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Dynamical approximation of internal transition layers in a bistable nonlocal reaction-diffusion equation via the averaged mean curvature flow

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Abstract. A singular perturbation problem for a scalar bistable nonlocal reaction-diffusion equation is treated. It is rigorously proved that the layer solutions of this nonlocal reaction-diffusion equation converge to solutions of the averaged mean curvature flow on a finite time interval as the singular perturbation parameter tends to zero.

1. Introduction and main results

1.1. Nonlocal reaction-diffusion equation. We consider in this paper the following scalar bistable nonlocal reaction-diffusion equation:

$$\text{(RD)} \qquad \begin{cases} u_t = \varepsilon^2 \Delta u + f(u) - \langle f(u) \rangle, & t > 0,\ x \in \Omega, \\ \partial u / \partial \mathbf{n} = 0, & t > 0,\ x \in \partial\Omega. \end{cases}$$

Here, Ω is a smooth bounded domain in $\mathbf{R}^N$ $(N \geq 2)$ with total volume $|\Omega|$ and the outward unit normal $\mathbf{n}$ on the boundary $\partial\Omega$; ε a small positive parameter; f a nonlinear function of bistable type, a typical example being $f(u) = u - u^3$; and the symbol $\langle \cdot \rangle$ stands for the spatial average over Ω, i.e.,

$$\langle \varphi \rangle := \frac{1}{|\Omega|} \int_\Omega \varphi \, dx.$$

Rubinstein and Sternberg [26] derived the nonlocal equation (RD) as a *shadow system* for the viscous Cahn-Hilliard equation (cf. [23, 24])

$$\text{(vCH)} \qquad \begin{cases} \tau u_t = -\Delta[\varepsilon^2 \Delta u + f(u) - u_t], & t > 0,\ x \in \Omega, \\ \partial u / \partial \mathbf{n} = \partial \Delta u / \partial \mathbf{n} = 0, & t > 0,\ x \in \partial\Omega \end{cases}$$

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with respect to the limit operation of the parameter $\tau \to 0$. The function $u = u(t,x)$ represents, e.g., an order parameter or a concentration of one of the components in the mixture at time $t > 0$ and position $x \in \Omega$, and the term Δu_t is regarded as a viscous effect. In particular, if the viscous effect is negligible, (vCH) is reduced to the Cahn-Hilliard equation

$$\text{(CH)} \qquad \begin{cases} \tau u_t = -\Delta[\varepsilon^2 \Delta u + f(u)], & t > 0,\ x \in \Omega, \\ \partial u/\partial \mathbf{n} = \partial \Delta u/\partial \mathbf{n} = 0, & t > 0,\ x \in \partial\Omega. \end{cases}$$

For (RD) with sufficiently small $\varepsilon > 0$, it is known that the dynamics of solution consists of several stages, and is roughly summarized as follows:

(S1) Generation of layers.
The solution with an appropriate initial condition generates sharp *internal transition layer* in a narrow region of $O(\varepsilon)$ near a hypersurface, called an *interface*. Such a solution is referred to as a *layer solution*.

(S2) Motion of interfaces (i).
The layer solution begins to move in such a way that the corresponding interface is driven according to a certain motion law.

(S3) Motion of interfaces (ii).
The layer solution then comes to evolve such that the motion of the corresponding interface is governed by another motion law, called the *averaged mean curvature flow*. The interface is driven in such a way that the volume of domain enclosed by itself is preserved and its surface area decreases. After a coarsening process, the interface evolves into a single sphere.

(S4) Motion of bubbles (i).
The layer solution with spherical shape is referred to as the *bubble solution*. The bubble solution drifts with exponentially slow speed, without changing shape, towards the closest point on $\partial\Omega$ from the center of the corresponding sphere.

(S5) Motion of bubbles (ii).
Once the bubble solution attaches to the boundary $\partial\Omega$, it intersects perpendicularly to $\partial\Omega$ with hemisphere-like shape, and evolves along $\partial\Omega$ by its geometric information.

The dynamics in (S1) through (S3) was discussed in [26] by using formal asymptotic analysis. For (S4), the existence of bubble motions was rigorously established by Alikakos et al. [3]. Ward gave in [36] an explicit asymptotic ordinary differential equation for the distance between the center of the bubble and the closest point on $\partial\Omega$ from it (see also [37]). Alikakos et al. derived in [5] such an ordinary differential equation for the Cahn-Hilliard equation (CH), and compared the bubble motions for (CH) with those for the nonlocal equation (RD). The dynamics in (S5) was studied by Alikakos et al. [4] and Ward [37].

Our result in this paper is concerned with the dynamics occurring in the stage (S3). We will rigorously demonstrate that the dynamics in (S3) of layer solutions to the nonlocal reaction-diffusion equation (RD) is approximated by the averaged mean curvature flow.

1.2. Dynamical approximation via interface equations. In the stages (S2) and (S3), the dynamics of layer solutions is approximately captured by a motion law of interface. Such a motion law is called an *interface equation*. Throughout the remaining part of this paper, an interface means

> a smooth, closed hypersurface embedded in $\Omega \subset \mathbf{R}^N$, staying uniformly away from $\partial\Omega$.

The interface Γ separates the whole domain Ω into two subdomains. We denote by Ω^+ one containing $\partial\Omega$ as a part of boundary, and by Ω^- the other:

$$\Omega = \Omega^- \cup \Gamma \cup \Omega^+, \qquad \partial\Omega^- = \Gamma, \qquad \partial\Omega^+ = \partial\Omega \cup \Gamma,$$

and by $\nu(x;\Gamma)$ the unit normal vector on Γ at $x \in \Gamma$ pointing toward the interior of the subdomain Ω^+. We also let the nonlinear function $f(u)$ satisfy the conditions listed below, in which the nonlinearity is regarded as $f(u) - v$, rather than $f(u)$ itself, by introducing an auxiliary variable v for the nonlocal term.

(A1) The function f is smooth on $\mathbf{R}$ and the nullcline $\{(u,v) \mid f(u) - v = 0\}$ has exactly three branches of solutions

$$\{(u,v) \mid u = h^-(v), v \in (\underline{v}, \infty)\},$$

$$\{(u,v) \mid u = h^+(v), v \in (-\infty, \bar{v})\},$$

$$\{(u,v) \mid u = h^0(v), v \in (\underline{v}, \bar{v})\},$$

satisfying the following inequalities for each $v \in (\underline{v}, \bar{v})$:

$$h^-(v) < h^0(v) < h^+(v),$$

$$f'(h^\pm(v)) < 0, \qquad \text{or equivalently,} \qquad h_v^\pm(v) < 0.$$

(A2) For each $v \in (\underline{v}, \bar{v})$, define $\mathbf{J}(v)$ by

$$\mathbf{J}(v) := \int_{h^-(v)}^{h^+(v)} (f(u) - v)du.$$

Then there exists a unique point $v^* \in (\underline{v}, \bar{v})$ such that $\mathbf{J}(v^*) = 0$ and $\mathbf{J}'(v^*) < 0$.

The dynamics of layer solutions to (RD) in (S2) is slow, and is of order $O(\varepsilon)$. In order to capture its dynamics in time scale of $O(1)$, we rescale the time t in (RD) as $t \to t/\varepsilon$:

$$(\mathrm{RD}') \qquad \begin{cases} \varepsilon u_t^\varepsilon = \varepsilon^2 \Delta u^\varepsilon + f(u^\varepsilon) - \langle f(u^\varepsilon)\rangle, & t > 0,\ x \in \Omega, \\ \partial u^\varepsilon/\partial \mathbf{n} = 0, & t > 0,\ x \in \partial\Omega. \end{cases}$$

Under the assumption **(A1)**, it is known that the following problem, called the *nonlinear eigenvalue problem*, has a unique smooth solution pair $(Q(z;v), c(v))$ for each $v \in (\underline{v}, \bar{v})$ acting as a parameter (cf. [13]):

$$(\mathrm{NEP}) \qquad \begin{cases} Q_{zz} + cQ_z + f(Q) - v = 0, \quad z \in (-\infty, \infty), \\ \lim\limits_{z \to \pm\infty} Q = h^\pm(v), \quad Q|_{z=0} = h^0(v). \end{cases}$$

The functions Q and c are referred to as the *profile* and the *speed* of the traveling wave, respectively. By employing the wave speed c, the interface equation in (S2) is expressed as

$$(\mathrm{IE}') \qquad \begin{cases} \mathbf{v}(x; \Gamma(t)) = c(v(t)), & t > 0,\ x \in \Gamma(t), \\ \dot{v}(t) = h(v(t); \Gamma(t)) c(v(t)) |\Gamma(t)|, & t > 0, \\ \Gamma(0) = \Gamma_0, \quad v(0) = v_0 \in (\underline{v}, \bar{v}) \end{cases}$$

with

$$h(v(t); \Gamma(t)) := \frac{h^+(v(t)) - h^-(v(t))}{h_v^-(v(t))|\Omega^+(t)| + h_v^+(v(t))|\Omega^+(t)|}.$$

Here, the scale of time is that of (RD$'$), the symbol $\mathbf{v}(x; \Gamma(t))$ stands for the normal velocity of $\Gamma(t)$ at $x \in \Gamma(t)$ in ν-direction; $|\Omega^\pm|$ and $|\Gamma|$ are the volume of $\Omega^\pm$ and the surface area of Γ, respectively. The motion law of interface in (S2) was earlier given as the equation (2.15) in [26]. The form, however, was implicit and unsuitable for the circumstantial examination. The explicit form by the interface equation (IE$'$) was later derived by Okada [25], in which the unique existence of smooth solutions and the stability of the equilibria to (IE$'$) were successfully established.

For $0 < \varepsilon \ll 1$, the dynamics of the solution u^ε and the nonlocal effect $\langle f(u^\varepsilon)\rangle$ to (RD$'$) are approximated by that of the solution pair $(\Gamma(t), v(t))$ to (IE$'$) in the sense that

$$\langle f(u^\varepsilon(t, \cdot)\rangle \approx v(t),$$

$$u^\varepsilon(t, x) \approx \begin{cases} h^-(v(t)), & t > 0,\ x \in \Omega^-(t), \\ h^+(v(t)), & t > 0,\ x \in \Omega^+(t). \end{cases}$$

Note, in particular, that the second property shows the sharp layer structure of u^ε near the interface $\Gamma(t)$. Such a characterization in (S2) was justified in [25] by the following

Theorem (Theorem 1.2 in [25]). *Assume that* **(A1)** *and* **(A2)** *are satisfied, and let* (Γ, v) *be the smooth solution pair to* (IE$'$) *on a time interval* $[0, T]$. *Then there exist* $\varepsilon^* > 0$ *and an* ε*-family of smooth solutions* u^ε *to* (RD$'$), *defined for* $\varepsilon \in (0, \varepsilon^*]$, *satisfying*

$$\lim_{\varepsilon\to 0} \langle f(u^\varepsilon)\rangle = v \qquad \textit{uniformly on } [0, T],$$

$$\lim_{\varepsilon\to 0} u^\varepsilon = \begin{cases} h^-(v) \\ h^+(v) \end{cases} \qquad \textit{uniformly on } \begin{cases} \bar{\Omega}_T^- \backslash \Gamma_T^\delta \\ \bar{\Omega}_T^+ \backslash \Gamma_T^\delta \end{cases} \quad \textit{for each } \delta > 0,$$

where

$$\Omega_T^\pm := \bigcup_{t\in[0,T]} \{t\} \times \Omega^\pm(t),$$

$$\Gamma_T^\delta := \bigcup_{t\in[0,T]} \{t\} \times \Gamma(t)^\delta$$

with $\Gamma(t)^\delta := \{x \in \Omega \mid \mathrm{dist}(x, \Gamma(t)) < \delta\}$, *the* δ*-neighborhood of the interface* $\Gamma(t)$.

In the next stage (S3), the dynamics of layer solutions to (RD) is much slower, compared with that in (S2), which is of order $O(\varepsilon^2)$. To capture this in time scale of $O(1)$, it is adequate to rescale the time t in (RD) as $t \to t/\varepsilon^2$, and to employ the rescaled version

$$(1.1) \qquad \begin{cases} \varepsilon^2 u_t^\varepsilon = \varepsilon^2 \Delta u^\varepsilon + f(u^\varepsilon) - \langle f(u^\varepsilon)\rangle, & t > 0,\ x \in \Omega, \\ \partial u^\varepsilon/\partial \mathbf{n} = 0, & t > 0,\ x \in \partial\Omega. \end{cases}$$

The corresponding interface equation is known to be the averaged mean curvature flow:

$$(1.2) \qquad \begin{cases} \mathbf{v}(x; \Gamma(t)) = -\kappa(x; \Gamma(t)) + \bar{\kappa}(t), & t > 0,\ x \in \Gamma(t), \\ \Gamma(0) = \Gamma_0. \end{cases}$$

Here, the scale of time is that of (1.1), the symbol $\kappa(x; \Gamma)$ stands for the sum of principal curvatures (the *mean curvature*, for short) of Γ at $x \in \Gamma$, and $\bar{\kappa}$ denotes the average of κ on Γ, i.e.,

$$\bar{\kappa}(t) := \frac{1}{|\Gamma(t)|} \int_{\Gamma(t)} \kappa(x; \Gamma(t)) dS_x^{\Gamma(t)},$$

dS_x^Γ being the surface element of Γ at $x \in \Gamma$. We notice that the sign of κ is chosen so that it is positive if the center of the curvature sphere lies in Ω^-.

The existence and uniqueness of smooth solutions to (1.2) are well established (cf. [14, 16, 11, 21]), and arise as the lowest compatibility condition in our approximations (cf. § 3.5 below).

