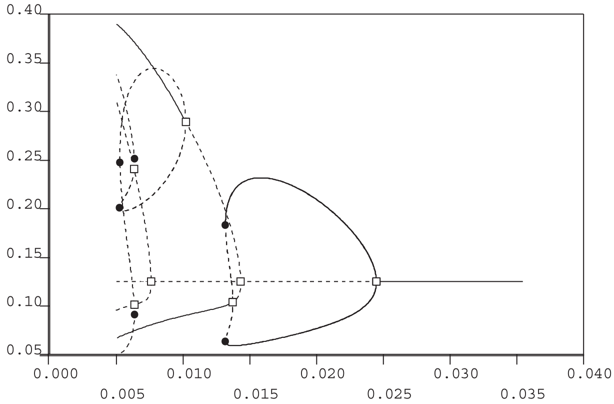


(b) $\varepsilon = 0.001$

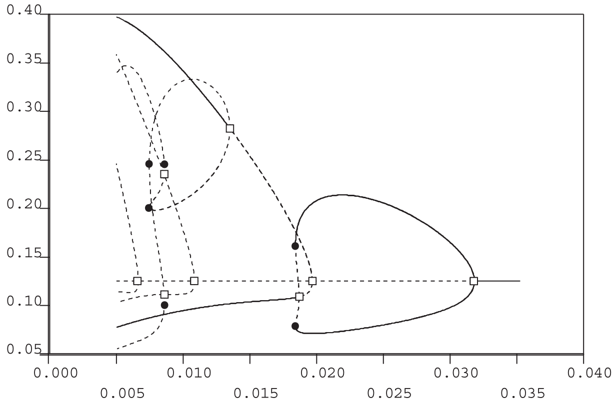


(c) $\varepsilon = 0.0001$

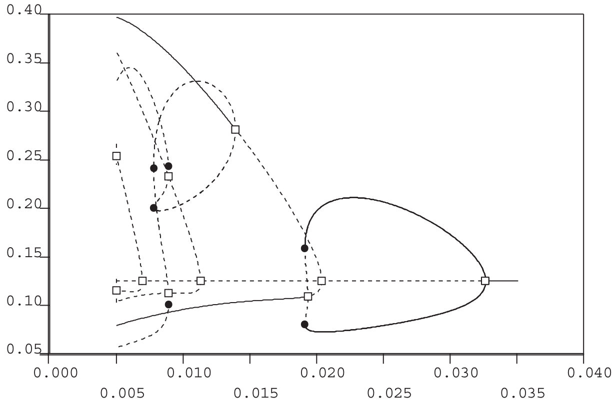
Fig. 4(a), 4(b) and 4(c). Global structure with a free parameter α, the vertical and horizontal axis imply the value of $v(0)$ and the free parameter α respectively. ● implies a limitting point. We put $k(V) = 1 - \frac{V}{M}$ and $h(V) = \frac{V}{M}$ and $M = 1$.



(a) $\varepsilon = 0.01$

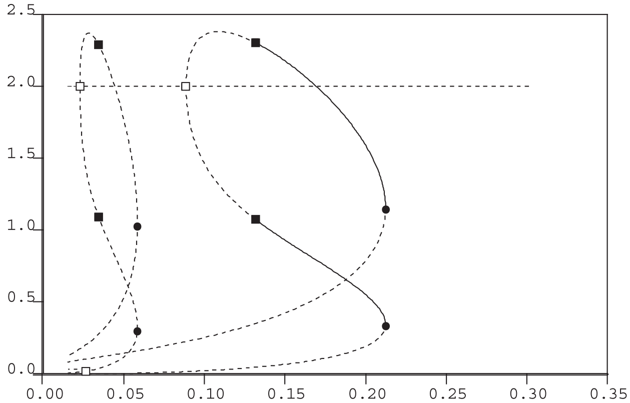


(b) $\varepsilon = 0.001$

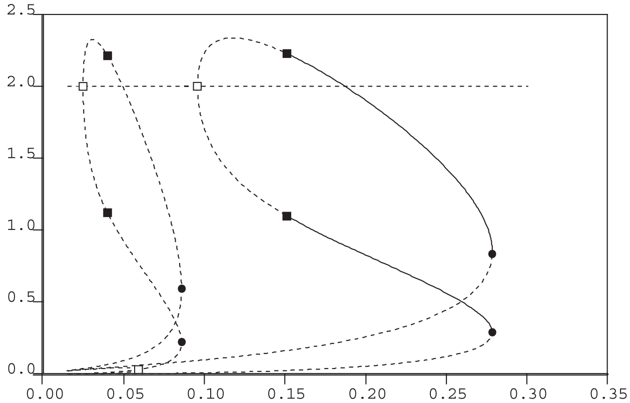


(c) $\varepsilon = 0.0001$

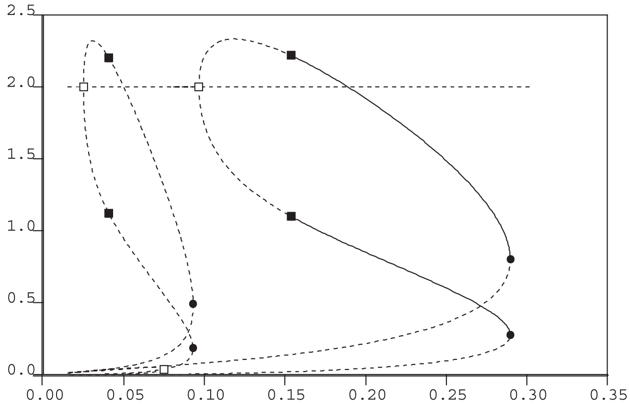
Fig. 5(a), 5(b) and 5(c). Global structure with a free parameter $d = d_u = d_v$, the vertical and horizontal axis imply the value of $v(0)$ and the free parameter d respectively. ● implies a limitting point. We put $k(V) = 1 - \frac{V}{M}$ and $h(V) = \frac{V}{M}$ and $M = 1$.



(a) $\varepsilon = 0.01$



(b) $\varepsilon = 0.001$



(c) $\varepsilon = 0.0001$

Fig. 6(a), 6(b) and 6(c). Global structure with a free parameter $d = d_u = d_v$, the vertical and horizontal axis imply the value of $v(0)$ and the free parameter d respectively. ■ implies a Hopf bifurcation point. ● implies a limitting point. We put $k(V) = 1 - \frac{V}{M}$ and $h(V) = \frac{V}{M}$ and $M = 7$.

3. Analytical results

The above numerical results motivate us to discuss the convergence problem between the stationary problems (1.12) with (1.13) and (1.14) with (1.15). We introduce the following function spaces to treat this problem:

$$W_N^{k,p}(\Omega) := \left\{ u \in W^{k,p}(\Omega); \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ x \in \partial\Omega \right\},$$

$$X := (W_N^{2,p}(\Omega))^3,$$

$$Y := (L^p(\Omega))^3,$$

where $p > n$ and $k \geq 2$. We obtain the following result:

Theorem 2. *Let $(u_0(x), v_0(x))$ be a sufficiently smooth positive solution of (1.12) and (1.13) such that the linearized operator of (1.12) around $(u_0(x), v_0(x))$ is bijective from $(W_N^{2,p}(\Omega))^2$ into $(L^p(\Omega))^2$. For example, $u_0(x), v_0(x) \in W_N^{6,p}(\Omega)$ is at least required. Suppose that the functions $k(s)$ and $h(s)$ are smooth on $(0, M]$ and there exists a positive constant $\beta > 0$ satisfying*

$$k(s) + h(s) \geq \beta > 0 \qquad \text{on } [0, M].$$

Then, there exist positive constants ε_0 and C such that for any $\varepsilon \in (0, \varepsilon_0]$ the problem (1.14) and (1.15) has a unique ε-family of equilibrium solutions $(U_A(x;\varepsilon), U_B(x;\varepsilon), V(x;\varepsilon))$ that satisfy

$$\|U_A(\cdot;\varepsilon) + U_B(\cdot;\varepsilon) - u_0(\cdot)\|_{W^{2,p}(\Omega)} \leq C\varepsilon,$$

$$\|V(\cdot;\varepsilon) - v_0(\cdot)\|_{W^{2,p}(\Omega)} \leq C\varepsilon.$$

In order to prove this theorem, we consider an equivalent problem by using several transformations. With $U = U_A + U_B$ and $W = U_B$, (1.14) and (1.15) are rewritten as

$$\begin{cases} 0 = \Delta(d_u U + \alpha M W) + (r_1 - a_1 U - b_1 V)U & \\ 0 = d_v \Delta V + (r_2 - b_2 U - a_2 V)V & x \in \Omega, \\ 0 = \varepsilon\{(d_u + \alpha M)\Delta W + (r_1 - a_1 U - b_1 V)W\} + Q(V)\left(\dfrac{1}{M} UV - W\right) & \\ 0 = \dfrac{\partial U}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = \dfrac{\partial W}{\partial \nu} & x \in \partial\Omega, \end{cases} \tag{3.1}$$

where $Q(V) = k(V) + h(V)$. Next, by using

$$\tilde{u} = \left(1 + \frac{\alpha}{d_u} v\right) u, \tag{3.2}$$

we transform (1.12) to obtain

$$\begin{cases} 0 = d_u \Delta \tilde{u} + \left(r_1 - a_1 \dfrac{d_u \tilde{u}}{d_u + \alpha v} - b_1 v\right) \dfrac{d_u \tilde{u}}{d_u + \alpha v} & \\ 0 = d_v \Delta v + \left(r_2 - b_2 \dfrac{d_u \tilde{u}}{d_u + \alpha v} - a_2 v\right) v & x \in \Omega, \\ 0 = \dfrac{\partial \tilde{u}}{\partial \nu} = \dfrac{\partial v}{\partial \nu} & x \in \partial\Omega. \end{cases} \tag{3.3}$$

By using the transformation

$$\tilde{U} = U + \frac{\alpha M}{d_u} W \tag{3.4}$$

which is a counterpart of (3.2) for (1.12), (3.1) becomes

$$\begin{cases} 0 = d_u \Delta \tilde{U} + \left(r_1 - a_1 \left(\tilde{U} - \dfrac{\alpha M}{d_u} W\right) - b_1 V\right) \left(\tilde{U} - \dfrac{\alpha M}{d_u} W\right) & \\ 0 = d_v \Delta V + \left(r_2 - b_2 \left(\tilde{U} - \dfrac{\alpha M}{d_u} W\right) - a_2 V\right) V & \\ 0 = \varepsilon \left\{ (d_u + \alpha M) \Delta W + \left(r_1 - a_1 \left(\tilde{U} - \dfrac{\alpha M}{d_u} W\right) - b_1 V\right) W \right\} & x \in \Omega, \\ \quad + Q(V) \left(\dfrac{1}{M} \left(\tilde{U} - \dfrac{\alpha M}{d_u} W\right) V - W\right) & \\ 0 = \dfrac{\partial \tilde{U}}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = \dfrac{\partial W}{\partial \nu} & x \in \partial\Omega. \end{cases} \tag{3.5}$$

Putting $\varepsilon = 0$ and rewriting $(\tilde{U}, V, W)$ as $(\tilde{u}_0, v_0, w_0)$ in (3.5), we can reduce the third equation in (3.5) to

$$w_0 = \frac{d_u \tilde{u}_0 v_0}{(d_u + \alpha v_0) M}$$

and see that

$$\tilde{u}_0 - \frac{\alpha M}{d_u} w_0 = \frac{d_u \tilde{u}_0}{d_u + \alpha v_0}.$$

Then $(\tilde{u}_0, v_0)$ formally becomes a solution of (3.3). We now consider the convergence problem between (3.3) and (3.5) because the transformations (3.2) and (3.4) are one-to-one. We show the following theorem:

THEOREM 3. *Let* $(\tilde{u}_0(x), v_0(x))$ *be a sufficiently smooth positive solution of* (3.3) *such that the linearized operator of* (3.3) *around* $(\tilde{u}_0(x), v_0(x))$ *is bijective from* $(W_N^{2,p}(\Omega))^2$ *into* $(L^p(\Omega))^2$. *For example,* $\tilde{u}_0(x), v_0(x) \in W_N^{6,p}(\Omega)$ *is at least required. Suppose that the functions* $k(s)$ *and* $h(s)$ *are smooth on* $(0, M]$ *and there exists a positive constant* $\beta > 0$ *satisfying*

$$Q(s) = k(s) + h(s) \geq \beta > 0 \qquad \text{on } [0, M]. \tag{3.6}$$

Then, there exist positive constants ε_0 *and* C *such that for any* $\varepsilon \in (0, \varepsilon_0]$ *the problem* (3.5) *has a unique* ε*-family of equilibrium solutions* $(\tilde{U}(x;\varepsilon), V(x;\varepsilon), W(x;\varepsilon))$ *that satisfy*

$$\begin{aligned} \|\tilde{U}(\cdot;\varepsilon) - \tilde{u}_0(\cdot)\|_{W^{2,p}(\Omega)} &\leq C\varepsilon, \\ \|V(\cdot;\varepsilon) - v_0(\cdot)\|_{W^{2,p}(\Omega)} &\leq C\varepsilon, \\ \|W(\cdot;\varepsilon) - w_0(\cdot)\|_{W^{2,p}(\Omega)} &\leq C\varepsilon, \end{aligned} \tag{3.7}$$

where $w_0 = \frac{d_u \tilde{u}_0 v_0}{(d_u + \alpha v_0)M}$.

We prove Theorem 3 along several steps. First we construct an ε-family of solutions of the problem (3.5) in the following form:

$$\begin{aligned} \tilde{U}(x;\varepsilon) &= \tilde{U}_2^{\varepsilon}(x) + \varphi_1(x;\varepsilon), \\ V(x;\varepsilon) &= V_2^{\varepsilon}(x) + \varphi_2(x;\varepsilon), \\ W(x;\varepsilon) &= W_2^{\varepsilon}(x) + \varphi_3(x;\varepsilon), \end{aligned} \tag{3.8}$$

where

$$\begin{pmatrix} \tilde{U}_2^{\varepsilon}(x) \\ V_2^{\varepsilon}(x) \\ W_2^{\varepsilon}(x) \end{pmatrix} = \sum_{n=0}^{2} \varepsilon^n \begin{pmatrix} \tilde{u}_n(x) \\ v_n(x) \\ w_n(x) \end{pmatrix}.$$

The reason why we take an approximate solution into account up to second order of ε lies in (3.16). If we consider the order up to ε^0 or ε^1 only, then we can not choose a and ε satisfying (3.16).

As outline of proof of Theorem 3, in subsection 3.1, we will obtain an approximate solution $(\tilde{U}_2^{\varepsilon}(x), V_2^{\varepsilon}(x), W_2^{\varepsilon}(x))$ and in subsection 3.2, we prove the existence and convergence of the correction term $(\varphi_1(x;\varepsilon), \varphi_2(x;\varepsilon), \varphi_3(x;\varepsilon))$. For this purpose, substituting $(\tilde{U}(x;\varepsilon), V(x;\varepsilon), W(x;\varepsilon))$ into (3.5), neglecting $(\varphi_1(x;\varepsilon), \varphi_2(x;\varepsilon), \varphi_3(x;\varepsilon))$ and equating like powers of ε, we obtain the following hierarchies of the problems:

Order ε^0:

$$\begin{cases} 0 = d_u \Delta \tilde{u}_0 + \left(r_1 - a_1\left(\tilde{u}_0 - \dfrac{\alpha M}{d_u} w_0\right) - b_1 v_0\right)\left(\tilde{u}_0 - \dfrac{\alpha M}{d_u} w_0\right), \\ 0 = d_v \Delta v_0 + \left(r_2 - b_2\left(\tilde{u}_0 - \dfrac{\alpha M}{d_u} w_0\right) - a_2 v_0\right) v_0, & x \in \Omega, \\ 0 = Q(v_0)\left(\dfrac{1}{M}\left(\tilde{u}_0 - \dfrac{\alpha M}{d_u} w_0\right) v_0 - w_0\right), \\ 0 = \dfrac{\partial \tilde{u}_0}{\partial v} = \dfrac{\partial v_0}{\partial v} = \dfrac{\partial w_0}{\partial v}, & x \in \partial\Omega. \end{cases} \tag{3.9}$$

Order ε^1:

$$L_0 \begin{pmatrix} \tilde{u}_1 \\ v_1 \\ w_1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ (d_u + \alpha M)\Delta w_0 + \left(r_1 - a_1\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right) - b_1 v_0\right) w_0 \end{pmatrix}, \qquad x \in \Omega, \tag{3.10}$$

$$\frac{\partial \tilde{u}_1}{\partial v} = \frac{\partial v_1}{\partial v} = \frac{\partial w_1}{\partial v} = 0, \qquad x \in \partial\Omega,$$

where

$$L_0 := \begin{pmatrix} A & -b_1\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right) & -\frac{\alpha M}{d_u}\left(r_1 - 2a_1\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right) - b_1 v_0\right) \\ -b_2 v_0 & B & b_2 \frac{\alpha M}{d_u} v_0 \\ v_0 \frac{Q(v_0)}{M} & \left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right) \frac{Q(v_0)}{M} & -\left(1 + \frac{\alpha}{d_u} v_0\right) Q(v_0) \end{pmatrix},$$

$$A = d_u \Delta + r_1 - 2a_1\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right) - b_1 v_0,$$

$$B = d_v \Delta + r_2 - b_2\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right) - 2a_2 v_0.$$

Order ε^2:

$$L_0 \begin{pmatrix} \tilde{u}_2 \\ v_2 \\ w_2 \end{pmatrix} = -\begin{pmatrix} \left(-a_1\left(\tilde{u}_1 - \frac{\alpha M}{d_u} w_1\right) - b_1 v_1\right)\left(\tilde{u}_1 - \frac{\alpha M}{d_u} w_1\right) \\ \left(-b_2\left(\tilde{u}_1 - \frac{\alpha M}{d_u} w_1\right) - a_2 v_1\right) v_1 \\ W_2 \end{pmatrix}, \qquad x \in \Omega, \tag{3.11}$$

$$\frac{\partial \tilde{u}_2}{\partial v} = \frac{\partial v_2}{\partial v} = \frac{\partial w_2}{\partial v} = 0, \qquad x \in \partial\Omega,$$

where

$$W_2 = (d_u + \alpha M)\Delta w_1 + \left(r_1 - a_1\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right) - b_1 v_0\right) w_1$$

$$+ \left(-a_1\left(\tilde{u}_1 - \frac{\alpha M}{d_u} w_1\right) - b_1 v_1\right) w_0 + \frac{1}{M}\left(\tilde{u}_1 - \frac{\alpha M}{d_u} w_1\right) v_1 Q(v_0)$$

$$+ \left(\frac{1}{M}\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right) v_1 + \frac{1}{M}\left(\tilde{u}_1 - \frac{\alpha M}{d_u} w_1\right) v_0 - w_1\right) Q'(v_0) v_1.$$

The remainder of (3.5) is expressed as a boundary value problem for $\Phi = {}^T(\varphi_1, \varphi_2, \varphi_3)$:

$$\begin{cases} F(\Phi, \varepsilon) = L_\varepsilon \Phi + N(\Phi, \varepsilon) + R(\varepsilon) = 0, & x \in \Omega, \\ \frac{\partial \Phi}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{3.12}$$

where L_ε is a linear operator defined by

$$L_\varepsilon = \begin{pmatrix} A_\varepsilon & -b_1\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u} W_2^\varepsilon\right) & L_{13} \\ -b_2 V_2^\varepsilon & B_\varepsilon & b_2 \frac{\alpha M}{d_u} V_2^\varepsilon \\ -\varepsilon a_1 W_2^\varepsilon + \frac{1}{M} V_2^\varepsilon Q(V_2^\varepsilon) & L_{32} & C_\varepsilon \end{pmatrix},$$

in which A_ε, B_ε, C_ε, L_{13} and L_{32} are given by

$$A_\varepsilon = d_u \Delta + r_1 - 2a_1\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u} W_2^\varepsilon\right) - b_1 V_2^\varepsilon,$$

$$B_\varepsilon = d_v \Delta + r_2 - b_2\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u} W_2^\varepsilon\right) - 2a_2 V_2^\varepsilon,$$

$$C_\varepsilon = \varepsilon\left\{(d_u + \alpha M)\Delta + r_1 - a_1\left(\tilde{U}_2^\varepsilon - 2\frac{\alpha M}{d_u} W_2^\varepsilon\right) - b_1 V_2^\varepsilon\right\} - \left(1 + \frac{\alpha}{d_u} V_2^\varepsilon\right) Q(V_2^\varepsilon),$$

$$L_{13} = -\left(r_1 - 2a_1\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u} W_2^\varepsilon\right) - b_1 V_2^\varepsilon\right)\frac{\alpha M}{d_u},$$

$$L_{32} = -\varepsilon b_1 W_2^\varepsilon + \frac{1}{M}\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u} W_2^\varepsilon\right) Q(V_2^\varepsilon)$$

$$+ \left(\frac{1}{M}\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u} W_2^\varepsilon\right) V_2^\varepsilon - W_2^\varepsilon\right) Q'(V_2^\varepsilon).$$