In the previous stage (S2), the interface dynamics by (IE$'$) approximates the dynamics of layer solution to (RD$'$) for small $\varepsilon > 0$. Then, in this stage (S3),

> does the averaged mean curvature flow (1.2) approximate the dynamics of layer solution to the nonlocal reaction-diffusion equation (1.1) for small $\varepsilon > 0$?

By a variational method, it was rigorously proved in Bronsard and Stoth [7] that the answer to this question is affirmative *for radially symmetric solutions in a spherically symmetric domain*. Our aim of this paper is to show, by means of an alternative method, that the answer remains affirmative *without* any restrictions of symmetricity.

1.3. Main result. We are now in a position to state our result. This ensures the existence of solutions to the nonlocal reaction-diffusion equation (1.1) which exhibit sharp transition layer near the interface driven by the averaged mean curvature flow (1.2).

Theorem 1.1. *Assume that* **(A1)** *and* **(A2)** *are satisfied, and let Γ be the smooth solution to* (1.2) *on a time interval* $[0, T]$. *Then there exist $\varepsilon^* > 0$ and an ε-family of smooth solutions u^ε to* (1.1), *defined for $\varepsilon \in (0, \varepsilon^*]$, satisfying*

$$\lim_{\varepsilon \to 0} \langle f(u^\varepsilon) \rangle = v^* \qquad \text{uniformly on } [0, T],$$

$$\lim_{\varepsilon \to 0} u^\varepsilon = \begin{cases} h^-(v^*) \\ h^+(v^*) \end{cases} \qquad \text{uniformly on } \begin{cases} \bar{\Omega}_T^- \backslash \Gamma_T^\delta \\ \bar{\Omega}_T^+ \backslash \Gamma_T^\delta \end{cases} \quad \text{for each } \delta > 0.$$

It is in general not so easy to establish this sort of convergence result for nonlocal problems. One reason for the difficulty is that the *method of sub- and super solutions* based on the maximum principle, or comparison principle, is not applicable. Situation is the same even for the most fundamental scalar equation (1.1), and so it is for the higher order equations such as (vCH) and (CH). To demonstalate Theorem 1.1, we will follow an alternative method, an *approximation method*. This method is based on the singular perturbation method and has been developed as a way to treat boundary/internal layers appearing in local elliptic problems [12, 20, 18, 15, 27, 28, 29, 22, 32, 33], and in local parabolic problems [2, 10, 30, 31]. For nonlocal problems, the application of this method to (RD$'$) was successfully established by Okada [25], following the argument developed in Sakamoto [31], Nefedov and Sakamoto [22].

This paper is organized as follows. We will prove Theorem 1.1 in §2. The proof consists of two steps; (i) construction of approximate solutions and (ii) perturbation argument. In the first step, a family of approximate solutions with high degree of accuracy is constructed (cf. Proposition 2.1 below). Since the construction is rather lengthly and involved, it will be postponed to §3. In the second step, we derive a true solution as a perturbation from the approximate solution (cf. Proposition 2.2 below). In this procedure, a certain estimate on the evolution operator associated with the linearized operator around the approximate solution plays a crucial role (cf. Proposition 2.3 below). The details will be developed in §4.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Several propositions are employed, the proofs for some of which are postponed to §3 and §4. In what follows, where no danger of confusion will arise, we employ the same symbol M to denote positive constants independent of $\varepsilon > 0$ which could be differ from line to line.

The first step is the construction of approximate solutions:

Proposition 2.1. *Assume that* **(A1)** *and* **(A2)** *are satisfied, and let Γ be the smooth solution of* (1.2) *on a time interval* $[0, T]$. *For each integer $k \geq 2$, there exist $\varepsilon^* > 0$ and an ε-family of smooth approximate solutions u_A^ε to* (1.1), *defined for $\varepsilon \in (0, \varepsilon^*]$, satisfying*

$$\max_{[0,T]} \left\| \varepsilon^2 \frac{\partial u_A^\varepsilon}{\partial t} - \varepsilon^2 \Delta u_A^\varepsilon - f(u_A^\varepsilon) + \langle f(u_A^\varepsilon) \rangle \right\|_{L^\infty(\Omega)} = O(\varepsilon^{k+1}), \tag{2.1a}$$

$$\frac{\partial u_A^\varepsilon}{\partial \mathbf{n}} = 0 \qquad \text{on } [0, T] \times \partial\Omega, \tag{2.1b}$$

$$\lim_{\varepsilon \to 0} \langle f(u_A^\varepsilon) \rangle = v^* \qquad \text{uniformly on } [0, T], \tag{2.1c}$$

$$\lim_{\varepsilon \to 0} u_A^\varepsilon = \begin{cases} h^-(v^*) \\ h^+(v^*) \end{cases} \qquad \text{uniformly on } \begin{cases} \bar{\Omega}_T^- \backslash \Gamma_T^\delta \\ \bar{\Omega}_T^+ \backslash \Gamma_T^\delta \end{cases} \quad \text{for each } \delta > 0. \tag{2.1d}$$

We will postpone the proof of Proposition 2.1 to §3.

Let us now move on to the second step. By means of perturbation argument, we will prove that there exists a true solution u^ε near the approximate solution u_A^ε constructed as in Proposition 2.1.

Proposition 2.2. *For each integer $k \geq 2$, let u_A^ε be the ε-family of smooth approximate solutions to* (1.1), *defined for $\varepsilon \in (0, \varepsilon^*]$, satisfying the properties* *stated in Proposition* 2.1. *Then there exists an* ε*-family of smooth solutions* u^ε *to* (1.1), *defined for* $\varepsilon \in (0, \varepsilon^*]$, *such that*

$$\max_{[0,T]} \|u^\varepsilon - u_A^\varepsilon\|_{L^\infty(\Omega)} = O(\varepsilon^{k-1}). \tag{2.2}$$

Proof. For each $t \in [0, T]$, let $\mathscr{L}^\varepsilon(t)$ be the linearized operator of (1.1) around the approximate solution u_A^ε:

$$\mathscr{L}^\varepsilon(t)\varphi := \Delta\varphi + \frac{1}{\varepsilon^2}[f'(u_A^\varepsilon(t,\cdot))\varphi - \langle f'(u_A^\varepsilon(t,\cdot))\varphi\rangle].$$

By introducing a scaling parameter $s \in \mathbf{R}$, which is to be determined, we rescale the time t in $\mathscr{L}^\varepsilon(t)$ by

$$t = \varepsilon^s \tau, \tag{2.3}$$

and seek a true solution u^ε of (1.1) with the following form:

$$u^\varepsilon(\varepsilon^s\tau,\cdot) = u_A^\varepsilon(\varepsilon^s\tau,\cdot) + \varphi^\varepsilon(\tau)(\cdot), \qquad \tau \in [0, T/\varepsilon^s]. \tag{2.4}$$

Our equation in (1.1) is recast as an evolution equation for $\varphi^\varepsilon(\tau)$

$$\dot{\varphi}^\varepsilon(\tau) = \mathscr{A}^\varepsilon(\tau)\varphi^\varepsilon(\tau) + \mathscr{N}^\varepsilon(\tau, \varphi^\varepsilon(\tau)) + \mathscr{R}^\varepsilon(\tau), \tag{2.5}$$

where "dot" stands for the derivative with respect to the variable τ; $\mathscr{A}^\varepsilon(\tau)\varphi$, $\mathscr{N}^\varepsilon(\tau,\varphi)$ and $\mathscr{R}^\varepsilon(\tau)$ are the linear, the nonlinear and the remainder parts, respectively, defined by

$$\begin{aligned} \mathscr{A}^\varepsilon(\tau)\varphi &:= \varepsilon^s \mathscr{L}^\varepsilon(\varepsilon^s\tau)\varphi \\ &= \varepsilon^s\Delta\varphi + \varepsilon^{s-2}[f'(u_A^\varepsilon(\varepsilon^s\tau,\cdot))\varphi - \langle f'(u_A^\varepsilon(\varepsilon^s\tau,\cdot))\varphi\rangle], \end{aligned} \tag{2.6a}$$

$$\begin{aligned} \mathscr{N}^\varepsilon(\tau,\varphi) := \varepsilon^{s-2}[&f(u_A^\varepsilon(\varepsilon^s\tau,\cdot)+\varphi) - f(u_A^\varepsilon(\varepsilon^s\tau,\cdot)) - f'(u_A^\varepsilon(\varepsilon^s\tau,\cdot))\varphi \\ &- \langle f(u_A^\varepsilon(\varepsilon^s\tau,\cdot)+\varphi) - f(u_A^\varepsilon(\varepsilon^s\tau,\cdot)) - f'(u_A^\varepsilon(\varepsilon^s\tau,\cdot))\varphi\rangle], \end{aligned} \tag{2.6b}$$

$$\begin{aligned} \mathscr{R}^\varepsilon(\tau) := \varepsilon^{s-2}\Bigg[&\varepsilon^2\Delta u_A^\varepsilon(\varepsilon^s\tau,\cdot) + f(u_A^\varepsilon(\varepsilon^s\tau,\cdot)) - \langle f(u_A^\varepsilon(\varepsilon^s\tau,\cdot))\rangle \\ &- \varepsilon^2\frac{\partial u_A^\varepsilon}{\partial t}(\varepsilon^s\tau,\cdot)\Bigg]. \end{aligned} \tag{2.6c}$$

Notice that the following estimates are valid for $\tau \in [0, T/\varepsilon^s]$ by virtue of (2.6b), (2.6c) and (2.1a) in Proposition 2.1:

$$\varepsilon^{2-s}\mathscr{N}^\varepsilon(\tau,\varphi) = O(|\varphi|^2), \tag{2.7a}$$

$$\|\mathscr{R}^\varepsilon(\tau)\|_{L^\infty(\Omega)} = O(\varepsilon^{s+k-1}). \tag{2.7b}$$

We now decompose $L^2(\Omega)$ as $L^2(\Omega) = \mathbf{M} \oplus \mathbf{M}^{\perp}$, where $\mathbf{M}$ and $\mathbf{M}^{\perp}$ stand for the space consisting of zero-average functions and the orthogonal complement of $\mathbf{M}$ spanned by the constant function $|\Omega|^{-1/2}$, respectively. According to this decomposition, we also decompose the function $\varphi^{\varepsilon}(\tau)$ in (2.5) as

$$\varphi^{\varepsilon}(\tau)(\cdot) = \varphi_1^{\varepsilon}(\tau)(\cdot) + \varphi_2^{\varepsilon}(\tau), \qquad \varphi_1^{\varepsilon}(\tau) \in \mathbf{M},\ \varphi_2^{\varepsilon}(\tau) \in \mathbf{M}^{\perp}. \tag{2.8}$$

Then, the equation (2.5) is equivalent to the following system:

$$\dot{\varphi}_1^{\varepsilon}(\tau) = \mathscr{A}^{\varepsilon}(\tau)\varphi_1^{\varepsilon}(\tau) + \mathscr{N}^{\varepsilon}(\tau, \varphi_1^{\varepsilon}(\tau) + \varphi_2^{\varepsilon}(\tau)) + \hat{\mathscr{R}}^{\varepsilon}(\tau, \varphi_2^{\varepsilon}(\tau)), \tag{2.9a}$$

$$\dot{\varphi}_2^{\varepsilon}(\tau) = \langle \mathscr{R}^{\varepsilon}(\tau) \rangle. \tag{2.9b}$$

Here, $\hat{\mathscr{R}}^{\varepsilon}(\tau, \varphi_2)$ $(\varphi_2 \in \mathbf{M}^{\perp})$ is defined by

$$\begin{aligned} \hat{\mathscr{R}}^{\varepsilon}(\tau, \varphi_2) &:= \mathscr{R}^{\varepsilon}(\tau) - \langle \mathscr{R}^{\varepsilon}(\tau) \rangle + \mathscr{A}^{\varepsilon}(\tau)\varphi_2 \\ &= \mathscr{R}^{\varepsilon}(\tau) - \langle \mathscr{R}^{\varepsilon}(\tau) \rangle + \varepsilon^{s-2}[f'(u_A^{\varepsilon}(\varepsilon^s\tau, \cdot)) - \langle f'(u_A^{\varepsilon}(\varepsilon^s\tau, \cdot)) \rangle]\varphi_2. \end{aligned} \tag{2.10}$$

Note that (2.9a) is the *evolution equation* and (2.9b) the *ordinary differential equation*.

In order to deal with the evolution equation (2.9a), let us now set up some appropriate function spaces. Let $p \geq 2$ and we define the basic space by

$$X_0^{\varepsilon} := L^p(\Omega) \cap \mathbf{M} \tag{2.11}$$

and the domain of $\mathscr{A}^{\varepsilon}(\tau)$ by

$$X_1^{\varepsilon} := W_{\varepsilon,\mathscr{B}}^{2,p}(\Omega) \cap \mathbf{M},$$

where $W_{\varepsilon,\mathscr{B}}^{2,p}(\Omega)$ is the same as the usual Sobolev space

$$W_{\mathscr{B}}^{2,p}(\Omega) := \{u \in W^{2,p}(\Omega) \mid \partial u/\partial \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

as a set, with the weighted norm

$$\|u\|_{W_{\varepsilon,\mathscr{B}}^{2,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \varepsilon^{s/2}\|Du\|_{L^p(\Omega)} + \varepsilon^s\|D^2u\|_{L^p(\Omega)}. \tag{2.12}$$

In the sequel, some weighted norms and embedding properties are employed. We notice here that the weighted norms are introduced to make the embedding constants *independent* of $\varepsilon > 0$.