$N(\Phi, \varepsilon)$ is given by

$$N(\Phi, \varepsilon) = \begin{pmatrix} N_1(\Phi, \varepsilon) \\ N_2(\Phi, \varepsilon) \\ N_3(\Phi, \varepsilon) \end{pmatrix},$$

where

$$N_1(\Phi,\varepsilon) = \left(-a_1\left(\varphi_1 - \frac{\alpha M}{d_u}\varphi_3\right) - b_1\varphi_2\right)\left(\varphi_1 - \frac{\alpha M}{d_u}\varphi_3\right),$$

$$N_2(\Phi,\varepsilon) = \left(-b_2\left(\varphi_1 - \frac{\alpha M}{d_u}\varphi_3\right) - a_2\varphi_2\right)\varphi_2,$$

$$\begin{aligned}N_3(\Phi,\varepsilon) = \varepsilon\Bigg(&-a_1\left(\varphi_1 - \frac{\alpha M}{d_u}\varphi_3\right) - b_1\varphi_2\Bigg)\varphi_3 + \left(1 + \frac{\alpha}{d_u}V_2^\varepsilon\right)Q(V_2^\varepsilon)\varphi_3\\
&+ \left(\frac{1}{M}\left(\tilde{U}_2^\varepsilon + \varphi_1 - \frac{\alpha M}{d_u}(W_2^\varepsilon + \varphi_3)\right)(V_2^\varepsilon + \varphi_2)\right.\\
&\left.- (W_2^\varepsilon + \varphi_3)\right)Q(V_2^\varepsilon + \varphi_2)\\
&- \left(\frac{1}{M}\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u}W_2^\varepsilon\right)V_2^\varepsilon - W_2^\varepsilon\right)Q(V_2^\varepsilon) - \frac{1}{M}V_2^\varepsilon Q(V_2^\varepsilon)\varphi_1\\
&- \left(\frac{1}{M}\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u}W_2^\varepsilon\right)Q(V_2^\varepsilon)\right.\\
&\left.+ \left(\frac{1}{M}\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u}W_2^\varepsilon\right)V_2^\varepsilon - W_2^\varepsilon\right)Q'(V_2^\varepsilon)\right)\varphi_2.\end{aligned}$$

$R(\varepsilon)$ depending only on ε is represented by

$$R(\varepsilon) = \begin{pmatrix} R_1(\varepsilon)\\ R_2(\varepsilon)\\ R_3(\varepsilon)\end{pmatrix},$$

where

$$\begin{aligned}R_1(\varepsilon) = \varepsilon^3&\left(-a_1\left(\tilde{u}_2 - \frac{\alpha M}{d_u}w_2\right) - b_1v_2\right)\left(\tilde{u}_1 - \frac{\alpha M}{d_u}w_1\right)\\
&+ \varepsilon^3\left(-a_1\left(\tilde{u}_1 + \varepsilon\tilde{u}_2 - \frac{\alpha M}{d_u}(w_1 + \varepsilon w_2)\right) - b_1(v_1 + \varepsilon v_2)\right)\left(\tilde{u}_2 - \frac{\alpha M}{d_u}w_2\right),\end{aligned}$$

$$\begin{aligned}R_2(\varepsilon) = \varepsilon^3&\left(-b_2\left(\tilde{u}_2 - \frac{\alpha M}{d_u}w_2\right) - a_2v_2\right)v_1\\
&+ \varepsilon^3\left(-b_2\left(\tilde{u}_1 + \varepsilon\tilde{u}_2 - \frac{\alpha M}{d_u}(w_1 + \varepsilon w_2)\right) - a_2(v_1 + \varepsilon v_2)\right)v_2,\end{aligned}$$

$$\begin{aligned}R_3(\varepsilon) = \varepsilon^3\Bigg\{&(d_u + \alpha M)\Delta w_2 + \left(-a_1\left(\tilde{u}_2 - \frac{\alpha M}{d_u}w_2\right) - b_1v_2\right)w_0\\
&+ \left(-a_1\left(\tilde{u}_1 + \varepsilon\tilde{u}_2 - \frac{\alpha M}{d_u}(w_1 + \varepsilon w_2)\right) - b_1(v_1 + \varepsilon v_2)\right)w_1\end{aligned}$$

$$
\begin{aligned}
&+\left(r_1 - a_1\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u} W_2^\varepsilon\right) - b_1 V_2^\varepsilon\right) w_2\Bigg\} \\
&+\left(\frac{1}{M}\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u} W_2^\varepsilon\right) V_2^\varepsilon - W_2^\varepsilon\right) Q(V_2^\varepsilon) \\
&-\varepsilon\left(\frac{1}{M}\left(\left(\tilde{u}_1 - \frac{\alpha M}{d_u} w_1\right) v_0 + \left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right) v_1\right) - w_1\right) Q(v_0) \\
&-\varepsilon^2\left(\frac{1}{M}\left(\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right) v_1 + \left(\tilde{u}_1 - \frac{\alpha M}{d_u} w_1\right) v_0\right) - w_1\right) Q'(v_0) v_1 \\
&-\varepsilon^2\left(\frac{1}{M}\left(\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right) v_2 + \left(\tilde{u}_1 - \frac{\alpha M}{d_u} w_1\right) v_1\right.\right. \\
&\left.\left.+\left(\tilde{u}_2 - \frac{\alpha M}{d_u} w_2\right) v_0\right) - w_2\right) Q(v_0).
\end{aligned}
$$

3.1. Construction of an approximate solution. (3.9) is equivalent to (3.3) since the third equation in (3.9) gives $w_0 = \frac{d_u \tilde{u}_0 v_0}{(d_u + \alpha v_0) M}$ due to $Q(v_0) > 0$. Therefore, it suffices to prove that

$$
L_0 : W_N^{2,p}(\Omega) \times W_N^{2,p}(\Omega) \times W_N^{2,p}(\Omega) \to L^p(\Omega) \times L^p(\Omega) \times W_N^{2,p}(\Omega)
$$

is invertible. By regularity result we know that $L_0^{-1}(W^{k,p}(\Omega) \times W^{k,p}(\Omega) \times W_N^{k+2,p}(\Omega)) = (W_N^{k+2,p}(\Omega) \times W_N^{k+2,p}(\Omega) \times W_N^{k+2,p}(\Omega))$ for $k = 1, 2$. Then the solutions $(\tilde{u}_1, v_1, w_1)$ and $(\tilde{u}_2, v_2, w_2)$ are obtained immediately, so that an approximate solution $(\tilde{U}_2^\varepsilon, V_2^\varepsilon, W_2^\varepsilon)$ of (3.5) is constructed. To prove the invertibility of L_0, we consider the linearlized operator $\mathscr{L}$ around $(\tilde{u}_0, v_0)$ of (3.3), which is given by

$$
\mathscr{L} := \begin{pmatrix} A' & -b_1 \frac{d_u \tilde{u}_0}{d_u + \alpha v_0} + \left(r_1 - 2a_1 \frac{d_u \tilde{u}_0}{d_u + \alpha v_0} - b_1 v_0\right) \frac{-d_u \tilde{u}_0 \alpha}{(d_u + \alpha v_0)^2} \\ -b_2 \frac{d_u v_0}{d_u + \alpha v_0} & B' \end{pmatrix},
$$

where

$$
A' = d_u \Delta + \left(r_1 - 2a_1 \frac{d_u \tilde{u}_0}{d_u + \alpha v_0} - b_1 v_0\right) \frac{d_u}{d_u + \alpha v_0},
$$

$$
B' = d_v \Delta + r_2 - b_2 \frac{d_u^2 \tilde{u}_0}{(d_u + \alpha v_0)^2} - 2a_2 v_0.
$$

Lemma 1. *Assume that* $\mathscr{L} : (W_N^{2,p}(\Omega))^2 \to (L^p(\Omega))^2$ *is invertible. Then* $L_0 : W_N^{k+2,p}(\Omega) \times W_N^{k+2,p}(\Omega) \times W_N^{k+2,p}(\Omega) \to W^{k,p}(\Omega) \times W^{k,p}(\Omega) \times W_N^{k+2,p}(\Omega)$ *is invertible for* $k = 0$ *and bijective for* $k = 1, 2$.

Proof. We first prove that for any $h_1, h_2 \in L^p(\Omega)$ and $h_3 \in W_N^{2,p}(\Omega)$

$$L_0 \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \tag{3.13}$$

has a unique solution $\xi_1, \xi_2, \xi_3 \in W_N^{2,p}(\Omega)$. Since the third equation is $\frac{v_0}{M} Q(v_0)\xi_1 + \frac{1}{M}\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right) Q(v_0)\xi_2 - \left(1 + \frac{\alpha}{d_u} v_0\right) Q(v_0)\xi_3 = h_3$, ξ_3 is described by

$$\xi_3 = \frac{\frac{1}{M} v_0}{1 + \frac{\alpha}{d_u} v_0} \xi_1 + \frac{\frac{1}{M}\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right)}{1 + \frac{\alpha}{d_u} v_0} \xi_2 - \frac{h_3}{\left(1 + \frac{\alpha}{d_u} v_0\right) Q(v_0)}.$$

Substituting this into the first two equations in (3.13), we obtain

$$\mathscr{L}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} h_1 - \left(r_1 - 2a_1\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right) - b_1 v_0\right) \frac{\alpha M}{(d_u + \alpha v_0) Q(v_0)} h_3 \\ h_2 + \frac{b_2 \alpha M v_0}{(d_u + \alpha v_0) Q(v_0)} h_3 \end{pmatrix}. \tag{3.14}$$

Now, since $\mathscr{L}$ is invertible and the right hand side belongs to $(L^p(\Omega))^2$, (3.14) has a unique solution (ξ_1, ξ_2) for any $h_1, h_2 \in L^p(\Omega)$, $h_3 \in W_N^{2,p}(\Omega)$. Moreover, since ξ_3 is obtained by $\xi_3 = \frac{\frac{1}{M} v_0}{1 + \frac{\alpha}{d_u} v_0} \xi_1 + \frac{\frac{1}{M}\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right)}{1 + \frac{\alpha}{d_u} v_0} \xi_2 - \frac{h_3}{\left(1 + \frac{\alpha}{d_u} v_0\right) Q(v_0)}$, we know that $L_0 : W_N^{2,p}(\Omega) \times W_N^{2,p}(\Omega) \times W_N^{2,p}(\Omega) \to L^p(\Omega) \times L^p(\Omega) \times W_N^{2,p}(\Omega)$ is invertible. By the regularity of the operator $\mathscr{L}$, we know that $\mathscr{L}^{-1}((W^{k,p}(\Omega))^2) = (W_N^{2,p}(\Omega) \cap W^{k+2,p}(\Omega))^2$ for $k = 1, 2$. Therefore we also find that $L_0 : W_N^{k+2,p}(\Omega) \times W_N^{k+2,p}(\Omega) \times W_N^{k+2,p}(\Omega) \to W^{k,p}(\Omega) \times W^{k,p}(\Omega) \times W_N^{k+2,p}(\Omega)$ is bijective for $k = 1, 2$. □

By using Lemma 1, we find that $(\tilde{u}_1, v_1, w_1)$ and $(\tilde{u}_2, v_2, w_2)$ are uniquely obtained.