For $\alpha \in (0,1)$, let X_{α}^{ε} be the real interpolation spaces between X_0^{ε} and X_1^{ε}

$$X_{\alpha}^{\varepsilon} := (X_0^{\varepsilon}, X_1^{\varepsilon})_{\alpha,p} \tag{2.13}$$

endowed with the norms $\|\cdot\|_{\alpha}$, where $(\cdot,\cdot)_{\alpha,p}$ stands for the standard real interpolation method (functor). Note that X_{α}^{ε} enjoy the continuous embedding properties

$$0 \le \alpha < \beta \le 1 \Rightarrow \begin{array}{l} X_\beta^\varepsilon \hookrightarrow X_\alpha^\varepsilon, \\ \|u\|_\alpha \le M\|u\|_\beta \qquad (u \in X_\beta^\varepsilon). \end{array}$$

We also set up weighted Hölder spaces $C_{\varepsilon,p}^{\alpha}(\bar{\Omega})$ for $\alpha \in (0,1)$, which are the same as the usual Hölder spaces $C^{\alpha}(\bar{\Omega})$ as sets, with the weighted norm

$$\|u\|_{C_{\varepsilon,p}^{\alpha}(\bar{\Omega})} := \varepsilon^{sN/2p}\|u\|_{L^\infty(\Omega)} + \varepsilon^{(s/2)(\alpha+N/p)}[u]_{C^\alpha(\bar{\Omega})}, \tag{2.14}$$

where $[u]_{C^\alpha(\bar{\Omega})}$ is the Hölder seminorm defined by

$$[u]_{C^\alpha(\bar{\Omega})} := \sup_{\substack{x,x' \in \bar{\Omega} \\ x \neq x'}} \frac{|u(x) - u(x')|}{|x - x'|^\alpha}.$$

Notice that, if the relation $2\alpha - N/p > \beta$ is valid for some α, $\beta \in (0,1)$, then X_α^ε is continuously embedded in $C_{\varepsilon,p}^{\beta}(\bar{\Omega})$:

$$2\alpha - \frac{N}{p} > \beta \Rightarrow \begin{array}{l} X_\alpha^\varepsilon \hookrightarrow C_{\varepsilon,p}^{\beta}(\bar{\Omega}), \\ \|u\|_{C_{\varepsilon,p}^{\beta}(\bar{\Omega})} \le M\|u\|_\alpha \qquad (u \in X_\alpha^\varepsilon). \end{array} \tag{2.15}$$

We simply denote by $\|B\|_{\alpha,\beta}$ the operator norm of a bounded linear operator $B : X_\alpha^\varepsilon \to X_\beta^\varepsilon$.

Let $\Phi^\varepsilon(\tau,\sigma) : X_\alpha^\varepsilon \to X_\beta^\varepsilon$ $(0 \le \sigma \le \tau \le T/\varepsilon^s)$ the evolution operator associated with the family $\{\mathscr{A}^\varepsilon(\tau)\}_{\tau \in [0, T/\varepsilon^s]}$. Applying the variation of constants formula to (2.9), we obtain

$$\varphi_1^\varepsilon(\tau) = \Phi^\varepsilon(\tau,0)\varphi_1^\varepsilon(0) + \int_0^\tau \Phi^\varepsilon(\tau,\sigma)\mathscr{N}^\varepsilon(\sigma, \varphi_1^\varepsilon(\sigma) + \varphi_2^\varepsilon(\sigma))d\sigma + \int_0^\tau \Phi^\varepsilon(\tau,\sigma)\hat{\mathscr{R}}^\varepsilon(\sigma, \varphi_2^\varepsilon(\sigma))d\sigma, \tag{2.16a}$$

$$\varphi_2^\varepsilon(\tau) = \varphi_2^\varepsilon(0) + \int_0^\tau \langle \mathscr{R}^\varepsilon(\sigma) \rangle d\sigma. \tag{2.16b}$$

The existence and uniqueness of smooth solutions is well established, and therefore our task is only to have an estimate for the solution φ^ε to (2.5) by employing those for φ_1^ε and φ_2^ε in (2.16). In estimating φ_1^ε, the estimate of evolution operator $\Phi^\varepsilon(\tau,\sigma)$ in the following proposition plays a crucial role.

Proposition 2.3. *For $0 \le \alpha \le \beta \le 1$ with $(\alpha,\beta) \neq (0,1)$, there exist some constants $K, M > 0$ such that the following estimate holds for small $\varepsilon > 0$, $s \ge 4$ and $0 \le \sigma \le \tau \le T/\varepsilon^s$:*

$$\|\Phi^\varepsilon(\tau,\sigma)\|_{\alpha,\beta} \le M(\tau - \sigma)^{\alpha-\beta} e^{\varepsilon^s(\lambda_* + K)(\tau-\sigma)} \tag{2.17}$$

with some $\lambda_* > 0$.

The constant $\lambda_* > 0$ in Proposition 2.3 is derived from some information of the principal eigenvalue of the linearized operator $\mathscr{L}^\varepsilon(t)$ via the spectral analysis developed in §4 below.

We postpone the proof of Proposition 2.3 to §4, and proceed with the argument to get the estimate (2.2). We recall that k, s, p and α are parameters related to the accuracy degree of approximations (cf. Proposition 2.1), the scaling of time (cf. (2.3)), the basic space (cf. (2.11)) and the interpolation spaces (cf. (2.13)), respectively. We now let

$$k \geq 2, \tag{2.18a}$$

$$s := 4, \tag{2.18b}$$

$$p > 2N, \tag{2.18c}$$

$$\alpha \in (3/4, 1), \tag{2.18d}$$

and choose $\varphi^\varepsilon(0) = \varphi_1^\varepsilon(0) + \varphi_2^\varepsilon(0)$ so small that

$$\|\varphi_1^\varepsilon(0)\|_\alpha = O(\varepsilon^{k+1}), \tag{2.19a}$$

$$|\varphi_2^\varepsilon(0)| = O(\varepsilon^{k+1}). \tag{2.19b}$$

Let us first treat (2.16b) together with (2.19b). The estimate (2.7b) with (2.18b) yields that

$$\begin{aligned} |\varphi_2^\varepsilon(\tau)| &\leq |\varphi_2^\varepsilon(0)| + \int_0^\tau \|\mathscr{R}^\varepsilon(\sigma)\|_{L^\infty(\Omega)} d\sigma \\ &= O(\varepsilon^{k+1}) + O(\varepsilon^{k+3}) \cdot T/\varepsilon^4. \end{aligned}$$

Therefore, the solution $\varphi_2^\varepsilon(\tau)$ of (2.9b) with (2.19b) satisfies

$$|\varphi_2^\varepsilon(\tau)| = O(\varepsilon^{k-1}), \qquad \tau \in [0, T/\varepsilon^4]. \tag{2.20}$$

Substituting the solution $\varphi_2^\varepsilon(\tau)$ with (2.20) into (2.16a), we move on to estimating φ_1^ε. Since p and α are chosen so that (2.18c) and (2.18d), respectively, it holds that

$$2\alpha - \frac{N}{p} > 2 \cdot \frac{3}{4} - \frac{1}{2} = 1,$$

and the embedding relations in (2.15) are fulfilled for $\beta \in (0, 1)$ chosen arbitrarily. Hence, by (2.7a) and (2.14) with (2.18b), (2.20) and $X_\alpha^\varepsilon \hookrightarrow X_0^\varepsilon$, we have the following estimates for $\sigma \in [0, T/\varepsilon^4]$:

$$\begin{aligned} &\|\mathscr{N}^\varepsilon(\sigma, \varphi_1^\varepsilon(\sigma) + \varphi_2^\varepsilon(\sigma))\|_0 \\ &\quad \leq M\varepsilon^2 \|\varphi_1^\varepsilon(\sigma) + \varphi_2^\varepsilon(\sigma)\|_{L^\infty(\Omega)} \|\varphi_1^\varepsilon(\sigma) + \varphi_2^\varepsilon(\sigma)\|_0 \end{aligned}$$

$$\le M\varepsilon^2(\|\varphi_1^\varepsilon(\sigma)\|_{L^\infty(\Omega)} + |\varphi_2^\varepsilon(\sigma)|)(\|\varphi_1^\varepsilon(\sigma)\|_0 + |\varphi_2^\varepsilon(\sigma)|)$$

$$\le M\varepsilon^2(\varepsilon^{-2N/p}\|\varphi_1^\varepsilon(\sigma)\|_\alpha + O(\varepsilon^{k-1}))(\|\varphi_1^\varepsilon(\sigma)\|_\alpha + O(\varepsilon^{k-1}))$$

$$\le M(\varepsilon^{2-2N/p}\|\varphi_1^\varepsilon(\sigma)\|_\alpha^2 + M\varepsilon^{2+(k-1)-2N/p}(1+\varepsilon^{2N/p})\|\varphi_1^\varepsilon(\sigma)\|_\alpha + M\varepsilon^{2(k-1)+2})$$

$$\le M(\varepsilon^{2(1-N/p)}\|\varphi_1^\varepsilon(\sigma)\|_\alpha^2 + \varepsilon^{k+1-2N/p}\|\varphi_1^\varepsilon(\sigma)\|_\alpha + \varepsilon^{2k}).$$

Moreover, employing (2.7b), (2.20) together with (2.18b) in (2.10), we have for $\sigma \in [0, T/\varepsilon^4]$,

$$\begin{aligned}\|\hat{\mathscr{R}}^\varepsilon(\sigma, \varphi_2^\varepsilon(\sigma))\|_0 &\le \|\mathscr{R}^\varepsilon(\sigma) - \langle \mathscr{R}^\varepsilon(\sigma)\rangle\|_0 \\ &\quad + \varepsilon^2\|f'(u_A^\varepsilon(\varepsilon^s\sigma,\cdot)) - \langle f'(u_A^\varepsilon(\varepsilon^s\sigma,\cdot))\rangle\|_0|\varphi_2^\varepsilon(\sigma)| \\ &\le 2(\|\mathscr{R}^\varepsilon(\sigma)\|_{L^\infty(\Omega)} + \varepsilon^2\|f'(u_A^\varepsilon(\varepsilon^2\sigma,\cdot))\|_{L^\infty(\Omega)}|\varphi_2^\varepsilon(\sigma)|) \\ &= O(\varepsilon^{k+3}) + \varepsilon^2 O(1)O(\varepsilon^{k-1}) \\ &\le M\varepsilon^{k+1}.\end{aligned}$$

Using these estimates in (2.16), we find that

$$\begin{aligned}\|\varphi_1^\varepsilon(\tau)\|_\alpha &\le \|\Phi^\varepsilon(\tau,0)\|_{\alpha,\alpha}\|\varphi_1^\varepsilon(0)\|_\alpha \\ &\quad + M\varepsilon^{2(1-N/p)}\int_0^\tau \|\Phi^\varepsilon(\tau,\sigma)\|_{0,\alpha}\|\varphi_1^\varepsilon(\sigma)\|_\alpha^2 d\sigma \\ &\quad + M\varepsilon^{k+1-2N/p}\int_0^\tau \|\Phi^\varepsilon(\tau,\sigma)\|_{0,\alpha}\|\varphi_1^\varepsilon(\sigma)\|_\alpha d\sigma \\ &\quad + M\varepsilon^{k+1}\int_0^\tau \|\Phi^\varepsilon(\tau,\sigma)\|_{0,\alpha} d\sigma,\end{aligned} \tag{2.21}$$

where the inequality $2k > k+1$ (under (2.18a)) has been employed to get the last term.

Let $r^\varepsilon(\tau)$ be the function defined by

$$r^\varepsilon(\tau) := \|\varphi_1^\varepsilon(\tau)\|_\alpha e^{-\varepsilon^4(\lambda_*+K)\tau}, \qquad \tau \in [0, T/\varepsilon^4]. \tag{2.22}$$

Then, by the estimates (2.17) with (2.18b), we can compute (2.21) in terms of $r^\varepsilon(\tau)$ so that

$$\begin{aligned}r^\varepsilon(\tau) \le M\biggl(& r^\varepsilon(0) + e^{(\lambda_*+K)T}\varepsilon^{2(1-2N/p)}\int_0^\tau (\tau-\sigma)^{-\alpha} r^\varepsilon(\sigma)^2 d\sigma \\ &+ \varepsilon^{k+1-2N/p}\int_0^\tau (\tau-\sigma)^{-\alpha} r^\varepsilon(\sigma) d\sigma + M\varepsilon^{k+1}\frac{T^{1-\alpha}}{1-\alpha}\varepsilon^{-4(1-\alpha)}\biggr)\end{aligned} \tag{2.23}$$

$$\le M\biggl(r^\varepsilon(0) + \varepsilon^{2(1-N/p)}\int_0^\tau (\tau-\sigma)^{-\alpha} r^\varepsilon(\sigma)^2 d\sigma + \varepsilon^{k+1-2N/p}\int_0^\tau (\tau-\sigma)^{-\alpha} r^\varepsilon(\sigma) d\sigma + \varepsilon^{k+4\alpha-3}\biggr).$$

By (2.19a), we have

$$r^\varepsilon(0) = \|\varphi_1^\varepsilon(0)\|_\alpha = O(\varepsilon^{k+1}). \tag{2.24}$$

Then from the continuity of $r^\varepsilon(\tau)$ it follows that

$$r^\varepsilon(\tau) \le \varepsilon^k \tag{2.25}$$

for small $\tau > 0$. Setting

$$T^\varepsilon := \sup\{\tau \in [0, T/\varepsilon^4] \mid r^\varepsilon(\sigma) \le \varepsilon^k \text{ for all } \sigma \in [0,\tau]\},$$

we have one of the alternatives

$$r^\varepsilon(T^\varepsilon) = \varepsilon^k \qquad \text{or} \qquad T^\varepsilon = T/\varepsilon^4.$$