3.2. Convergence problem. In this subsection, we prove that the correction term $\Phi = {}^T(\varphi_1, \varphi_2, \varphi_3)$ of (3.12) exists. In order to do it, we need the following two lemmas:

Lemma 2. *The operator $L_\varepsilon : X \to Y$ is invertible, and the norm of inverse operator is estimated as*

$$\|L_\varepsilon^{-1}\|_{Y \to X} \leq \frac{C_1}{\varepsilon}.$$

The proof will be shown later.

Lemma 3. (i) *Define a domain D by $D = \{\Phi \in X; \|\Phi\|_X \leq \varepsilon\}$. There exists a positive constant C_2 for $F : D \to Y$ such that*

$$\|D_\Phi F(\Phi, \varepsilon) - D_\Phi F(\tilde{\Phi}, \varepsilon)\|_{X \to Y} \leq C_2 \|\Phi - \tilde{\Phi}\|_X.$$

(ii) *There exists a positive constant* C_3 *such that*

$$\|F(0,\varepsilon)\|_Y \le \varepsilon^3 C_3.$$

Proof. (i) Let us define $D_\Phi N(\Phi,\varepsilon)$ by

$$D_\Phi N(\Phi,\varepsilon) = \begin{pmatrix} N_{11}(\Phi) & N_{12}(\Phi) & N_{13}(\Phi) \\ N_{21}(\Phi) & N_{22}(\Phi) & N_{23}(\Phi) \\ N_{31}(\Phi) & N_{32}(\Phi) & N_{33}(\Phi) \end{pmatrix},$$

where

$$N_{11}(\Phi) = -2a_1\left(\varphi_1 - \frac{\alpha M}{d_u}\varphi_3\right) - b_1\varphi_2, \qquad N_{12}(\Phi) = -b_1\left(\varphi_1 - \frac{\alpha M}{d_u}\varphi_3\right),$$

$$N_{13}(\Phi) = -\frac{\alpha M}{d_u}\left(-2a_1\left(\varphi_1 - \frac{\alpha M}{d_u}\varphi_3\right) - b_1\varphi_2\right), \quad N_{21}(\Phi) = -b_2\varphi_2,$$

$$N_{22}(\Phi) = -b_2\left(\varphi_1 - \frac{\alpha M}{d_u}\varphi_3\right) - 2a_2\varphi_2, \qquad N_{23}(\Phi) = b_2\frac{\alpha M}{d_u}\varphi_2,$$

$$N_{31}(\Phi) = -\varepsilon a_1\varphi_3 + \frac{1}{M}(V_2^\varepsilon + \varphi_2)Q(V_2^\varepsilon + \varphi_2) - \frac{1}{M}V_2^\varepsilon Q(V_2^\varepsilon),$$

$$\begin{aligned} N_{32}(\Phi) = {}& -\varepsilon b_1\varphi_3 + \frac{1}{M}\left(\tilde{U}_2^\varepsilon + \varphi_1 - \frac{\alpha M}{d_u}(W_2^\varepsilon + \varphi_3)\right)Q(V_2^\varepsilon + \varphi_2) \\ &+ \left\{\frac{1}{M}\left(\tilde{U}_2^\varepsilon + \varphi_1 - \frac{\alpha M}{d_u}(W_2^\varepsilon + \varphi_3)\right)(V_2^\varepsilon + \varphi_2)\right. \\ &\quad \left. - (W_2^\varepsilon + \varphi_3)\right\}Q'(V_2^\varepsilon + \varphi_2) \\ &- \left(\frac{1}{M}\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u}W_2^\varepsilon\right)Q(V_2^\varepsilon)\right. \\ &\quad \left. + \left(\frac{1}{M}\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u}W_2^\varepsilon\right)V_2^\varepsilon - W_2^\varepsilon\right)\right)Q'(V_2^\varepsilon), \end{aligned}$$

$$\begin{aligned} N_{33}(\Phi) = {}& \varepsilon\left(-a_1\left(\varphi_1 - 2\frac{\alpha M}{d_u}\varphi_3\right) - b_1\varphi_2\right) \\ &- \left(\frac{\alpha}{d_u}(V_2^\varepsilon + \varphi_2) + 1\right)Q(V_2^\varepsilon + \varphi_2) + \left(1 + \frac{\alpha}{d_u}V_2^\varepsilon\right)Q(V_2^\varepsilon). \end{aligned}$$

We write

$$\left\|\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}\right\|_Y = \|(D_\Phi F(\Phi,\varepsilon) - D_\Phi F(\tilde{\Phi},\varepsilon))\xi\|_Y$$

for $\xi \in X$, where

$$\|g_1\|_{L^p(\Omega)} \leq \|(N_{11}(\Phi) - N_{11}(\tilde{\Phi}))\xi_1\|_{L^p} + \|(N_{12}(\Phi) - N_{12}(\tilde{\Phi}))\xi_2\|_{L^p} + \|(N_{13}(\Phi) - N_{13}(\tilde{\Phi}))\xi_3\|_{L^p},$$

$$\|g_2\|_{L^p(\Omega)} \leq \|(N_{21}(\Phi) - N_{21}(\tilde{\Phi}))\xi_1\|_{L^p} + \|(N_{22}(\Phi) - N_{22}(\tilde{\Phi}))\xi_2\|_{L^p} + \|(N_{23}(\Phi) - N_{23}(\tilde{\Phi}))\xi_3\|_{L^p},$$

$$\|g_3\|_{L^p(\Omega)} \leq \|(N_{31}(\Phi) - N_{31}(\tilde{\Phi}))\xi_1\|_{L^p} + \|(N_{32}(\Phi) - N_{32}(\tilde{\Phi}))\xi_2\|_{L^p} + \|(N_{33}(\Phi) - N_{33}(\tilde{\Phi}))\xi_3\|_{L^p}.$$

Using Hölder's inequality and Sobolev's imbedding theorem, we obtain

$$\begin{aligned}
&\|(N_{11}(\Phi) - N_{11}(\tilde{\Phi}))\xi_1\|_{L^p} \\
&\quad \leq 2a_1\|\varphi_1 - \tilde{\varphi}_1\|_{L^{2p}}\|\xi_1\|_{L^{2p}} \\
&\qquad + 2a_1\frac{\alpha M}{d_u}\|\varphi_3 - \tilde{\varphi}_3\|_{L^{2p}}\|\xi_1\|_{L^{2p}} + b_1\|\varphi_2 - \tilde{\varphi}_2\|_{L^{2p}}\|\xi_1\|_{L^{2p}} \\
&\quad \leq C(\|\varphi_1 - \tilde{\varphi}_1\|_{W^{2,p}}\|\xi_1\|_{W^{2,p}} \\
&\qquad + \|\varphi_2 - \tilde{\varphi}_2\|_{W^{2,p}}\|\xi_1\|_{W^{2,p}} + \|\varphi_3 - \tilde{\varphi}_3\|_{W^{2,p}}\|\xi_1\|_{W^{2,p}}).
\end{aligned}$$

Since $\|(N_{12}(\Phi) - N_{12}(\tilde{\Phi}))\xi_2\|_{L^p}$ and $\|(N_{13}(\Phi) - N_{13}(\tilde{\Phi}))\xi_3\|_{L^p}$ possess similar inequalities to the above, we obtain

$$\|g_1\|_{L^p(\Omega)} \leq C\|\Phi - \tilde{\Phi}\|_X\|\xi\|_X,$$

and similarly

$$\|g_2\|_{L^p(\Omega)} \leq C\|\Phi - \tilde{\Phi}\|_X\|\xi\|_X.$$

For $\|g_3\|_{L^p(\Omega)}$, we obtain

$$\begin{aligned}
&\|(N_{31}(\Phi) - N_{31}(\tilde{\Phi}))\xi_1\|_{L^p} \\
&\quad \leq \varepsilon a_1\|\varphi_3 - \tilde{\varphi}_3\|_{L^{2p}}\|\xi_1\|_{L^{2p}} + \frac{1}{M}\|(\varphi_2 - \tilde{\varphi}_2)Q(V_2^\varepsilon + \varphi_2)\xi_1\|_{L^p} \\
&\qquad + \frac{1}{M}\|(V_2^\varepsilon + \tilde{\varphi}_2)(Q(V_2^\varepsilon + \varphi_2) - Q(V_2^\varepsilon + \tilde{\varphi}_2))\xi_1\|_{L^p} \\
&\quad \leq C(\varepsilon a_1\|\varphi_3 - \tilde{\varphi}_3\|_{W^{2,p}}\|\xi_1\|_{W^{2,p}} + \|\varphi_2 - \tilde{\varphi}_2\|_{W^{2,p}}\|\xi_1\|_{W^{2,p}}).
\end{aligned}$$

Here we used $\|Q(V_2^\varepsilon + \varphi_2) - Q(V_2^\varepsilon + \tilde{\varphi}_2)\|_{L^p} \le C\|\varphi_2 - \tilde{\varphi}_2\|_{W^{2,p}}$ and $\|Q(V_2^\varepsilon + \varphi_2)\|_{L^\infty} \le C_0$.