Assuming the former situation is realized, we can compute by employing (2.24) in (2.23) so that

$$\begin{aligned} \varepsilon^k = r^\varepsilon(T^\varepsilon) &\le M\biggl(\varepsilon^{k+1} + \varepsilon^{2(1-2N/p)}\varepsilon^{2k}\frac{T^{1-\alpha}}{1-\alpha}\varepsilon^{-4(1-\alpha)} \\ &\qquad + \varepsilon^{k+1-2N/p}\varepsilon^k\frac{T^{1-\alpha}}{1-\alpha}\varepsilon^{-4(1-\alpha)} + \varepsilon^{k+4\alpha-3}\biggr) \\ &\le \varepsilon^k\biggl(M\varepsilon + \frac{MT^{1-\alpha}}{1-\alpha}\varepsilon^{k-2+4\alpha-4N/p} \\ &\qquad + \frac{MT^{1-\alpha}}{1-\alpha}\varepsilon^{k-3+4\alpha-2N/p} + M\varepsilon^{4\alpha-3}\biggr). \end{aligned} \tag{2.26}$$

Noting our choice of parameters in (2.18), we have

$$k - 2 + 4\alpha - \frac{4N}{p} > 2 - 2 + 4\cdot\frac{3}{4} - 4\cdot\frac{1}{2} = 1 > 0,$$

$$k - 3 + 4\alpha - \frac{2N}{p} > 2 - 3 + 4\cdot\frac{3}{4} - 2\cdot\frac{1}{2} = 1 > 0,$$

$$4\alpha - 3 > 4\cdot\frac{3}{4} - 3 = 0.$$

Thus for sufficiently small $\varepsilon > 0$, (2.26) implies

$$\varepsilon^k \le \frac{\varepsilon^k}{2},$$

which is a contradiction. Hence, the latter case is realized, namely, (2.25) is valid for $\tau \in [0, T/\varepsilon^4]$, and by (2.22) we have

$$\|\varphi_1^\varepsilon(\tau)\|_\alpha \le Me^{(\lambda_*+K)T}\varepsilon^k = O(\varepsilon^k), \qquad \tau \in [0, T/\varepsilon^4].$$

By employing (2.14) and (2.15), it follows that

$$\|\varphi_1^\varepsilon(\tau)\|_{L^\infty(\Omega)} = O(\varepsilon^{k-2N/p}), \qquad \tau \in [0, T/\varepsilon^4]. \tag{2.27}$$

Combining (2.20) and (2.27) in (2.8), we have

$$\begin{aligned}\|\varphi^\varepsilon(\tau)\|_{L^\infty(\Omega)} &\le \|\varphi_1^\varepsilon(\tau)\|_{L^\infty(\Omega)} + |\varphi_2^\varepsilon(\tau)| \\ &= O(\varepsilon^{k-2N/p}) + O(\varepsilon^{k-1}) \\ &= O(\varepsilon^{k-1}), \qquad \tau \in [0, T/\varepsilon^4].\end{aligned}$$

This estimate and (2.4) lead to (2.2), which completes the proof of Proposition 2.2. □

Theorem 1.1 immediately follows from Proposition 2.1 and Proposition 2.2 with $k = 2$, and the following inequalities

$$\begin{aligned}\|u^\varepsilon - h^\pm(v^*)\|_{L^\infty(\bar\Omega^\pm\setminus\Gamma^\delta)} &\le \|u^\varepsilon - u_A^\varepsilon\|_{L^\infty(\Omega)} + \|u_A^\varepsilon - h^\pm(v^*)\|_{L^\infty(\bar\Omega^\pm\setminus\Gamma^\delta)}, \\ |\langle f(u^\varepsilon)\rangle - v^*| &\le |\langle f(u^\varepsilon)\rangle - \langle f(u_A^\varepsilon)\rangle| + |\langle f(u_A^\varepsilon)\rangle - v^*| \\ &\le M\|u^\varepsilon - u_A^\varepsilon\|_{L^\infty(\Omega)} + |\langle f(u_A^\varepsilon)\rangle - v^*|.\end{aligned}$$

This completes the proof of Theorem 1.1.

3. Proof of Proposition 2.1

In this section, we construct the approximate solutions satisfying the properties stated in Proposition 2.1. For this purpose, we recast the equation in (1.1), by introducing an auxiliary variable, as

$$\varepsilon^2 u_t^\varepsilon = \varepsilon^2 \Delta u^\varepsilon + f(u^\varepsilon) - v^\varepsilon, \qquad t > 0,\ x \in \Omega, \tag{3.1a}$$

$$v^\varepsilon - \langle f(u^\varepsilon)\rangle = 0, \qquad t \ge 0. \tag{3.1b}$$

The procedure of construction consists of five parts; outer expansion (§3.1), inner expansion (§3.2), C^1-matching (§3.3), nonlocal expansion (§3.4), and uniform approximation (§3.5). §3.1 through §3.3 are devoted to the first equation (3.1a), in which (3.1a) is treated as a scalar equation with a parameter

v^ε and the method of matched asymptotic expansions for local problems is employed. In §3.1 and §3.2, we determine the *outer* and *inner solutions* to (3.1a), respectively. In §3.3, we derive a series of equalities equivalent to the C^1-*matching conditions* which guarantee that the inner solution is smooth across the level-set interface $\Gamma^\varepsilon(t) := \{x \in \Omega \,|\, u^\varepsilon(t,x) = h^0(v^*)\}$. In §3.4, we substitute all information for the outer and inner solutions obtained in the previous procedures into $v^\varepsilon - \langle f(u^\varepsilon)\rangle$ and regulate it as in (3.1b), which gives rise to another series of equalities. In the last section §3.5, we construct the desired approximate solution by solving these two series of equalities obtained in §3.3 and 3.4.

3.1. Outer expansion. We assume that $u^\varepsilon(t,x)$ in (3.1a) has the formal expansions

$$u^\varepsilon(t,x) = U^\varepsilon(t,x) \sim \sum_{j\geq 0} \varepsilon^j U^{j,\pm}(t,x) \qquad \text{in } \Omega\backslash\Gamma(t)^\delta, \tag{3.2}$$

and determine the coefficients $U^{j,\pm}$ $(j \geq 0)$ so that they asymptotically solve (3.1a) on the respective domain $\Omega^\pm$. Substituting (3.2) together with the formal expansion

$$v^\varepsilon(t) \sim \sum_{j\geq 0} \varepsilon^j v^j(t) \tag{3.3}$$

into (3.1a) and equating the coefficient of each power of ε in the resulting equation, we have the following series of equations for $U^{j,\pm}$ in $\Omega^\pm$.

$$\begin{aligned} &(j=0) \qquad f(U^{0,\pm}) = v^0, \\ &(j=1) \qquad f'(U^{0,\pm})U^{1,\pm} = v^1, \\ &(j\geq 2) \qquad f'(U^{0,\pm})U^{j,\pm} = v^j + F_j^\pm, \end{aligned} \tag{3.4}$$

where $F_j^\pm$ stand for the terms depending only on $U^{0,\pm}, \dots, U^{j-1,\pm}$, explicitly given by

$$F_j^\pm = U_t^{j-2,\pm} - \Delta U^{j-2,\pm} - \left[\frac{1}{j!}\frac{d^j}{d\varepsilon^j} f\left(\sum_{m\geq 0} \varepsilon^m U^{m,\pm}\right)\bigg|_{\varepsilon=0} - f'(U^{0,\pm})U^{j,\pm} \right].$$

As a solution of (3.4) for $j=0$, we choose (cf. **(A1)**)

$$(j=0) \qquad U^{0,\pm}(t,x) = U^{0,\pm}(t) := h^\pm(v^0(t)) \qquad \text{on } \Omega^\pm(t). \tag{3.5a}$$

If we make this choice, $U^{j,\pm}$ $(j \geq 1)$ can be successively expressed, by (3.4) and **(A1)**, as

(3.5b) $(j \geq 1)$ $$U^{j,\pm}(t,x) = U^{j,\pm}(t) := h_v^{\pm}(v^0(t))v^j(t) + V_j^{\pm}(t) \quad \text{on } \Omega^{\pm}(t),$$

where $V_j^{\pm}$ are some functions depending only on $v^0, \dots, v^{j-1}$ satisfying $V_1^{\pm} \equiv 0$. Once $v^0, \dots, v^j$ are known, $U^{j,\pm}$ are completely determined via (3.5), although v^j $(j \geq 0)$ are unknown at this stage. These will be determined later, by the procedure developed in § 3.5 below.

By setting formally

$$U^{\varepsilon,\pm}(t) := \sum_{j \geq 0} \varepsilon^j U^{j,\pm}(t) \qquad \text{on } \Omega^{\pm}(t),$$

the outer solution U^{ε} is asymptotically characterized by

$$U^{\varepsilon}(t,x) = \begin{cases} U^{\varepsilon,-}(t), & t > 0,\ x \in \Omega^{-}(t), \\ U^{\varepsilon,+}(t), & t > 0,\ x \in \Omega^{+}(t). \end{cases}$$

3.2. Inner expansion. Transition layers cannot be captured by the outer solution U^{ε} because it has a jump between $U^{\varepsilon,-}(t)$ and $U^{\varepsilon,+}(t)$ on the interface $\Gamma(t)$. We now introduce a local coodinate system in $\Gamma(t)^{\delta}$ adapted to describing layer phenomena.

For each $t \geq 0$, we assume that $\Gamma(t)$ is expressed as a smooth embedding from a fixed $(N-1)$-dimensional reference manifold $\mathcal{M}$ to $\mathbf{R}^N$:

$$\begin{aligned} \gamma(t,\cdot) : \mathcal{M} &\to \Gamma(t) \\ y &\mapsto x = \gamma(t,y). \end{aligned} \tag{3.6}$$

We denote by $\nu(t,y) \in \mathbf{R}^N$ the unit normal vector on $\Gamma(t)$ at $x = \gamma(t,y)$ pointing into the interior of $\Omega^{+}(t)$, and normalize the parametrization (3.6) in such a way that γ_t is always parallel to ν (cf. [10]). A point $x \in \Gamma(t)^{\delta}$ is uniquely represented as

$$x = \Phi(t,r,y) := \gamma(t,y) + r\nu(t,y) \tag{3.7}$$

by the diffeomorphism $\Phi(t,\cdot,\cdot) : (-\delta,\delta) \times \mathcal{M} \to \Gamma(t)^{\delta}$, which gives the transformation between the coordinate systems (t,x) and (t,r,y). By virtue of this, a function $u(t,x)$ for $x \in \Gamma(t)^{\delta}$ is also denoted by $u(t,r,y)$.

In terms of (t,r,y), the differential operators $\partial/\partial t$ and Δ_x in (t,x) then transform as follows:

$$\begin{aligned} \frac{\partial}{\partial t} &\to \frac{\partial}{\partial t} - \gamma_t \cdot \nu \frac{\partial}{\partial r} - r\nu_t \cdot \nabla_{\mathcal{M}}^{\Gamma^r}, \\ \Delta_x &= \frac{\partial^2}{\partial r^2} + K\frac{\partial}{\partial r} + \Delta_{\mathcal{M}}^{\Gamma^r}. \end{aligned} \tag{3.8}$$

Here, $K(t,r,y)$ is the sum of pricipal curvatures (mean curvature, for short) of the r-shifted interface

$$\Gamma^r(t) := \Phi(t,r,\mathscr{M}) = \{x \in \Omega \mid x = \gamma(t,y) + r\nu(t,y), y \in \mathscr{M}\}$$

at $x = \Phi(t,r,y) \in \Gamma^r(t)$. Let $\kappa_i(t,y)$ $(i = 1,\dots,N-1)$ be the principal curvatures of $\Gamma(t)$ at $x = \gamma(t,y) \in \Gamma(t)$. Then K is explicitly expressed as

$$K(t,r,y) = \sum_{i=1}^{N-1} \frac{\kappa_i(t,y)}{1 + r\kappa_i(t,y)}.$$

The symbols $\nabla_{\mathscr{M}}^{\Gamma^r}$ and $\Delta_{\mathscr{M}}^{\Gamma^r}$ denote the gradient operator and the Laplace-Beltrami operator on $\mathscr{M}$ induced from ∇^{Γ^r} and Δ^{Γ^r}, those on $\Gamma^r(t)$, by $\Phi(t,r,\cdot)$, respectively. Let $(G_{ij})(t,r,y)$ $(i,j = 1,\dots,N-1)$ be the covariant metric tensor on $\Gamma^r(t)$ at $x = \Phi(t,r,y)$ induced from the Euclidean metric in $\mathbf{R}^N$, and denote $(G^{ij}) = (G_{ij})^{-1}$ and $G = \det(G_{ij})$. Then $\nabla_{\mathscr{M}}^{\Gamma^r}$ and $\Delta_{\mathscr{M}}^{\Gamma^r}$ are explicitly represented as

$$(3.9) \qquad \nabla_{\mathscr{M}}^{\Gamma^r}(t,r,y) = \sum_{i,j=1}^{N-1} \frac{\partial \Phi(t,r,y)}{\partial y^i} G^{ij}(t,r,y) \frac{\partial}{\partial y^j},$$

$$\Delta_{\mathscr{M}}^{\Gamma^r}(t,r,y) = \frac{1}{\sqrt{G(t,r,y)}} \sum_{i,j=1}^{N-1} \frac{\partial}{\partial y^i}\left(\sqrt{G(t,r,y)}G^{ij}(t,r,y)\frac{\partial}{\partial y^j}\right).$$

We note that the following equalities are valid:

$$(3.10) \qquad K(t,0,y) = \sum_{i=1}^{N-1} \kappa_i(t,y) =: \kappa(t,y), \qquad \nabla_{\mathscr{M}}^{\Gamma^r}(t,r,y)|_{r=0} = \nabla_{\mathscr{M}}^{\Gamma}(t,y),$$

$$K_r(t,0,y) = -\sum_{i=1}^{N-1} \kappa_i^2(t,y), \qquad \Delta_{\mathscr{M}}^{\Gamma^r}(t,r,y)|_{r=0} = \Delta_{\mathscr{M}}^{\Gamma}(t,y),$$

where $\nabla_{\mathscr{M}}^{\Gamma}$ and $\Delta_{\mathscr{M}}^{\Gamma}$ are the gradient operator and the Laplace-Beltrami operator on $\mathscr{M}$ induced from ∇^{Γ} and Δ^{Γ}, those on $\Gamma(t)$, by $\gamma(t,\cdot)$, respectively.