Since $\|(N_{32}(\Phi) - N_{32}(\tilde{\Phi}))\xi_2\|_{L^p}$ and $\|(N_{33}(\Phi) - N_{33}(\tilde{\Phi}))\xi_3\|_{L^p}$ are treated similarly, we obtain

$$\|g_3\|_{L^p(\Omega)} \le C\|\Phi - \tilde{\Phi}\|_X \|\xi\|_X.$$

Therefore, we have

$$\|(D_\Phi F(\Phi, \varepsilon) - D_\Phi F(\tilde{\Phi}, \varepsilon))\xi\|_Y \le C_2\|\Phi - \tilde{\Phi}\|_X \|\xi\|_X$$

and then

$$\|D_\Phi F(\Phi, \varepsilon) - D_\Phi F(\tilde{\Phi}, \varepsilon)\|_{X \to Y} \le C_2\|\Phi - \tilde{\Phi}\|_X.$$

(ii) is obvious because $R(\varepsilon)$ consists of only ε^3 or more. □

The above-mentioned lemmas can show that a function of Φ satisfying

$$F(\Phi, \varepsilon) = L_\varepsilon \Phi + N(\Phi, \varepsilon) + R(\varepsilon) = 0$$

is obtained as a limit of $\{\Phi_n\}$ constructed by the following successive approximation:

$$\begin{cases} F(\Phi_n, \varepsilon) + L_\varepsilon(\Phi_{n+1} - \Phi_n) = 0, \quad n = 0, 1, 2, \dots \\ \Phi_0 = 0. \end{cases} \tag{3.15}$$

To show this, we require the following lemma:

LEMMA 4. *Define a closed sphere B_a as $B_a = \{\Phi \mid \|\Phi\|_X \le a\}$. Then, for a sufficiently small a, Φ_n satisfies*

$$\Phi_n \in B_a \qquad (n = 0, 1, 2, \dots).$$

PROOF. Let $\Phi_n \in B_a$. We rewrite (3.15) as

$$\begin{aligned} \Phi_{n+1} &= \Phi_n - L_\varepsilon^{-1} F(\Phi_n, \varepsilon) \\ &= L_\varepsilon^{-1}\{L_\varepsilon \Phi_n - F(\Phi_n, \varepsilon) + F(0, \varepsilon)\} - L_\varepsilon^{-1} F(0, \varepsilon) \\ &= L_\varepsilon^{-1}\left\{L_\varepsilon \Phi_n - \int_0^1 D_\Phi F(t\Phi_n, \varepsilon)\Phi_n \, dt\right\} - L_\varepsilon^{-1} F(0, \varepsilon) \\ &= L_\varepsilon^{-1} \int_0^1 (L_\varepsilon - D_\Phi F(t\Phi_n, \varepsilon))\Phi_n \, dt - L_\varepsilon^{-1} F(0, \varepsilon). \end{aligned}$$

Therefore,

$$\|\Phi_{n+1}\|_X \le \|L_\varepsilon^{-1}\|_{Y\to X}\left\{\int_0^1 \|L_\varepsilon - D_\Phi F(t\Phi_n,\varepsilon)\|_{X\to Y}\|\Phi_n\|_X dt + \|F(0,\varepsilon)\|_Y\right\}$$

$$\le \|L_\varepsilon^{-1}\|_{Y\to X}(C_2\|\Phi_n\|_X^2 + \|F(0,\varepsilon)\|_Y).$$

By choosing a and ε so as to satisfy

$$C_2\|L_\varepsilon^{-1}\|_{Y\to X} a \le \frac{1}{2}, \qquad \|L_\varepsilon^{-1}\|_{Y\to X}\|F(0,\varepsilon)\|_Y \le \frac{a}{2}, \tag{3.16}$$

we find

$$\|L_\varepsilon^{-1}\|_{Y\to X}(C_2\|\Phi_n\|_X^2 + \|F(0,\varepsilon)\|_Y) \le \frac{\|\Phi_n\|_X^2}{2a} + \frac{a}{2} \le \frac{a}{2} + \frac{a}{2} = a.$$

Consequently, if $\Phi_n \in B_a$, then $\Phi_{n+1} \in B_a$. Since $\Phi_0 = 0 \in B_a$, $\Phi_n \in B_a$ for $n \in \mathbf{N}$. □

Proof (Theorem 3). We now show the existence and convergence of $\{\Phi_n\}$ as follows

$$\Phi_{n+1} - \Phi_n = L_\varepsilon^{-1}\left\{L_\varepsilon(\Phi_n - \Phi_{n-1}) - \int_0^1 D_\Phi F(t\Phi_n,\varepsilon)\Phi_n\, dt + \int_0^1 D_\Phi F(t\Phi_{n-1},\varepsilon)\Phi_{n-1}\, dt\right\}$$

$$= L_\varepsilon^{-1}\left[\int_0^1 \{L_\varepsilon - D_\Phi F(t\Phi_n,\varepsilon)\}(\Phi_n - \Phi_{n-1})dt - \int_0^1 \{D_\Phi F(t\Phi_n,\varepsilon) - D_\Phi F(t\Phi_{n-1},\varepsilon)\}\Phi_{n-1}\, dt\right]$$

and then

$$\|\Phi_{n+1} - \Phi_n\|_X \le \|L_\varepsilon^{-1}\|_{Y\to X} C_2\{\|\Phi_n\|_X + \|\Phi_{n-1}\|_X\}\|\Phi_n - \Phi_{n-1}\|_X.$$

By choosing a so as to satisfy $C_2\|L_\varepsilon^{-1}\|a \le \frac{1}{4}$, we find

$$\|\Phi_{n+1} - \Phi_n\|_X \le \frac{1}{2}\|\Phi_n - \Phi_{n-1}\|_X.$$

Consequently, by Banach's fixed point theorem, $\Phi = \Phi(\varepsilon)$ which satisfies $F(\Phi,\varepsilon) = 0$ exists uniquely if a and ε are chosen sufficiently small. Finally, since $\|\Phi(\varepsilon)\|_X \le a \le \frac{\varepsilon}{4C_1C_2}$, $\|\Phi(\varepsilon)\|_X \to 0$ as $\varepsilon \to 0$.

We thus know that a solution of the problem (3.5) approximates the solution of the problem (3.3) for a sufficiently small ε. Namely, there exists a unique solution $(\tilde{U}(x;\varepsilon), V(x;\varepsilon), W(x;\varepsilon))$ of (3.5) which is given by the following form:

$$\tilde{U}(x;\varepsilon) = \tilde{u}_0(x) + \varepsilon\tilde{u}_1(x) + \varepsilon^2\tilde{u}_2(x) + \varphi_1(x;\varepsilon),$$

$$V(x;\varepsilon) = v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x) + \varphi_2(x;\varepsilon),$$

$$W(x;\varepsilon) = w_0(x) + \varepsilon w_1(x) + \varepsilon^2 w_2(x) + \varphi_3(x;\varepsilon),$$

satisfying

$$\lim_{\varepsilon\to+0} \|\tilde{U}(\cdot;\varepsilon) - \tilde{u}_0(\cdot)\|_{W^{2,p}(\Omega)} = 0,$$

$$\lim_{\varepsilon\to+0} \|V(\cdot;\varepsilon) - v_0(\cdot)\|_{W^{2,p}(\Omega)} = 0,$$

$$\lim_{\varepsilon\to+0} \|W(\cdot;\varepsilon) - w_0(\cdot)\|_{W^{2,p}(\Omega)} = 0. \qquad \square$$

3.3. Proof of Lemma 2. In this subsection, we will prove Lemma 2. In order to do it, we use some transformations. We transform the linear operator $L_\varepsilon\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix}$ by using

$$\tilde{\varphi}_3 = \frac{1}{M}v_0\varphi_1 + \frac{1}{M}\left(\tilde{u}_0 - \frac{\alpha M}{d_u}w_0\right)\varphi_2 - \left(1 + \frac{\alpha}{d_u}v_0\right)\varphi_3. \tag{3.17}$$

Then, it is transformed into

$$\tilde{L}_\varepsilon = \begin{pmatrix} \tilde{A}_\varepsilon & \tilde{L}_{12} & \left(r_1 - 2a_1\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u}W_2^\varepsilon\right) - b_1 V_2^\varepsilon\right)\frac{\alpha M}{d_u+\alpha v_0} \\ -b_2\frac{d_u V_2^\varepsilon}{d_u+\alpha v_0} & \tilde{B}_\varepsilon & -b_2\frac{\alpha M V_2^\varepsilon}{d_u+\alpha v_0} \\ \varepsilon\tilde{L}_{31} & \varepsilon\tilde{L}_{32} & \tilde{C}_\varepsilon + \varepsilon\tilde{L}_{33} \end{pmatrix},$$

where

$$\tilde{A}_\varepsilon = d_u\Delta + \left(r_1 - 2a_1\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u}W_2^\varepsilon\right) - b_1V_2^\varepsilon\right)\frac{d_u}{d_u + \alpha v_0},$$

$$\tilde{B}_\varepsilon = d_v\Delta + r_2 - b_2\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u}W_2^\varepsilon\right) - 2a_2V_2^\varepsilon + \frac{b_2\alpha V_2^\varepsilon\left(\tilde{u}_0 - \frac{\alpha M}{d_u}w_0\right)}{d_u + \alpha v_0},$$

$$\tilde{C}_\varepsilon = -\varepsilon(d_u + \alpha M)\Delta\left(\frac{d_u}{d_u + \alpha v_0}\cdot\right) + Q(V_2^\varepsilon),$$

$$\tilde{L}_{12} = -b_1\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u} W_2^\varepsilon\right) - \left(r_1 - 2a_1\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u} W_2^\varepsilon\right) - b_1 V_2^\varepsilon\right)\frac{\alpha\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right)}{d_u + \alpha v_0},$$

$$\tilde{L}_{31} = -a_1 W_2^\varepsilon + \frac{d_u(v_1 + \varepsilon v_2)}{(d_u + \alpha v_0)M} Q(V_2^\varepsilon) + (d_u + \alpha M)\Delta\left(\frac{d_u v_0}{(d_u + \alpha v_0)M}\cdot\right)$$

$$+ \left(r_1 - a_1\left(\tilde{U}_2^\varepsilon - 2\frac{\alpha M}{d_u} W_2^\varepsilon\right) - b_1 V_2^\varepsilon\right)\frac{d_u v_0}{(d_u + \alpha v_0)M},$$

$$\tilde{L}_{32} = -b_1 W_2^\varepsilon + \frac{1}{M}\left(\tilde{u}_1 + \varepsilon\tilde{u}_2 - \frac{\alpha M}{d_u}(w_1 + \varepsilon w_2)\right.$$

$$\left. - \frac{v_1 + \varepsilon v_2}{d_u + \alpha v_0}\alpha\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right)\right) Q(V_2^\varepsilon)$$

$$+ \frac{1}{\varepsilon}\left(\frac{1}{M}\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u} W_2^\varepsilon\right) V_2^\varepsilon - W_2^\varepsilon - \frac{1}{M}\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right) v_0 + w_0\right) Q'(V_2^\varepsilon)$$

$$+ (d_u + \alpha M)\Delta\left(\frac{d_u\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right)}{(d_u + \alpha v_0)M}\cdot\right)$$

$$+ \left(r_1 - a_1\left(\tilde{U}_2^\varepsilon - 2\frac{\alpha M}{d_u} W_2^\varepsilon\right) - b_1 V_2^\varepsilon\right)\frac{d_u\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right)}{(d_u + \alpha v_0)M},$$

$$\tilde{L}_{33} = -\left(r_1 - a_1\left(\tilde{U}_2^\varepsilon - 2\frac{\alpha}{d_u} W_2^\varepsilon\right) - b_1 V_2^\varepsilon\right)\frac{d_u}{d_u + \alpha v_0} + (v_1 + \varepsilon v_2)\frac{\alpha}{d_u + \alpha v_0} Q(V_2^\varepsilon).$$