For $t \geq 0$, we define the ε-dependent interface $\Gamma^\varepsilon(t)$ as a level set of the solution u^ε. The transition layer is expected to develop near $\{x \in \Omega \mid u^\varepsilon(t,x) \approx h^0(v^*)\}$, and without loss of generality, we may assume

$$(3.11) \qquad h^0(v^*) := 0$$

by an appropriate translation. From this, we set

$$(3.12) \qquad \Gamma^\varepsilon(t) := \{x \in \Omega \mid u^\varepsilon(t,x) = 0\}.$$

We also expect that $\Gamma^{\varepsilon}(t)$ is expressed as the graph of a smooth function over $\Gamma(t)$:

$$\Gamma^{\varepsilon}(t) = \{x \in \Omega \mid x = \gamma(t, y) + \varepsilon R^{\varepsilon}(t, y)\nu(t, y),\ y \in \mathcal{M}\}, \tag{3.13}$$

where R^{ε} is a priori unknown and to be determined. To describe the layer phenomena near $\Gamma^{\varepsilon}(t)$, i.e., near $r = \varepsilon R^{\varepsilon}(t, y)$ in (t, r, y), let us introduce a stretched variable

$$z := \frac{r - \varepsilon R^{\varepsilon}(t, y)}{\varepsilon}. \tag{3.14}$$

Then the equation (3.1a) is recast in terms of z as follows, in which t and y are regarded as parameters:

Lemma 3.1. *The equation* (3.1a) *is recast as*

$$u^{\varepsilon}_{zz} + D^{\varepsilon}u^{\varepsilon} + f(u^{\varepsilon}) - v^{\varepsilon} = 0 \qquad \text{in } (-\delta/\varepsilon - R^{\varepsilon}, \delta/\varepsilon - R^{\varepsilon}). \tag{3.15}$$

Here, D^{ε} is the differential operator defined by

$$\begin{aligned} D^{\varepsilon} := {} & \varepsilon(\gamma_t \cdot \nu + K^{\varepsilon})\frac{\partial}{\partial z} + \varepsilon^2\biggl(R^{\varepsilon}_t \frac{\partial}{\partial z} - \Delta^{\varepsilon}_{\mathcal{M}} R^{\varepsilon} \frac{\partial}{\partial z} \\ & + |\nabla^{\varepsilon}_{\mathcal{M}} R^{\varepsilon}|^2 \frac{\partial^2}{\partial z^2} - 2\nabla^{\varepsilon}_{\mathcal{M}} R^{\varepsilon} \cdot \nabla^{\varepsilon}_{\mathcal{M}} \frac{\partial}{\partial z} + \Delta^{\varepsilon}_{\mathcal{M}} - \frac{\partial}{\partial t}\biggr) \\ & + \varepsilon^3\biggl[(z + R^{\varepsilon})\nu_t \cdot \biggl(\nabla^{\varepsilon}_{\mathcal{M}} - \nabla^{\varepsilon}_{\mathcal{M}} R^{\varepsilon} \frac{\partial}{\partial z}\biggr) \\ & - \frac{1}{\sqrt{G^{\varepsilon}}} \sum_{i,j=1}^{N-1} \biggl(\frac{\partial}{\partial r}(\sqrt{G}G^{ij})\biggr)^{\varepsilon} R^{\varepsilon}_{y^i} \frac{\partial}{\partial y^j} \\ & + \frac{1}{\sqrt{G^{\varepsilon}}} \sum_{i,j=1}^{N-1} \biggl(\frac{\partial}{\partial r}(\sqrt{G}G^{ij})\biggr)^{\varepsilon} R^{\varepsilon}_{y^i} R^{\varepsilon}_{y^j} \frac{\partial}{\partial z}\biggr], \end{aligned} \tag{3.16}$$

in which

$$\nabla^{\varepsilon}_{\mathcal{M}}(t, z, y) := \nabla^{\Gamma^r}_{\mathcal{M}}(t, r, y)|_{r = \varepsilon z + \varepsilon R^{\varepsilon}(t, y)},$$

$$\Delta^{\varepsilon}_{\mathcal{M}}(t, z, y) := \Delta^{\Gamma^r}_{\mathcal{M}}(t, r, y)|_{r = \varepsilon z + \varepsilon R^{\varepsilon}(t, y)},$$

and other functions $K^{\varepsilon}(t, z, y)$ etc. are defined by $K(t, r, y)$ etc. with $r = \varepsilon z + \varepsilon R^{\varepsilon}(t, y)$.

Proof. By virtue of (3.14), the differential operators $\partial/\partial t$, $\partial/\partial r$ and $\partial/\partial y^i$ in (t, r, y) are represented in (t, z, y) as

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - R_t^\varepsilon \frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial r} = \frac{1}{\varepsilon}\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial y^i} \to \frac{\partial}{\partial y^i} - R_{y^i}^\varepsilon \frac{\partial}{\partial z}. \tag{3.17}$$

Therefore, the differential operator $\varepsilon^2\Delta - \varepsilon^2\partial/\partial t$ in the equation (3.1a)

$$\varepsilon^2\left(\Delta - \frac{\partial}{\partial t}\right)u^\varepsilon + f(u^\varepsilon) - v^\varepsilon = 0$$

is recast, by (3.8), (3.9) and (3.17), as

$$\frac{\partial^2}{\partial z^2} + \left[\varepsilon K^\varepsilon \frac{\partial}{\partial z} + \varepsilon^2 \underline{\Delta}_{\mathscr{M}}^\varepsilon \right. \tag{3.18a}$$

$$\left. - \varepsilon^2 \frac{\partial}{\partial t} + \varepsilon^2 R_t^\varepsilon \frac{\partial}{\partial z} + \varepsilon\gamma_t \cdot v \frac{\partial}{\partial z} + \varepsilon^2(\varepsilon z + \varepsilon R^\varepsilon) v_t \cdot \underline{\nabla}_{\mathscr{M}}^\varepsilon \right], \tag{3.18b}$$

where

$$\underline{\nabla}_{\mathscr{M}}^\varepsilon := \sum_{i,j=1}^{N-1} \left(\frac{\partial \Phi^\varepsilon}{\partial y^i} - R_{y^i}^\varepsilon \frac{\partial \Phi^\varepsilon}{\partial z}\right)(G^{ij})^\varepsilon \left(\frac{\partial}{\partial y^j} - R_{y^j}^\varepsilon \frac{\partial}{\partial z}\right), \tag{3.19}$$

$$\underline{\Delta}_{\mathscr{M}}^\varepsilon := \frac{1}{\sqrt{G^\varepsilon}} \sum_{i,j=1}^{N-1} \left(\frac{\partial}{\partial y^i} - R_{y^i}^\varepsilon \frac{\partial}{\partial z}\right)\left[(\sqrt{G}G^{ij})^\varepsilon \left(\frac{\partial}{\partial y^j} - R_{y^j}^\varepsilon \frac{\partial}{\partial z}\right)\right].$$

Note that (3.18a) and (3.18b) come from $\varepsilon^2\Delta$ and $-\varepsilon^2\partial/\partial t$, respectively. In (3.19), we explicitly compute as

$$\begin{aligned} v_t \cdot \underline{\nabla}_{\mathscr{M}}^\varepsilon &= v_t \cdot \left(\nabla_{\mathscr{M}}^\varepsilon - \nabla_{\mathscr{M}}^\varepsilon R^\varepsilon \frac{\partial}{\partial z}\right) - \varepsilon(v_t \cdot v)\nabla_{\mathscr{M}}^\varepsilon R^\varepsilon \cdot \nabla_{\mathscr{M}}^\varepsilon - \varepsilon(v_t \cdot v)|\nabla_{\mathscr{M}}^\varepsilon R^\varepsilon|^2 \frac{\partial}{\partial z} \\ &= v_t \cdot \left(\nabla_{\mathscr{M}}^\varepsilon - \nabla_{\mathscr{M}}^\varepsilon R^\varepsilon \frac{\partial}{\partial z}\right), \end{aligned}$$

$$\begin{aligned} \underline{\Delta}_{\mathscr{M}}^\varepsilon &= \Delta_{\mathscr{M}}^\varepsilon - \Delta_{\mathscr{M}}^\varepsilon R^\varepsilon \frac{\partial}{\partial z} - 2\nabla_{\mathscr{M}}^\varepsilon R^\varepsilon \cdot \nabla_{\mathscr{M}}^\varepsilon \frac{\partial}{\partial z} + |\nabla_{\mathscr{M}}^\varepsilon R^\varepsilon|^2 \frac{\partial^2}{\partial z^2} \\ &\quad - \frac{\varepsilon}{\sqrt{G^\varepsilon}} \sum_{i,j=1}^{N-1} \left(\frac{\partial}{\partial r}(\sqrt{G}G^{ij})\right)^\varepsilon R_{yi}^\varepsilon \frac{\partial}{\partial y^j} + \frac{\varepsilon}{\sqrt{G^\varepsilon}} \sum_{i,j=1}^{N-1} \left(\frac{\partial}{\partial r}(\sqrt{G}G^{ij})\right)^\varepsilon R_{y^i}^\varepsilon R_{y^j}^\varepsilon \frac{\partial}{\partial z}. \end{aligned}$$

Then, in (3.18), we find that the differential operators in the brackets $[\cdot]$ form the operator D^ε defined in (3.15). This completes the proof. □

Substituting the formal expansion

$$R^\varepsilon(t,y) \sim \sum_{j\geq 0} \varepsilon^j R^{j+1}(t,y) \tag{3.20}$$

into (3.16) and noting (3.10), we can easily verify that the coefficients D^j in the formal expansion

$$D^\varepsilon \sim \sum_{j\geq 0} \varepsilon^j D^j \tag{3.21}$$

are given as follows:

$$\begin{aligned}
(3.22)\quad (j=0)\qquad & D^0 = 0,\\
(j=1)\qquad & D^1 = (\gamma_t \cdot \nu + \kappa)\frac{\partial}{\partial z},\\
(j=2)\qquad & D^2 = \left(\frac{\partial}{\partial t} - \Delta^\Gamma_{\mathscr{M}} - \sum_{i=1}^{N-1}\kappa_i^2\right) R^1 \frac{\partial}{\partial z}\\
& \quad + |\nabla^\Gamma_{\mathscr{M}} R^1|^2 \frac{\partial^2}{\partial z^2} - 2\nabla^\Gamma_{\mathscr{M}} R^1 \cdot \nabla^\Gamma_{\mathscr{M}} \frac{\partial}{\partial z}\\
& \quad + \Delta^\Gamma_{\mathscr{M}} - z\left(\sum_{i=1}^{N-1}\kappa_i^2\right)\frac{\partial}{\partial z} - \frac{\partial}{\partial t},\\
(j\geq 3)\qquad & D^j = \left(\frac{\partial}{\partial t} - \Delta^\Gamma_{\mathscr{M}} - \sum_{i=1}^{N-1}\kappa_i^2\right) R^{j-1} \frac{\partial}{\partial z}\\
& \quad + 2\nabla^\Gamma_{\mathscr{M}} R^{j-1} \cdot \nabla^\Gamma_{\mathscr{M}} R^1 \frac{\partial^2}{\partial z^2} - 2\nabla^\Gamma_{\mathscr{M}} R^{j-1} \cdot \nabla^\Gamma_{\mathscr{M}} \frac{\partial}{\partial z}\\
& \quad + \text{differential operator including } \Gamma,\ R^1,\dots,R^{j-2}.
\end{aligned}$$

Assuming that $u^\varepsilon(t,z,y)$ in (3.15) has the formal expansions

$$\begin{aligned}
(3.23)\quad u^\varepsilon(t,z,y) &= \tilde{u}^\varepsilon(t,z,y)\\
&\sim \sum_{j\geq 0}\varepsilon^j \tilde{u}^{j,\pm}(t,z,y) \qquad \text{in } \left(-\frac{\delta}{\varepsilon} - R^\varepsilon(t,y), \frac{\delta}{\varepsilon} - R^\varepsilon(t,y)\right),
\end{aligned}$$

we will determine the coefficients $\tilde{u}^{j,\pm}(t,z,y)$ $(j\geq 0)$ so that $\tilde{u}^{j,-}$ and $\tilde{u}^{j,+}$ asymptotically solve the equation in (3.15) in the intervals $(-\infty,0)$ and $(0,+\infty)$, respectively. Substituting (3.23) together with (3.21) and (3.3) into (3.15), and equating the coefficient of each power of ε in the resulting equation, we obtain the series of equations for $\tilde{u}^{j,\pm}$ in $\pm z \in (0,\infty)$.

$$\text{(3.24)}\qquad \begin{aligned} &(j=0) \qquad \tilde{u}^{0,\pm}_{zz} + f(\tilde{u}^{0,\pm}) - v^0 = 0,\\ &(j \geq 1) \qquad \tilde{u}^{j,\pm}_{zz} + f'(\tilde{u}^{0,\pm})\tilde{u}^{j,\pm} + \tilde{F}^{\pm}_j = 0, \end{aligned}$$

where, $\tilde{F}^{\pm}_j$ are the functions given by

$$\text{(3.25)}\quad \tilde{F}^{\pm}_j := \sum_{m=0}^{j} D^m \tilde{u}^{j-m,\pm} - v^j + \left[\frac{1}{j!}\frac{d^j}{d\varepsilon^j} f\left(\sum_{m\geq 0} \varepsilon^m \tilde{u}^{m,\pm} \right)\bigg|_{\varepsilon=0} - f'(\tilde{u}^{0,\pm})\tilde{u}^{j,\pm} \right],$$

depending only on Γ, $R^1, \dots, R^{j-1}$; v^j; $\tilde{u}^{0,\pm}, \dots, \tilde{u}^{j-1,\pm}$ (cf. (3.22)).

We treat (3.24) together with the following boundary conditions.

$$\text{(3.26a)}\quad \tilde{u}^{j,-}(t,0,y) = 0 = \tilde{u}^{j,+}(t,0,y),$$

$$\text{(3.26b)}\quad |\tilde{u}^{j,\pm}(t,z,y) - \tilde{U}^{j,\pm}(t,z,y)| = O(e^{-\eta|z|}) \quad \text{for some } \eta > 0 \text{ as } \pm z \to \infty.$$

The first condition (3.26a), the *interface condition*, comes from the definition of the level-set interface $\Gamma^{\varepsilon}(t)$ (cf. (3.12)). The second condition (3.26b) is called the *inner-outer matching condition*, in which $\tilde{U}^{j,\pm}(t,z,y)$ stand for the coefficients of ε^j in the expansion for

$$\tilde{U}^{\varepsilon}(t,z,y) = U^{\varepsilon}(t, \gamma(t,y) + (\varepsilon z + \varepsilon R^{\varepsilon}(t,y))v(t,y)),$$

the inner expression of outer solution $U^{\varepsilon}(t,x)$ in terms of (t,z,y). More precisely, $\tilde{U}^{j,\pm}(t,z,y)$ are, in general, the functions defined by

$$\tilde{U}^{j,\pm}(t,z,y) := \sum_{k=0}^{j} \frac{1}{k!}\frac{d^k}{d\varepsilon^k} U^{j-k,\pm}\left(t, \varepsilon z + \sum_{m\geq 1} \varepsilon^m R^m(t,y), y \right)\bigg|_{\varepsilon=0},$$

where we employed the expression $U^{j-k,\pm}(t,r,y)$. In our situation, $\tilde{U}^{j,\pm}(t,z,y)$ are nothing but the functions given by

$$\text{(3.27)}\qquad \tilde{U}^{j,\pm}(t,z,y) = U^{j,\pm}(t), \qquad \pm z \in (0,\infty),$$

thanks to the fact that $U^{j,\pm}(t,x)$ $(j \geq 0)$ are spatially homogeneous (cf. (3.5)). The function $\tilde{U}^{\varepsilon}(t,z,y)$ is asymptotically characterized by

$$\tilde{U}^{\varepsilon}(t,z,y) = \begin{cases} U^{\varepsilon,-}(t), & z \in (-\infty, 0),\\ U^{\varepsilon,+}(t), & z \in (0,\infty). \end{cases}$$

In determing the solutions $\tilde{u}^{j,\pm}$ to the equations (3.24) satisfying (3.26), the smooth solution pair $(Q(z;v), c(v))$ $(v \in (\underline{v}, \bar{v}))$ to the nonlinear eigenvalue problem (NEP) (cf. §1)

$$\begin{cases} Q_{zz} + cQ_z + f(Q) - v = 0, & z \in (-\infty,\infty),\\ \lim_{z\to\pm\infty} Q = h^{\pm}(v), & Q|_{z=0} = h^0(v) := 0 \end{cases}$$

plays an important role. Here, we note that $Q|_{z=0}$ is normalized as $Q|_{z=0} = 0$ by virtue of (3.11). Under the assumptions **(A1)** and **(A2)**, the solution pair $(Q(z;v), c(v))$ satisfies the following properties:

(i) The function $Q(z;v)$ and its derivatives of any order with respect to z converge to the limits $h^{\pm}(v)$ and 0 with an exponential order of $O(e^{-\eta|z|})$ for some $\eta > 0$ as $z \to \pm\infty$, respectively. Furthermore, $Q_z(z;v) > 0$ for $z \in (-\infty, \infty)$.

(ii) The function $c(v)$ is given by $c(v) = -\mathbf{J}(v)/m(v)$. Moreover, there exists unique point $v^* \in (\underline{v}, \bar{v})$ such that

$$c(v^*) = 0, \tag{3.28a}$$

$$c'(v^*) = -\frac{\mathbf{J}'(v^*)}{m^*} = \frac{[h]^*}{m^*} > 0, \tag{3.28b}$$

where

$$m(v) := \int_{-\infty}^{\infty} [Q_z(z;v)]^2 dz > 0,$$

$m^* := m(v^*) > 0$, and $[h]^* := h^+(v^*) - h^-(v^*) > 0$.

We simply denote the function $Q(z;v^*)$ by $Q^*(z)$. The equations in (3.24) $(j=0)$ have unique solutions $\tilde{u}^{0,\pm}$ satisfying (3.26a) $(j=0)$ if and only if

$$v^0 \equiv v^*. \tag{3.29}$$

Once (3.29) is realized, the solutions $\tilde{u}^{0,\pm}$ are determined as

$$\begin{aligned} &\tilde{u}^{0,-}(t,z,y) = Q^*(z), \qquad z \in (-\infty, 0], \\ &\tilde{u}^{0,+}(t,z,y) = Q^*(z), \qquad z \in [0, \infty), \end{aligned} \tag{3.30}$$

and the condition (3.26b) $(j=0)$ is satisfied by (3.27) $(j=0)$, (3.5a) and the property of Q^*. Moreover, it is proved that for each $j \geq 1$ the equations (3.24)

$$\tilde{u}^{j,\pm}_{zz} + f'(Q^*)\tilde{u}^{j,\pm} + \tilde{F}^{\pm}_j = 0$$

with (3.26) have unique solutions $\tilde{u}^{j,\pm}$ $(\pm z \in (0,\infty))$ explicitly expressed as

$$\tilde{u}^{j,\pm}(t,z,y) = -Q^*_z(z) \int_0^z \frac{1}{[Q^*_z(z')]^2} \int_{\pm\infty}^{z'} Q^*_z(z'')\tilde{F}^{\pm}_j(t,z'',y)dz''dz', \tag{3.31}$$

and that the derivatives for $\tilde{u}^{j,\pm} - \tilde{U}^{j,\pm}$ of any order with respect to t, z and y also decay with exponential rate of $O(e^{-\eta|z|})$ for some $\eta > 0$ as $z \to \pm\infty$.

If v^0 is chosen so that (3.29) and $(\Gamma, R^1, \ldots, R^{j-1}; v^1, \ldots, v^j)$ are known, the functions $\tilde{u}^{j,\pm}$ $(j \geq 0)$ are completely determined via (3.30) and (3.31), although $(\Gamma, R^1, \dots; v^0, v^1, \dots)$ are still unknown at this stage. By setting formally

$$\tilde{u}^{\varepsilon,\pm}(t,z,y) := \sum_{j\geq 0} \varepsilon^j \tilde{u}^{j,\pm}(t,z,y), \qquad \pm z \in [0,\infty),$$

the inner solution $\tilde{u}^\varepsilon$ is asymptotically characterized by

$$\tilde{u}^\varepsilon(t,z,y) = \begin{cases} \tilde{u}^{\varepsilon,-}(t,z,y), & z \in (-\infty, 0], \\ \tilde{u}^{\varepsilon,+}(t,z,y), & z \in [0,\infty). \end{cases}$$

3.3. C^1-matching. We imposed in the previous subsection the interface condition (3.26a) on the functions $\tilde{u}^{j,\pm}$ $(j \geq 0)$. In this subsection, we deal with additional conditions for their derivatives, called C^1-*matching conditions*:

$$\tilde{u}_z^{j,-}(t,0,y) = \tilde{u}_z^{j,+}(t,0,y), \qquad j \geq 0. \tag{3.32}$$

We will derive a series of equations for $(\Gamma, R^1, \dots; v^0, v^1, \dots)$ which is equivalent to (3.32). For the functions $\tilde{u}^{j,\pm}$ satisfying (3.32), we also employ the simple notation $\tilde{u}^j$ without superscripts "$\pm$" in the sense that

$$\tilde{u}^j(t,z,y) = \begin{cases} \tilde{u}^{j,-}(t,z,y), & z \in (-\infty, 0], \\ \tilde{u}^{j,+}(t,z,y), & z \in [0,\infty). \end{cases}$$

Lemma 3.2. *The C^1-matching conditions* (3.32) *are equivalent to*

$$(j = 0) \qquad v^0 = v^*,$$

$$(j = 1) \qquad \gamma_t \cdot \nu = -\kappa + c'(v^*) v^1, \tag{3.33a}$$

$$(j \geq 2) \qquad R_t^{j-1} = \left(\Delta_{\mathscr{M}}^{\Gamma} + \sum_{i=1}^{N-1} \kappa_i^2 \right) R^{j-1} + c'(v^*) v^j + \rho_{j-1}, \tag{3.33b}$$

where ρ_{j-1} is a function depending only on $(\Gamma, R^1, \dots, R^{j-2}; v^*, v^1, \dots, v^{j-1})$.

Proof. The statement for the 0-th order condition immediately follows from the last part in the previous subsection (cf. (3.30)).

By virtue of (3.31), we find that the conditions (3.32) for $j \geq 1$ are equivalent to

$$\int_{-\infty}^{\infty} Q_z^*(z) \tilde{F}_j(t,z,y)\,dz = 0, \qquad j \geq 1, \tag{3.34}$$

where $\tilde{F}_j$ is the function defined by two functions $\tilde{F}_j^{\pm}$ (cf. (3.25)) as

$$\tilde{F}_j(t,z,y) := \begin{cases} \tilde{F}_j^-(t,z,y), & z \in (-\infty, 0], \\ \tilde{F}_j^+(t,z,y), & z \in [0,\infty). \end{cases}$$

For $j = 1$, we have from (3.22) and (3.30),

$$\tilde{F}_1 = D^1 \tilde{u}^0 - v^1 = (\gamma_t \cdot v + \kappa) Q_z^* - v^1.$$

Then, the condition (3.34) with $j = 1$ reads as

$$0 = \int_{-\infty}^{\infty} Q_z^* \tilde{F}_1 \, dz = (\gamma_t \cdot v + \kappa) m^* - [h]^* v^1,$$

where m^* and $[h]^*$ are the positive constants introduced in (3.28b). This implies (3.33a).

For $j = 2$, the function $\tilde{F}_2$ turns out to be

$$\tilde{F}_2 = D^2 Q^* + D^1 \tilde{u}^1 - v^2 + \frac{1}{2} f''(Q^*)(\tilde{u}^1)^2.$$

Noting (3.22), we can recast (3.34) as

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} Q_z^* \tilde{F}_2 \, dz \\ &= \left(\frac{\partial}{\partial t} - \Delta_{\mathscr{M}}^{\Gamma} - \sum_{i=1}^{N-1} \kappa_i^2 \right) R^1 m^* + |\nabla_{\mathscr{M}}^{\Gamma} R^1|^2 \int_{-\infty}^{\infty} Q_{zz}^* Q_z^* \, dz - [h]^* v^2 \\ &\quad + \int_{-\infty}^{\infty} \left[-z \left(\sum_{i=1}^{N-1} \kappa_i^2 \right) Q_z^* + (\gamma_t \cdot v + \kappa) \tilde{u}_z^1 + \frac{1}{2} f''(Q^*)(\tilde{u}^1)^2 \right] Q_z^* \, dz. \end{aligned}$$

By (3.28b) and the fact that $\int_{-\infty}^{\infty} Q_{zz}^* Q_z^* \, dz = 0$, we arrive at (3.33b) with $j = 2$, where ρ_1 is a function depending only on $(\Gamma; v^*, v^1)$, explicitly given by

$$\rho_1 := \frac{1}{m^*} \int_{-\infty}^{\infty} \left[\left(\sum_{i=1}^{N-1} \kappa_i^2 \right) z Q_z^* - (\gamma_t \cdot v + \kappa) \tilde{u}_z^1 - \frac{1}{2} f''(Q^*)(\tilde{u}^1)^2 \right] Q_z^* \, dz. \tag{3.35}$$

Proceeding with the same argument as above, we get the equalities (3.33b) for $j \geq 3$. Indeed, the function $\tilde{F}_j$ $(j \geq 3)$ is

$$\tilde{F}_j = D^j Q^* - v^j + \cdots,$$

where we employed the expression "$\cdots$" to denote the lower order terms depending only on $(\Gamma, R^1, \ldots, R^{j-2}; v^*, v^1, \ldots, v^{j-1})$. Then (3.34) gives

$$
\begin{aligned}
0 &= \int_{-\infty}^{\infty} Q_z^* \tilde{F}_j \, dz \\
&= \left(\frac{\partial}{\partial t} - \Delta_{\mathscr{M}}^{\Gamma} - \sum_{i=1}^{N-1} \kappa_i^2 \right) R^{j-1} m^* \\
&\quad + 2 \nabla_{\mathscr{M}}^{\Gamma} R^{j-1} \cdot \nabla_{\mathscr{M}}^{\Gamma} R^1 \int_{-\infty}^{\infty} Q_{zz}^* Q_z^* \, dz - [h]^* v^j + \cdots .
\end{aligned}
$$

Using $\int_{-\infty}^{\infty} Q_{zz}^* Q_z^* \, dz = 0$ and (3.28b) again, we obtain (3.33b). This completes the proof. □

We cannot determine $(\Gamma, R^1, \ldots; v^1, v^2, \ldots)$ by the series of equalities (3.33) alone. In order to determine them, we need another series of equalities, which will be derived in the next subsection from the second equation (3.1b).