We express the 2×2 block at the upper left in $\tilde{L}_\varepsilon$ as follows:

$$\mathscr{L}_\varepsilon := \begin{pmatrix} \tilde{A}_\varepsilon & \tilde{L}_{12} \\ -b_2 \frac{d_u V_2^\varepsilon}{d_u + \alpha v_0} & \tilde{B}_\varepsilon \end{pmatrix} = \mathscr{L} + \varepsilon\tilde{\mathscr{L}}.$$

Here, we note that $\mathscr{L}$ is the linear operator of (3.3) around $(\tilde{u}_0, v_0)$ and $\tilde{\mathscr{L}}$ denote

$$\tilde{\mathscr{L}} = \begin{pmatrix} \tilde{\mathscr{L}}_{11} & \tilde{\mathscr{L}}_{12} \\ \tilde{\mathscr{L}}_{21} & \tilde{\mathscr{L}}_{22} \end{pmatrix},$$

where

$$\tilde{\mathscr{L}}_{11} = \left(-2a_1\left(\tilde{u}_1 + \varepsilon\tilde{u}_2 - \frac{\alpha M}{d_u}(w_1 + \varepsilon w_2)\right) - b_1(v_1 + \varepsilon v_2)\right)\frac{d_u}{d_u + \alpha v_0},$$

$$\tilde{\mathscr{L}}_{12} = -b_1\left(\tilde{u}_1 + \varepsilon\tilde{u}_2 - \frac{\alpha M}{d_u}(w_1 + \varepsilon w_2)\right)$$

$$- \left(-2a_1\left(\tilde{u}_1 + \varepsilon\tilde{u}_2 - \frac{\alpha M}{d_u}(w_1 + \varepsilon w_2)\right) - b_1(v_1 + \varepsilon v_2)\right)\frac{d_u\alpha\tilde{u}_0}{(d_u + \alpha v_0)^2},$$

$$\tilde{\mathscr{L}}_{21} = -b_2 \frac{d_u}{d_u + \alpha v_0}(v_1 + \varepsilon v_2),$$

$$\tilde{\mathscr{L}}_{22} = -b_2\left(\tilde{u}_1 + \varepsilon \tilde{u}_2 - \frac{\alpha M}{d_u}(w_1 + \varepsilon w_2)\right) - 2a_2(v_1 + \varepsilon v_2) + \frac{b_2 \alpha\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right)}{d_u + \alpha v_0}(v_1 + \varepsilon v_2).$$

So, if ε is sufficiently small, then $\varepsilon\tilde{\mathscr{L}}$ is regarded as a perturbation. Since the operator $\mathscr{L}$ is assumed to be invertible, we can show that $\mathscr{L}_\varepsilon$ is also invertible, if ε is sufficiently small.

Lemma 5. *The operator* $\mathscr{L}_\varepsilon : (W_N^{2,p}(\Omega))^2 \to (L^p(\Omega))^2$ *is invertible and satisfies*

$$\|\mathscr{L}_\varepsilon^{-1}\|_{(L^p(\Omega))^2 \to (W_N^{2,p}(\Omega))^2} \le C_4.$$

In addition, $\mathscr{L}_\varepsilon : (W_N^{4,p}(\Omega))^2 \to (W^{2,p}(\Omega))^2$ *is invertible and satisfies*

$$\|\mathscr{L}_\varepsilon^{-1}\|_{(W^{2,p}(\Omega))^2 \to (W_N^{4,p}(\Omega))^2} \le C_5.$$

Proof. We find that $\tilde{\mathscr{L}} : (W_N^{2,p}(\Omega))^2 \to (L^p(\Omega))^2$ is a bounded linear operator by applying Hölder's inequality and Sobolev's embedding theorem to $\tilde{\mathscr{L}}$. We know

$$\left\|\tilde{\mathscr{L}}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\right\|_{(L^p(\Omega))^2} \le C\left\|\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\right\|_{(W_N^{2,p}(\Omega))^2}.$$

$\tilde{\mathscr{L}} : (W_N^{4,p}(\Omega))^2 \to (W^{2,p}(\Omega))^2$ is also a bounded linear operator in a similar way

$$\left\|\tilde{\mathscr{L}}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\right\|_{(W^{2,p}(\Omega))^2} \le C'\left\|\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\right\|_{(W_N^{4,p}(\Omega))^2}.$$

Now, as we assume that $\mathscr{L}$ is invertible, there exist positive constants $M_1 > 0$ and $M_2 > 0$ such that

$$\|\mathscr{L}^{-1}\|_{(L^p(\Omega))^2 \to (W_N^{2,p}(\Omega))^2} \le M_1$$

$$\|\mathscr{L}^{-1}\|_{(W^{2,p}(\Omega))^2 \to (W_N^{4,p}(\Omega))^2} \le M_2.$$

We choose ε sufficiently small to satisfy the following inequalities:

$$\|\mathscr{L}^{-1}\varepsilon\tilde{\mathscr{L}}\|_{(W_N^{2,p})^2 \to (W_N^{2,p})^2} \le \varepsilon\|\mathscr{L}^{-1}\|_{(L^p)^2 \to (W_N^{2,p})^2}\|\tilde{\mathscr{L}}\|_{(W_N^{2,p})^2 \to (L^p)^2} \le \varepsilon M_1 C \le \frac{1}{2},$$

$$\|\mathscr{L}^{-1}\varepsilon\tilde{\mathscr{L}}\|_{(W_N^{4,p})^2\to(W_N^{4,p})^2} \le \varepsilon\|\mathscr{L}^{-1}\|_{(W^{2,p})^2\to(W_N^{4,p})^2}\|\tilde{\mathscr{L}}\|_{(W_N^{4,p})^2\to(W^{2,p})^2}$$
$$\le \varepsilon M_2 C' \le \frac{1}{2}.$$

By Banach's perturbation theorem ([4]I-§4.4 P.31), we find that $\mathscr{L}_\varepsilon = \mathscr{L} + \varepsilon\tilde{\mathscr{L}}$ are invertible and

$$\|\mathscr{L}_\varepsilon^{-1}\|_{(L^p)^2\to(W_N^{2,p})^2} \le \frac{\|\mathscr{L}^{-1}\|_{(L^p)^2\to(W_N^{2,p})^2}}{1-\|\mathscr{L}^{-1}\varepsilon\tilde{\mathscr{L}}\|_{(W_N^{2,p})^2\to(W_N^{2,p})^2}}$$
$$\le 2\|\mathscr{L}^{-1}\|_{(L^p)^2\to(W_N^{2,p})^2} \le C_4,$$
$$\|\mathscr{L}_\varepsilon^{-1}\|_{(W^{2,p})^2\to(W_N^{4,p})^2} \le \frac{\|\mathscr{L}^{-1}\|_{(W^{2,p})^2\to(W_N^{4,p})^2}}{1-\|\mathscr{L}^{-1}\varepsilon\tilde{\mathscr{L}}\|_{(W_N^{4,p})^2\to(W_N^{4,p})^2}}$$
$$\le 2\|\mathscr{L}^{-1}\|_{(W^{2,p})^2\to(W_N^{4,p})^2} \le C_5,$$

respectively. □

We write

$$\tilde{L}_\varepsilon\begin{pmatrix}\varphi_1\\ \varphi_2\\ \tilde{\varphi}_3\end{pmatrix} = \begin{pmatrix}h_1\\ h_2\\ h_3\end{pmatrix} \tag{3.18}$$

for any $h_1, h_2, h_3 \in L^p(\Omega)$. Then, by Lemma 5, (φ_1, φ_2) can be solved as follows:

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \mathscr{L}_\varepsilon^{-1}\begin{pmatrix}h_1\\ h_2\end{pmatrix} + \mathscr{L}_\varepsilon^{-1}\begin{pmatrix} -\left(r_1 - 2a_1\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u}W_2^\varepsilon\right) - b_1 V_2^\varepsilon\right)\frac{\alpha M}{d_u+\alpha v_0}\tilde{\varphi}_3 \\ \frac{b_2\alpha M V_2^\varepsilon}{d_u+\alpha v_0}\tilde{\varphi}_3\end{pmatrix}$$
$$=: \begin{pmatrix}\tilde{h}_1\\ \tilde{h}_2\end{pmatrix} + \begin{pmatrix}\tilde{\varphi}_1(\tilde{\varphi}_3)\\ \tilde{\varphi}_2(\tilde{\varphi}_3)\end{pmatrix}. \tag{3.19}$$

Substituting this into the third equation of (3.18), we obtain

$$-\varepsilon(d_u + \alpha M)\Delta\left(\frac{d_u}{d_u + \alpha v_0}\tilde{\varphi}_3\right) + Q(V_2^\varepsilon)\tilde{\varphi}_3 + \varepsilon K\tilde{\varphi}_3 = \tilde{h}_3, \tag{3.20}$$

where

$$\tilde{h}_3 = h_3 - [\varepsilon\tilde{L}_{31}\tilde{h}_1 + \varepsilon\tilde{L}_{32}\tilde{h}_2],$$
$$K\tilde{\varphi}_3 = \tilde{L}_{31}\tilde{\varphi}_1(\tilde{\varphi}_3) + \tilde{L}_{32}\tilde{\varphi}_2(\tilde{\varphi}_3) + \tilde{L}_{33}\tilde{\varphi}_3.$$

Thus, if we could prove that there exists uniquely $\tilde{\varphi}_3 \in W_N^{2,p}(\Omega)$ for any $\tilde{h}_3 \in L^p(\Omega)$, then we know that $\tilde{L}_\varepsilon$ is invertible. Taking εK as a perturbation in (3.20), we consider the main part $-\varepsilon(d_u + \alpha M)\Delta\big(\frac{d_u}{d_u+\alpha v_0}\cdot\big) + Q(V_2^\varepsilon)$. We write it as $T\tilde{\varphi}_3 = -\varepsilon(d_u + \alpha M)\Delta\big(\frac{d_u}{d_u+\alpha v_0}\tilde{\varphi}_3\big) + Q(V_2^\varepsilon)\tilde{\varphi}_3$, and moreover by $\psi = \frac{d_u}{d_u+\alpha v_0}\tilde{\varphi}_3$ write it as

$$\tilde{T}\psi = -\varepsilon(d_u + \alpha M)\Delta\psi + \frac{d_u + \alpha v_0}{d_u}Q(V_2^\varepsilon)\psi.$$

By using (3.6) and Theorem 2.4.2.7 in [5], one finds that $\tilde{T}$ becomes a bijection map from $W_N^{2,p}(\Omega)$ to $L^p(\Omega)$. Therefore, T becomes also a bijection map from $W_N^{2,p}(\Omega)$ to $L^p(\Omega)$. Consequently T is a Fredholm operator with index 0. If the operator K is compact, then Fredholm stability theorem indicates that the operator $T + \varepsilon K : W_N^{2,p}(\Omega) \to L^p(\Omega)$ is a Fredholm operator with index 0.