3.4. Nonlocal expansion. So far, we have dealed with the equation (3.1a). In turns, we treat in this section the nonlocal equality (3.1b). Note that the outer and inner solutions U^ε and $\tilde{u}^\varepsilon$, which were obtained only from (3.1a), depend on the data $\chi^\varepsilon = (\Gamma, R^\varepsilon; v^\varepsilon)$. We will substitute $U^\varepsilon(\chi^\varepsilon)$ and $\tilde{u}^\varepsilon(\chi^\varepsilon)$ satisfying (3.1a) into $v^\varepsilon - \langle f(u^\varepsilon) \rangle$, and constrain this so that (3.1b) is satisfied. It is expected that such a constraint gives rise to another series of equalities among $(\Gamma, R^1, \ldots; v^0, v^1, \ldots)$ which couples with (3.33).

To materialize this idea, let us first recast the volume element dx in terms of (t, r, y) and (t, z, y). We define

$$
J(t, r, y) := \prod_{i=1}^{N-1} (1 + r\kappa_i(t, y)). \tag{3.36}
$$

Then, by virtue of (3.7) and (3.14), the Euclidean volume element dx is expressed in terms of (t, r, y) and (t, z, y) as

$$
dx = J(t, r, y) dr dS_y^{\Gamma(t)} = \varepsilon J^\varepsilon(t, z, y) dz dS_y^{\Gamma(t)}, \tag{3.37}
$$

where $dS_y^{\Gamma(t)}$ stands for the $(N-1)$-dimensional volume element on $\mathscr{M}$ induced from the surface element $dS_x^{\Gamma(t)}$ on $\Gamma(t)$ at $x = \gamma(t, y)$ by $\gamma(t, \cdot)$, and J^ε is a function defined by

$$
J^\varepsilon(t, z, y) := J(t, \varepsilon z + \varepsilon R^\varepsilon(t, y), y). \tag{3.38}
$$

We note that $dS_y^{\Gamma(t)}$ is explicitly expressed as

$$
dS_y^{\Gamma(t)} = \sqrt{G(t, 0, y)} dy. \tag{3.39}
$$

Lemma 3.3. *The equation* (3.1a) *implies the nonlocal equality*

$$(3.40)\qquad v^{\varepsilon}-\langle f(u^{\varepsilon})\rangle=\frac{\varepsilon}{|\Omega|}[I^{\varepsilon}-\varepsilon(\dot{U}^{\varepsilon,-}|\Omega^{-}|+\dot{U}^{\varepsilon,+}|\Omega^{+}|)+O(e^{-\eta/\varepsilon})]$$

with

$$(3.41)\qquad I^{\varepsilon}:=\int_{\mathscr{M}}\int_{-\infty}^{\infty}[(\tilde{u}^{\varepsilon}-\tilde{U}^{\varepsilon})_{zz}+D^{\varepsilon}(\tilde{u}^{\varepsilon}-\tilde{U}^{\varepsilon})]J^{\varepsilon}\,dzdS_{y}^{\Gamma}.$$

Proof. A direct computation by employing (3.1a) implies

$$\begin{aligned}
v^{\varepsilon}-\langle f(u^{\varepsilon})\rangle&=\frac{1}{|\Omega|}\int_{\Omega}[v^{\varepsilon}-f(u^{\varepsilon})]dx\\
&=\frac{1}{|\Omega|}\int_{\Omega}(\varepsilon^{2}\Delta u^{\varepsilon}-\varepsilon^{2}u_{t}^{\varepsilon})dx\\
&=\frac{1}{|\Omega|}\int_{\Gamma(t)^{\delta}}(\varepsilon^{2}\Delta u^{\varepsilon}-\varepsilon^{2}u_{t}^{\varepsilon})dx+\frac{1}{|\Omega|}\int_{\Omega\backslash\Gamma(t)^{\delta}}(\varepsilon^{2}\Delta u^{\varepsilon}-\varepsilon^{2}u_{t}^{\varepsilon})dx,
\end{aligned}$$

and by using the inner expression and (3.37), we have

$$(3.42)\qquad \begin{aligned}
v^{\varepsilon}-\langle f(u^{\varepsilon})\rangle=&\frac{\varepsilon}{|\Omega|}\int_{\mathscr{M}}\int_{-\delta/\varepsilon-R^{\varepsilon}}^{\delta/\varepsilon-R^{\varepsilon}}(u_{zz}^{\varepsilon}+D^{\varepsilon}u^{\varepsilon})J^{\varepsilon}\,dzdS_{y}^{\Gamma}\\
&+\frac{\varepsilon^{2}}{|\Omega|}\int_{\Omega\backslash\Gamma(t)^{\delta}}(\Delta u^{\varepsilon}-u_{t}^{\varepsilon})dx.
\end{aligned}$$

Note that $u^{\varepsilon}=U^{\varepsilon}$ on $\Omega\backslash\Gamma(t)^{\delta}$ and $u^{\varepsilon}=\tilde{u}^{\varepsilon}$ on $\Gamma(t)^{\delta}$. Therefore, we may replace u^{ε} in the first and the second nonlocal terms on the right-hand side of (3.42) by $\tilde{u}^{\varepsilon}$ and U^{ε}, respectively. Employing the inner expression $\tilde{U}^{\varepsilon}$, we have

$$\begin{aligned}
v^{\varepsilon}-\langle f(u^{\varepsilon})\rangle&=\frac{\varepsilon}{|\Omega|}\int_{\mathscr{M}}\int_{-\delta/\varepsilon-R^{\varepsilon}}^{\delta/\varepsilon-R^{\varepsilon}}(\tilde{u}_{zz}^{\varepsilon}+D^{\varepsilon}\tilde{u}^{\varepsilon})J^{\varepsilon}\,dzdS_{y}^{\Gamma}+\frac{\varepsilon^{2}}{|\Omega|}\int_{\Omega\backslash\Gamma(t)^{\delta}}(\Delta U^{\varepsilon}-U_{t}^{\varepsilon})dx\\
&=\frac{\varepsilon}{|\Omega|}\int_{\mathscr{M}}\int_{-\delta/\varepsilon-R^{\varepsilon}}^{\delta/\varepsilon-R^{\varepsilon}}[(\tilde{u}^{\varepsilon}-\tilde{U}^{\varepsilon})_{zz}+D^{\varepsilon}(\tilde{u}^{\varepsilon}-\tilde{U}^{\varepsilon})]J^{\varepsilon}\,dzdS_{y}^{\Gamma}\\
&\quad+\frac{\varepsilon^{2}}{|\Omega|}\int_{\Omega}(\Delta U^{\varepsilon}-U_{t}^{\varepsilon})dx.
\end{aligned}$$

Moreover, U^{ε} is expressed as $U^{\varepsilon}(t,x)=U^{\varepsilon,\pm}(t)$ on $\Omega^{\pm}(t)$ by the spatial homogeneous functions $U^{\varepsilon,\pm}(t)$. This implies

$$\begin{aligned}
v^{\varepsilon}-\langle f(u^{\varepsilon})\rangle=&\frac{\varepsilon}{|\Omega|}\int_{\mathscr{M}}\int_{-\delta/\varepsilon-R^{\varepsilon}}^{\delta/\varepsilon-R^{\varepsilon}}[(\tilde{u}^{\varepsilon}-\tilde{U}^{\varepsilon})_{zz}+D^{\varepsilon}(\tilde{u}^{\varepsilon}-\tilde{U}^{\varepsilon})]J^{\varepsilon}\,dzdS_{y}^{\Gamma}\\
&+\frac{\varepsilon^{2}}{|\Omega|}\int_{\Omega^{-}}(\Delta U^{\varepsilon,-}-U_{t}^{\varepsilon,-})dx+\frac{\varepsilon^{2}}{|\Omega|}\int_{\Omega^{+}}(\Delta U^{\varepsilon,+}-U_{t}^{\varepsilon,+})dx
\end{aligned}$$

$$= \frac{\varepsilon}{|\Omega|} \int_{\mathscr{M}} \int_{-\delta/\varepsilon - R^\varepsilon}^{\delta/\varepsilon - R^\varepsilon} [(\tilde{u}^\varepsilon - \tilde{U}^\varepsilon)_{zz} + D^\varepsilon(\tilde{u}^\varepsilon - \tilde{U}^\varepsilon)] J^\varepsilon \, dz dS_y^\Gamma$$
$$- \frac{\varepsilon^2}{|\Omega|} (\dot{U}^{\varepsilon,-}|\Omega^-| + \dot{U}^{\varepsilon,+}|\Omega^+|).$$

The error caused by replacing the integral interval $(-\delta/\varepsilon - R^\varepsilon, \delta/\varepsilon - R^\varepsilon)$ by $(-\infty, \infty)$ is of order $O(e^{-\eta/\varepsilon})$ for some $\eta > 0$ because of the decay properties for the difference $\tilde{u}^{\varepsilon,\pm} - \tilde{U}^{\varepsilon,\pm}$ as $z \to \pm\infty$. Hence we have (3.40). □

Let us now constrain $v^\varepsilon - \langle f(u^\varepsilon) \rangle$ in (3.40) so that (3.1b) is satisfied:

$$I^\varepsilon - \varepsilon(\dot{U}^{\varepsilon,-}|\Omega^-| + \dot{U}^{\varepsilon,+}|\Omega^+|) + O(e^{-\eta/\varepsilon}) := 0. \tag{3.43}$$

We will expand the equality (3.43) and calculate the coefficient of each power of ε. In order to do so, we need to know the coefficients J^j $(j \geq 0)$ in the formal power series

$$J^\varepsilon(t, z, y) \sim \sum_{j \geq 0} \varepsilon^j J^j(t, z, y). \tag{3.44}$$

Substituting the expansion (3.20) into J^ε, let us compute the coefficients J^j.

We first express $J(t, r, y)$ in (3.36) as

$$J(t, r, y) = \prod_{i=1}^{N-1} (1 + r\kappa_i(t, y)) =: \sum_{i \geq 0} H_i(t, y) r^i, \tag{3.45}$$

in which $H_0, \dots, H_{N-1}$ are the fundamental symmetric functions of $\kappa_1, \dots, \kappa_{N-1}$ and $H_i \equiv 0$ for all $i \geq N$:

$$H_0 \equiv 1, \qquad H_1 = \sum_{i=1}^{N-1} \kappa_i = \kappa, \qquad H_2 = \sum_{1 \leq i < j \leq N-1} \kappa_i \kappa_j, \tag{3.46}$$
$$\cdots\cdots, \qquad H_{N-1} = \prod_{i=1}^{N-1} \kappa_i, \qquad H_i \equiv 0 \quad (i \geq N).$$

Hence, from (3.20), (3.38), (3.44) and (3.45), we have

$$\sum_{j \geq 0} \varepsilon^j J^j(t, z, y) = J^\varepsilon(t, z, y) = \sum_{i \geq 0} H_i(t, y) \left(\varepsilon z + \sum_{m \geq 1} \varepsilon^m R^m(t, y) \right)^i.$$

Noting (3.46) and equating the coefficient of each power of ε, we find that the coefficients J^j are as follows:

$$(3.47)\qquad \begin{aligned} &(j=0) && J^0 = 1,\\ &(j=1) && J^1 = \kappa(z+R^1),\\ &(j=2) && J^2 = \kappa R^2 + H_2(z+R^1)^2,\\ &(j\geq 3) && J^j = \kappa R^j + 2H_2(z+R^1)R^{j-1}\\ &&& \qquad\qquad + \text{terms including } \Gamma,\ R^1,\dots,R^{j-2}. \end{aligned}$$

Substituting the expansions (3.2), (3.23), (3.21) and (3.44) into (3.43) with (3.41), we obtain the following series of equalities:

$$(3.48)\qquad \begin{aligned} &(j=0) && I^0 = 0,\\ &(j\geq 1) && I^j - (\dot{U}^{j-1,-}|\Omega^-| + \dot{U}^{j-1,+}|\Omega^+|) = 0. \end{aligned}$$

Here, I^j $(j\geq 0)$ are coefficients in the expansion $I^\varepsilon(t)\sim\sum_{j\geq 0}\varepsilon^j I^j(t)$ for (3.41) defined by

$$(3.49)\qquad \begin{aligned} I^j := {} & \sum_{m=0}^{j}\int_{\mathscr{M}}\left[\int_{-\infty}^{0}(\tilde{u}^{j-m,-}-\tilde{U}^{j-m,-})_{zz}J^m\,dz\right.\\ & \left. + \int_0^\infty(\tilde{u}^{j-m,+}-\tilde{U}^{j-m,+})_{zz}J^m\,dz\right]dS_y^\Gamma\\ & + \sum_{m=0}^{j}\sum_{l=0}^{m}\int_{\mathscr{M}}\left[\int_{-\infty}^{0}D^{j-m}(\tilde{u}^{m-l,-}-\tilde{U}^{m-l,-})J^l\,dz\right.\\ & \left. + \int_0^\infty D^{j-m}(\tilde{u}^{m-l,+}-\tilde{U}^{m-l,+})J^l\,dz\right]dS_y^\Gamma. \end{aligned}$$

Lemma 3.4. *The nonlocal equalities* (3.48) *are satisfied for* $j=0$, *and are recast as follows for* $j\geq 1$.

$$(3.50\text{a})\qquad (j=1)\qquad c'(v^*)v^1 = \frac{1}{|\Gamma|}\int_{\mathscr{M}}\kappa\,dS_y^\Gamma,$$

$$(3.50\text{b})\qquad (j\geq 2)\qquad c'(v^*)v^j = -\frac{1}{|\Gamma|}\int_{\mathscr{M}}\left(\sum_{i=1}^{N-1}\kappa_i^2+\kappa(\gamma_t\cdot\nu)\right)R^{j-1}\,dS_y^\Gamma + \sigma_{j-1},$$

where σ_{j-1} *is a function depending only on* $(\Gamma,R^1,\dots,R^{j-2};v^*,\dots,v^{j-1})$.

Proof. In the proof, we use the decay properties for $\tilde{u}^{\varepsilon,\pm}-\tilde{U}^{\varepsilon,\pm}$, (3.22), (3.27) and (3.47), (3.32) and their equivalent expressions listed in Lemma 3.2.