Lemma 6. *K is a bounded linear operator: $L^p(\Omega) \to L^p(\Omega)$ and a bounded linear operator: $W_N^{2,p}(\Omega) \to W^{2,p}(\Omega)$. Futhermore, K is a compact operator: $W_N^{2,p}(\Omega) \to L^p(\Omega)$, and*

$$\|K\tilde{\varphi}_3\|_{L^p(\Omega)} \le C_6\|\tilde{\varphi}_3\|_{L^p(\Omega)},$$

$$\|K\tilde{\varphi}_3\|_{W^{2,p}(\Omega)} \le C_7\|\tilde{\varphi}_3\|_{W_N^{2,p}(\Omega)},$$

$$\|K\tilde{\varphi}_3\|_{L^p(\Omega)} \le C_8\|\tilde{\varphi}_3\|_{W_N^{2,p}(\Omega)}.$$

Proof. By (3.19) and Lemma 5, we find

$$\begin{aligned}
&\left\|\begin{pmatrix}\tilde{\varphi}_1(\tilde{\varphi}_3)\\ \tilde{\varphi}_2(\tilde{\varphi}_3)\end{pmatrix}\right\|_{(W_N^{2,p}(\Omega))^2} \\
&\quad\le C_4\left\|\mathscr{L}_\varepsilon\begin{pmatrix}\tilde{\varphi}_1(\tilde{\varphi}_3)\\ \tilde{\varphi}_2(\tilde{\varphi}_3)\end{pmatrix}\right\|_{(L^p(\Omega))^2} \\
&\quad\le C_4\Bigg(\left\|\left(r_1 - 2a_1\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u}W_2^\varepsilon\right) - b_1V_2^\varepsilon\right)\frac{\alpha M}{d_u + \alpha v_0}\right\|_{L^\infty}\|\tilde{\varphi}_3\|_{L^p} \\
&\qquad\qquad + \left\|\frac{b_2\alpha MV_2^\varepsilon}{d_u + \alpha v_0}\right\|_{L^\infty}\|\tilde{\varphi}_3\|_{L^p}\Bigg) \\
&\quad\le C_4'\|\tilde{\varphi}_3\|_{L^p(\Omega)}.
\end{aligned}$$

By similar argument to the above, we also find

$$\left\|\begin{pmatrix}\tilde{\varphi}_1(\tilde{\varphi}_3)\\ \tilde{\varphi}_2(\tilde{\varphi}_3)\end{pmatrix}\right\|_{(W_N^{4,p}(\Omega))^2}$$
$$\leq C_5\left\|\mathscr{L}_\varepsilon\begin{pmatrix}\tilde{\varphi}_1(\tilde{\varphi}_3)\\ \tilde{\varphi}_2(\tilde{\varphi}_3)\end{pmatrix}\right\|_{(W^{2,p}(\Omega))^2}$$
$$\leq C_5\left\|\left(r_1-2a_1\left(\tilde{U}_2^\varepsilon-\frac{\alpha M}{d_u}W_2^\varepsilon\right)-b_1V_2^\varepsilon\right)\frac{\alpha M}{d_u+\alpha v_0}\right\|_{W^{2,p}}\|\tilde{\varphi}_3\|_{W^{2,p}}$$
$$+\left\|\frac{b_2\alpha M V_2^\varepsilon}{d_u+\alpha v_0}\right\|_{W^{2,p}}\|\tilde{\varphi}_3\|_{W^{2,p}}$$
$$\leq C_5'\|\tilde{\varphi}_3\|_{W^{2,p}(\Omega)}.$$

Thus, we know

$$\|\tilde{\varphi}_1(\tilde{\varphi}_3)\|_{W_N^{2,p}(\Omega)}\leq C_4'\|\tilde{\varphi}_3\|_{L^p(\Omega)},$$
$$\|\tilde{\varphi}_2(\tilde{\varphi}_3)\|_{W_N^{2,p}(\Omega)}\leq C_4'\|\tilde{\varphi}_3\|_{L^p(\Omega)},$$
$$\|\tilde{\varphi}_1(\tilde{\varphi}_3)\|_{W_N^{4,p}(\Omega)}\leq C_5'\|\tilde{\varphi}_3\|_{W^{2,p}(\Omega)},$$
$$\|\tilde{\varphi}_2(\tilde{\varphi}_3)\|_{W_N^{4,p}(\Omega)}\leq C_5'\|\tilde{\varphi}_3\|_{W^{2,p}(\Omega)}.$$

By using them, we obtain the following inequalities for some positive constants C_6 and C_7:

$$\begin{aligned}\|K\tilde{\varphi}_3\|_{L^p(\Omega)}&\leq\|\tilde{L}_{31}\tilde{\varphi}_1(\tilde{\varphi}_3)\|_{L^p(\Omega)}+\|\tilde{L}_{32}\tilde{\varphi}_2(\tilde{\varphi}_3)\|_{L^p(\Omega)}+\|\tilde{L}_{33}\tilde{\varphi}_3\|_{L^p(\Omega)}\\&\leq C\|\tilde{\varphi}_1(\tilde{\varphi}_3)\|_{W_N^{2,p}(\Omega)}+C\|\tilde{\varphi}_2(\tilde{\varphi}_3)\|_{W_N^{2,p}(\Omega)}+C\|\tilde{\varphi}_3\|_{L^p(\Omega)}\\&\leq C_6\|\tilde{\varphi}_3\|_{L^p(\Omega)},\\\|K\tilde{\varphi}_3\|_{W^{2,p}(\Omega)}&\leq\|\tilde{L}_{31}\tilde{\varphi}_1(\tilde{\varphi}_3)\|_{W^{2,p}(\Omega)}+\|\tilde{L}_{32}\tilde{\varphi}_2(\tilde{\varphi}_3)\|_{W^{2,p}(\Omega)}+\|\tilde{L}_{33}\tilde{\varphi}_3\|_{W^{2,p}(\Omega)}\\&\leq C\|\tilde{\varphi}_1(\tilde{\varphi}_3)\|_{W_N^{4,p}(\Omega)}+C\|\tilde{\varphi}_2(\tilde{\varphi}_3)\|_{W_N^{4,p}(\Omega)}+C\|L_{33}\|_{W^{2,p}(\Omega)}\|\tilde{\varphi}_3\|_{W^{2,p}(\Omega)}\\&\leq C_7\|\tilde{\varphi}_3\|_{W^{2,p}(\Omega)}.\end{aligned}$$

Therefore, we find that K is a bounded linear oparator: $L^p(\Omega)\to L^p(\Omega)$ and $W^{2,p}(\Omega)\to W^{2,p}(\Omega)$. Since the injection of $W^{2,p}(\Omega)$ into $L^p(\Omega)$ is compact, K is a compact operator: $W_N^{2,p}(\Omega)\to L^p(\Omega)$. Consequently, we obtain

$$\|K\tilde{\varphi}_3\|_{L^p(\Omega)}\leq C\|K\tilde{\varphi}_3\|_{W^{2,p}(\Omega)}\leq C_8\|\tilde{\varphi}_3\|_{W_N^{2,p}(\Omega)}.$$ □

Therefore, the Fredholm stability theorem shows that $T + \varepsilon K : W_N^{2,p}(\Omega) \to L^p(\Omega)$ is a Fredholm operator with index 0. Thus, it suffices to show that $T + \varepsilon K$ is injection.

LEMMA 7. *$T + \varepsilon K : W_N^{2,p}(\Omega) \to L^p(\Omega)$ is bijective and there exists a positive constant $C_9 > 0$ such that*

$$\|(T + \varepsilon K)^{-1}\|_{L^p(\Omega) \to W_N^{2,p}(\Omega)} \le \frac{C_9}{\varepsilon}.$$

PROOF. Let us consider

$$\tilde{T}\psi = -\varepsilon(d_u + \alpha M)\Delta\psi + \frac{d_u + \alpha v_0}{d_u} Q(V_2^\varepsilon)\psi.$$

By multiplying $\psi_\delta^*(x) = (\psi(x)^2 + \delta)^{(p-2)/2}\psi(x)$, $\delta > 0$ and using integration by parts and the Lebesgue dominated convergence theorem, we obtain

$$\|\psi\|_{L^p(\Omega)} \le \frac{1}{\beta}\|\tilde{T}\psi\|_{L^p(\Omega)}.$$

Because of $\psi = \frac{d_u}{d_u + \alpha v_0}\tilde{\varphi}_3$, we know

$$\|\tilde{\varphi}_3\|_{L^p(\Omega)} \le C\left\|-\varepsilon(d_u + \alpha M)\Delta\left(\frac{d_u}{d_u + \alpha v_0}\tilde{\varphi}_3\right) + Q(V_2^\varepsilon)\tilde{\varphi}_3\right\|_{L^p(\Omega)}.$$

Also, we know

$$\begin{aligned}\|T\tilde{\varphi}_3 + \varepsilon K\tilde{\varphi}_3\|_{L^p(\Omega)} &\ge \|T\tilde{\varphi}_3\|_{L^p(\Omega)} - \varepsilon\|K\tilde{\varphi}_3\|_{L^p(\Omega)} \\ &\ge \left(\frac{1}{C} - \varepsilon C_6\right)\|\tilde{\varphi}_3\|_{L^p(\Omega)}.\end{aligned}$$

Therefore, by choosing ε sufficiently small, we obtain

$$\|\tilde{\varphi}_3\|_{L^p(\Omega)} \le C\|T\tilde{\varphi}_3 + \varepsilon K\tilde{\varphi}_3\|_{L^p(\Omega)}. \tag{3.21}$$

Consequently, $T + \varepsilon K : W_N^{2,p}(\Omega) \to L^p(\Omega)$ is injection. By property of Fredholm operator, $T + \varepsilon K$ is bijective.

Next, we estimate the inverse operator of $T + \varepsilon K$. Applying Theorem 2.3.3.2 in [5] to $-(d_u + \alpha M)\nabla\left(\frac{d_u}{d_u + \alpha v_0}\nabla\tilde{\varphi}_3\right)$, we find for any $\tilde{\varphi}_3 \in W_N^{2,p}(\Omega)$,

$$\begin{aligned}\|\tilde{\varphi}_3\|_{W_N^{2,p}(\Omega)} &\le C\left(\left\|-(d_u + \alpha M)\nabla\left(\frac{d_u}{d_u + \alpha v_0}\nabla\tilde{\varphi}_3\right)\right\|_{L^p(\Omega)} + \|\tilde{\varphi}_3\|_{W^{1,p}(\Omega)}\right) \\ &= C\left(\left\|-(d_u + \alpha M)\nabla\left(\frac{d_u}{d_u + \alpha v_0}\nabla\tilde{\varphi}_3\right) + (d_u + \alpha M)\nabla\left(\tilde{\varphi}_3\nabla\frac{d_u}{d_u + \alpha v_0}\right)\right.\right. \\ &\quad \left.\left. - (d_u + \alpha M)\nabla\left(\tilde{\varphi}_3\nabla\frac{d_u}{d_u + \alpha v_0}\right)\right\|_{L^p(\Omega)} + \|\tilde{\varphi}_3\|_{W^{1,p}(\Omega)}\right)\end{aligned}$$

$$\leq C\left(\left\|-(d_u+\alpha M)\Delta\left(\frac{d_u}{d_u+\alpha v_0}\tilde{\varphi}_3\right)\right\|_{L^p(\Omega)} + C\|\tilde{\varphi}_3\|_{W^{1,p}(\Omega)}\right)$$

$$\leq C'\left(\left\|-(d_u+\alpha M)\Delta\left(\frac{d_u}{d_u+\alpha v_0}\tilde{\varphi}_3\right)\right\|_{L^p(\Omega)} + \|\tilde{\varphi}_3\|_{W^{1,p}(\Omega)}\right).$$

Using the interporation inequality $\|u\|_{W^{1,p}(\Omega)} \leq \delta\|u\|_{W^{2,p}(\Omega)} + \frac{K}{\delta}\|u\|_{L^p(\Omega)}$, we know

$$\|\tilde{\varphi}_3\|_{W_N^{2,p}(\Omega)} \leq C'\left(\left\|-(d_u+\alpha M)\Delta\left(\frac{d_u}{d_u+\alpha v_0}\tilde{\varphi}_3\right)\right\|_{L^p(\Omega)} + \|\tilde{\varphi}_3\|_{L^p(\Omega)}\right)$$

$$\leq C'\left(\left\|-(d_u+\alpha M)\Delta\left(\frac{d_u}{d_u+\alpha v_0}\tilde{\varphi}_3\right) + K\tilde{\varphi}_3 + \frac{1}{\varepsilon}Q(V_2^\varepsilon)\tilde{\varphi}_3\right\|_{L^p(\Omega)}\right.$$

$$\left. + \|K\tilde{\varphi}_3\|_{L^p(\Omega)} + \frac{1}{\varepsilon}\|Q(V_2^\varepsilon)\|_{L^\infty(\Omega)}\|\tilde{\varphi}_3\|_{L^p(\Omega)} + \|\tilde{\varphi}_3\|_{L^p(\Omega)}\right)$$

$$\leq C'\left(\frac{1}{\varepsilon}\|T\tilde{\varphi}_3 + \varepsilon K\tilde{\varphi}_3\|_{L^p(\Omega)} + \left(C_6 + \frac{C}{\varepsilon} + 1\right)\|\tilde{\varphi}_3\|_{L^p(\Omega)}\right).$$

Here, applying L^p estimate (3.21), we obtain

$$\|\tilde{\varphi}_3\|_{W_N^{2,p}(\Omega)} \leq \frac{C_9}{\varepsilon}\|T\tilde{\varphi}_3 + \varepsilon K\tilde{\varphi}_3\|_{L^p(\Omega)}.$$

Therefore we arrive at

$$\|(T+\varepsilon K)^{-1}\|_{L^p(\Omega)\to W_N^{2,p}(\Omega)} \leq \frac{C_9}{\varepsilon}. \qquad \square$$

Proof (Proof of Lemma 2). By the above lemma, we know

$$\|\tilde{\varphi}_3\|_{W_N^{2,p}(\Omega)} \leq \frac{C_9}{\varepsilon}\|T\tilde{\varphi}_3 + \varepsilon K\tilde{\varphi}_3\|_{L^p(\Omega)} = \frac{C_9}{\varepsilon}\|\tilde{h}_3\|_{L^p(\Omega)}$$

$$\leq \frac{C_9}{\varepsilon}(\|h_3\|_{L^p(\Omega)} + \varepsilon\|\tilde{L}_{31}\tilde{h}_1\|_{L^p(\Omega)} + \varepsilon\|\tilde{L}_{32}\tilde{h}_2\|_{L^p(\Omega)})$$

$$\leq \frac{C_9}{\varepsilon}(\|h_3\|_{L^p(\Omega)} + \varepsilon C\|\tilde{h}_1\|_{W_N^{2,p}(\Omega)} + \varepsilon C\|\tilde{h}_2\|_{W_N^{2,p}(\Omega)})$$

Here, considering

$$\begin{pmatrix}\tilde{h}_1\\ \tilde{h}_2\end{pmatrix} = \mathscr{L}_\varepsilon^{-1}\begin{pmatrix}h_1\\ h_2\end{pmatrix},$$

we know

$$\left\| \begin{pmatrix} \tilde{h}_1 \\ \tilde{h}_2 \end{pmatrix} \right\|_{(W_N^{2,p})^2} \leq C \left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|_{(L^p)^2}$$

and therefore,

$$\begin{aligned} \|\tilde{\varphi}_3\|_{W_N^{2,p}(\Omega)} &\leq \frac{C_9}{\varepsilon} \left(\|h_3\|_{L^p(\Omega)} + \varepsilon C \left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|_{(L^p)^2} + \varepsilon C \left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|_{(L^p)^2} \right) \\ &\leq \frac{C_9'}{\varepsilon} \left\| \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \right\|_Y . \end{aligned}$$

On the other hand, using

$$\begin{aligned} &\left\| \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right\|_{(W_N^{2,p}(\Omega))^2} \\ &\quad \leq C_4 \left\| \begin{pmatrix} h_1 - \left(r_1 - 2a_1\left(\tilde{U}_2^\varepsilon - \frac{\alpha M}{d_u} W_2^\varepsilon\right) - b_1 V_2^\varepsilon\right) \frac{\alpha M}{d_u + \alpha v_0} \tilde{\varphi}_3 \\ h_2 + \frac{b_2 \alpha M V_2^\varepsilon}{d_u + \alpha v_0} \tilde{\varphi}_3 \end{pmatrix} \right\|_{(L^p(\Omega))^2} \\ &\quad \leq C_4 \left(\left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|_{(L^p(\Omega))^2} + C \|\tilde{\varphi}_3\|_{L^p(\Omega)} \right) \\ &\quad \leq C_4 \left(\left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|_{(L^p(\Omega))^2} + C \|\tilde{h}_3\|_{L^p(\Omega)} \right) \qquad (\text{by } L^p \text{ estimate } (3.21)) \\ &\quad \leq C_4 \left(\left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|_{(L^p(\Omega))^2} + C \left\| \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \right\|_Y \right) \\ &\quad \leq C \left\| \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \right\|_Y , \end{aligned}$$

we have

$$\left\| \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \tilde{\varphi}_3 \end{pmatrix} \right\|_X = \left\| \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right\|_{(W_N^{2,p}(\Omega))^2} + \|\tilde{\varphi}_3\|_{(W_N^{2,p}(\Omega))} \leq \frac{C}{\varepsilon} \left\| \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \right\|_Y .$$

By using these inequalities, we find $\|\tilde{L}_\varepsilon^{-1}\|_{Y \to X} \leq \frac{C}{\varepsilon}$. Since $\tilde{L}_\varepsilon$ is a operator that transfromed L_ε by $\tilde{\varphi}_3 = \frac{1}{M} v_0 \varphi_1 + \frac{1}{M}\left(\tilde{u}_0 - \frac{\alpha M}{d_u} w_0\right)\varphi_2 - \left(1 + \frac{\alpha}{d_u} v_0\right)\varphi_3$, $\tilde{L}_\varepsilon$ returns to L_ε by transforming again by

$$\varphi_3 = \frac{\frac{1}{M}v_0}{1+\frac{\alpha}{d_u}v_0}\varphi_1 + \frac{\frac{1}{M}\left(\tilde{u}_0 - \frac{\alpha M}{d_u}w_0\right)}{1+\frac{\alpha}{d_u}v_0}\varphi_2 - \frac{1}{1+\frac{\alpha}{d_u}v_0}\tilde{\varphi}_3.$$

Therefore, we obtain

$$\left\|\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix}\right\|_X \le C'\left\|\begin{pmatrix}\varphi_1\\ \varphi_2\\ \tilde{\varphi}_3\end{pmatrix}\right\|_X,$$

and then

$$\left\|\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix}\right\|_X \le C'\left\|\begin{pmatrix}\varphi_1\\ \varphi_2\\ \tilde{\varphi}_3\end{pmatrix}\right\|_X \le C'\frac{C}{\varepsilon}\left\|\begin{pmatrix}h_1\\ h_2\\ h_3\end{pmatrix}\right\|_Y = \frac{C_1}{\varepsilon}\left\|L_\varepsilon\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix}\right\|_Y.$$

Consequently we arrive at $\|L_\varepsilon^{-1}\|_{Y\to X} \le \frac{C_1}{\varepsilon}$. □

4. Concluding remarks

We have concerned with the two-component cross-diffusion competition-system (1.5) for (u, v) arising in the field of mathematical ecology. It is numerically revealed that the structure of equilibrium solutions of (1.5) is so complex when the cross diffusion effects are included, as was shown in Figs. 1 and 2. In this paper, we have considered the convergence problem between (1.5) and the three component normal diffusion-reaction systems (1.8) with a small parameter ε for (U_A, U_B, V). We have shown the following two results on the stationary problems (1.12) and (1.14) for the corresponding to (1.5) and (1.8), (i) As ε tends to zero, we numerically showed in Figs. 4–6 that the global structure of 1 dimensional equilibrium solutions of (1.8) converges to the one of the cross-diffusion competition system (1.5) when some parameters are globally varied; (ii) we analytically showed that for any equilibrium solutions $(u(x), v(x))$ of (1.12), there exists a (unique) solution $(U_A(x;\varepsilon), U_B(x;\varepsilon), V(x;\varepsilon))$ of (1.14) satisfying $\lim_{\varepsilon\to+0}(U_A(x;\varepsilon)+U_B(x;\varepsilon), V(x;\varepsilon)) = (u(x), v(x))$ when the linearized operator of (1.8) around $(u(x), v(x))$ has no zero eigenvalue. As far as numerical results, it is surprising that the structures of equilibrium solutions of (1.12) and (1.14) near bifurcation points are qualitatively similar. Its analytical understanding has been unsolved and it is our future work.

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