Let us begin with $j=0$. The term I^0 in (3.48) is recast as

$$
\begin{aligned}
I^0 &= \int_{\mathscr{M}} \int_{-\infty}^{0} (\tilde{u}^{0,-} - \tilde{U}^{0,-})_{zz} dz dS_y^{\Gamma} + \int_{\mathscr{M}} \int_{0}^{\infty} (\tilde{u}^{0,+} - \tilde{U}^{0,+})_{zz} dz dS_y^{\Gamma} \\
&= \int_{\mathscr{M}} \left(\int_{-\infty}^{\infty} Q_{zz}^* \, dz \right) dS_y^{\Gamma} = [Q_z^*(\infty) - Q_z^*(-\infty)] |\Gamma|,
\end{aligned}
$$

which vanishes due to the property of Q^*. Therefore, the equation (3.48) with $j = 0$ is valid.

We move on to the case where $j = 1$. The function I^1 is as follows:

$$
I^1 = \int_{\mathscr{M}} \left[\int_{-\infty}^{0} (\tilde{u}^{1,-} - \tilde{U}^{1,-})_{zz} J^0 \, dz + \int_{0}^{\infty} (\tilde{u}^{1,+} - \tilde{U}^{1,+})_{zz} J^0 \, dz \right] dS_y^{\Gamma} \tag{3.51a}
$$

$$
+ \int_{\mathscr{M}} \left[\int_{-\infty}^{0} (\tilde{u}^{0,-} - \tilde{U}^{0,-})_{zz} J^1 \, dz + \int_{0}^{\infty} (\tilde{u}^{0,+} - \tilde{U}^{0,+})_{zz} J^1 \, dz \right] dS_y^{\Gamma} \tag{3.51b}
$$

$$
+ \int_{\mathscr{M}} \left[\int_{-\infty}^{0} D^1 (\tilde{u}^{0,-} - \tilde{U}^{0,-}) J^0 \, dz + \int_{0}^{\infty} D^1 (\tilde{u}^{0,+} - \tilde{U}^{0,+}) J^0 \, dz \right] dS_y^{\Gamma}. \tag{3.51c}
$$

We can compute so that

$$
\begin{aligned}
[\cdot] \text{ in } (3.51\text{a}) &= \tilde{u}_z^{1,-}|_{z=0} - \tilde{u}_z^{1,+}|_{z=0} + (\tilde{u}^{1,+} - \tilde{U}^{1,+})_z|_{z=\infty} - (\tilde{u}^{1,-} - \tilde{U}^{1,-})_z|_{z=-\infty} \\
&= 0, \\
[\cdot] \text{ in } (3.51\text{b}) &= \int_{-\infty}^{\infty} Q_{zz}^* \kappa (z + R^1) dz = \kappa \int_{-\infty}^{\infty} Q_{zz}^* (z + R^1) dz = -\kappa \int_{-\infty}^{\infty} Q_z^* \, dz \\
&= -[h]^* \kappa, \\
[\cdot] \text{ in } (3.51\text{c}) &= (\gamma_t \cdot v + \kappa) \int_{-\infty}^{\infty} Q_z^* \, dz = c'(v^*) v^1 \int_{-\infty}^{\infty} Q_z^* \, dz = [h]^* c'(v^*) v^1.
\end{aligned}
$$

Hence, I^1 becomes

$$
I^1 = [h]^* |\Gamma| \left(c'(v^*) v^1 - \frac{1}{|\Gamma|} \int_{\mathscr{M}} \kappa \, dS_y^{\Gamma} \right).
$$

On the other hand, the second term in (3.48) with $j = 1$ vanishes since $U^{0,\pm} = h^{\pm}(v^*)$ are independent of t, which implies that (3.48) with $j = 1$ is

$$
[h]^* |\Gamma| \left(c'(v^*) v^1 - \frac{1}{|\Gamma|} \int_{\mathscr{M}} \kappa \, dS_y^{\Gamma} \right) = 0.
$$

Since $[h]^* |\Gamma| > 0$, we have (3.50a).

We next treat the case where $j = 2$. The term I^2 turns out to be as follows:

$$(3.52a)\qquad I^2 = \int_{\mathscr{M}} \left[\int_{-\infty}^{0} (\tilde{u}^{2,-} - \tilde{U}^{2,-})_{zz} J^0 \, dz + \int_0^{\infty} (\tilde{u}^{2,+} - \tilde{U}^{2,+})_{zz} J^0 \, dz \right] dS_y^{\Gamma}$$

$$(3.52b)\qquad + \int_{\mathscr{M}} \left[\int_{-\infty}^{0} (\tilde{u}^{1,-} - \tilde{U}^{1,-})_{zz} J^1 \, dz + \int_0^{\infty} (\tilde{u}^{1,+} - \tilde{U}^{1,+})_{zz} J^1 \, dz \right] dS_y^{\Gamma}$$

$$(3.52c)\qquad + \int_{\mathscr{M}} \left[\int_{-\infty}^{0} (\tilde{u}^{0,-} - \tilde{U}^{0,-})_{zz} J^2 \, dz + \int_0^{\infty} (\tilde{u}^{0,+} - \tilde{U}^{0,+})_{zz} J^2 \, dz \right] dS_y^{\Gamma}$$

$$(3.52d)\qquad + \int_{\mathscr{M}} \left[\int_{-\infty}^{0} D^2 (\tilde{u}^{0,-} - \tilde{U}^{0,-}) J^0 \, dz + \int_0^{\infty} D^2 (\tilde{u}^{0,+} - \tilde{U}^{0,+}) J^0 \, dz \right] dS_y^{\Gamma}$$

$$(3.52e)\qquad + \int_{\mathscr{M}} \left[\int_{-\infty}^{0} D^1 (\tilde{u}^{0,-} - \tilde{U}^{0,-}) J^1 \, dz + \int_0^{\infty} D^1 (\tilde{u}^{0,+} - \tilde{U}^{0,+}) J^1 \, dz \right] dS_y^{\Gamma}$$

$$(3.52f)\qquad + \int_{\mathscr{M}} \left[\int_{-\infty}^{0} D^1 (\tilde{u}^{1,-} - \tilde{U}^{1,-}) J^0 \, dz + \int_0^{\infty} D^1 (\tilde{u}^{1,+} - \tilde{U}^{1,+}) J^0 \, dz \right] dS_y^{\Gamma}.$$

We can calculate so that

$$\begin{aligned}
[\cdot] \text{ in } (3.52a) &= \tilde{u}_z^{2,-}|_{z=0} - \tilde{u}_z^{2,+}|_{z=0} + (\tilde{u}^{2,+} - \tilde{U}^{2,+})_z|_{z=\infty} - (\tilde{u}^{2,-} - \tilde{U}^{2,-})_z|_{z=-\infty} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
[\cdot] \text{ in } (3.52b) &= [(\tilde{u}^{1,-} - \tilde{U}^{1,-})_z J^1]_{-\infty}^{0} + [(\tilde{u}^{1,+} - \tilde{U}^{1,+})_z J^1]_0^{\infty} \\
&\quad - \int_{-\infty}^{0} (\tilde{u}^{1,-} - \tilde{U}^{1,-})_z J_z^1 \, dz - \int_0^{\infty} (\tilde{u}^{1,+} - \tilde{U}^{1,+})_z J_z^1 \, dz \\
&= - \int_{-\infty}^{0} (\tilde{u}^{1,-} - \tilde{U}^{1,-})_z \kappa \, dz - \int_0^{\infty} (\tilde{u}^{1,+} - \tilde{U}^{1,+})_z \kappa \, dz \\
&= -\kappa [(\tilde{u}^{1,-} - \tilde{U}^{1,-})]_{-\infty}^{0} - \kappa [(\tilde{u}^{1,+} - \tilde{U}^{1,+})]_0^{\infty} \\
&= -\kappa (U^{1,+} - U^{1,-}),
\end{aligned}$$

$$\begin{aligned}
[\cdot] \text{ in } (3.52c) &= \int_{-\infty}^{\infty} Q_{zz}^* J^2 \, dz = - \int_{-\infty}^{\infty} Q_z^* J_z^2 \, dz = - \int_{-\infty}^{\infty} Q_z^* (2H_2 (z + R^1)) dz \\
&= -2[h]^* H_2 R^1 - 2H_2 \left(\int_{-\infty}^{\infty} z Q_z \, dz \right),
\end{aligned}$$

$$\begin{aligned}
[\cdot] \text{ in } (3.52d) &= \int_{-\infty}^{\infty} \left[\left(\frac{\partial}{\partial t} - \Delta_{\mathscr{M}}^{\Gamma} - \sum_{i=1}^{N-1} \kappa_i^2 \right) R_1 \right] Q_z^* \, dz + \int_{-\infty}^{\infty} |\nabla_{\mathscr{M}}^{\Gamma} R^1|^2 Q_{zz}^* \, dz \\
&\quad - \int_{-\infty}^{\infty} \left(\sum_{i=1}^{N-1} \kappa_i^2 \right) z Q_z^* \, dz
\end{aligned}$$

$$= [h]^*(c'(v^*)v^2 + \rho_1) + |\nabla_{\mathscr{M}}^{\Gamma} R^1|^2 [Q_z^*(\infty) - Q_z^*(-\infty)]$$

$$- \sum_{i=1}^{N-1} \kappa_i^2 \int_{-\infty}^{\infty} z Q_z^* \, dz$$

$$= [h]^* c'(v^*)v^2 + \left([h^*]\rho_1 - \sum_{i=1}^{N-1} \kappa_i^2 \int_{-\infty}^{\infty} z Q_z^* \, dz \right),$$

where ρ_1 is the function depending only on $(\Gamma; v^*, v^1)$ defined as in (3.35),

$$[\cdot] \text{ in } (3.52e) = \int_{-\infty}^{\infty} (\gamma_t \cdot \nu + \kappa) Q_z^* \kappa (z + R^1) dz$$

$$= (\gamma_t \cdot \nu + \kappa)\kappa \int_{-\infty}^{\infty} z Q_z^* \, dz + (\gamma_t \cdot \nu + \kappa)\kappa R^1 \int_{-\infty}^{\infty} Q_z^* \, dz$$

$$= [h]^*(\gamma_t \cdot \nu + \kappa)\kappa R^1 + (\gamma_t \cdot \nu + \kappa)\kappa \int_{-\infty}^{\infty} z Q_z^* \, dz,$$

$$[\cdot] \text{ in } (3.52f) = \int_{-\infty}^{0} (\gamma_t \cdot \nu + \kappa)(\tilde{u}^{1,-} - \tilde{U}^{1,-})_z \, dz + \int_{0}^{\infty} (\gamma_t \cdot \nu + \kappa)(\tilde{u}^{1,+} - \tilde{U}^{1,+})_z \, dz$$

$$= (\gamma_t \cdot \nu + \kappa)[\tilde{u}^{1,-} - \tilde{U}^{1,-}]_{-\infty}^{0} + (\gamma_t \cdot \nu + \kappa)[\tilde{u}^{1,+} - \tilde{U}^{1,+}]_{0}^{\infty}$$

$$= (\gamma_t \cdot \nu + \kappa)(U^{1,+} - U^{1,-}).$$

The second term in (3.48) with $j = 2$ is

$$\dot{U}^{1,-}|\Omega^-| + \dot{U}^{1,+}|\Omega^+|,$$

which is lower order since $U^{1,\pm}$ depend only on v^*, v^1. Therefore, (3.48) with $j = 2$ is recast as

$$(3.53) \quad [h]^*|\Gamma| \left(c'(v^*)v^2 - \frac{1}{|\Gamma|} \int_{\mathscr{M}} (2H_2 - (\gamma_t \cdot \nu + \kappa)\kappa) R^1 \, dS_y^{\Gamma} - \sigma_1 \right) = 0,$$

where σ_1 is a lower order term depending only on $(\Gamma; v^*, v^1)$, explicitly given by

$$\sigma_1 := \frac{1}{[h]^*|\Gamma|} \left[(\dot{U}^{1,-}|\Omega^-| + \dot{U}^{1,+}|\Omega^+|) - (U^{1,+} - U^{1,-}) \int_{\mathscr{M}} \gamma_t \cdot \nu \, dS_y^{\Gamma} \right]$$

$$+ \frac{1}{[h]^*|\Gamma|} \left(\int_{-\infty}^{\infty} z Q_z^* \, dz \right) \int_{\mathscr{M}} \left(2H_2 + \sum_{i=1}^{N-1} \kappa_i^2 - (\gamma_t \cdot \nu + \kappa)\kappa \right) dS_y^{\Gamma}$$

$$- \frac{1}{|\Gamma|} \int_{\mathscr{M}} \rho_1 \, dS_y^{\Gamma}.$$

From the definition of H_2 (cf. (3.46)), the integral kernel in (3.53)

$$2H_2 - (\gamma_t \cdot v + \kappa)\kappa$$

is recast as

$$\begin{aligned}2H_2 - (\gamma_t \cdot v + \kappa)\kappa &= 2H_2 - \kappa^2 - \kappa(\gamma_t \cdot v)\\ &= 2 \sum_{1 \le i < j \le N-1} \kappa_i \kappa_j - \left(\sum_{i=1}^{N-1} \kappa_i\right)^2 - \kappa(\gamma_t \cdot v)\\ &= -\sum_{i=1}^{N-1} \kappa_i^2 - \kappa(\gamma_t \cdot v).\end{aligned}$$

Thus, from (3.53) and $[h]^*|\Gamma| > 0$, we arrive at (3.50b) with $j = 2$.

The same computation as above implies the results for all $j \ge 3$. Indeed, the term I^j is:

$$I^j = \int_{\mathscr{M}} \left[\int_{-\infty}^{0} (\tilde{u}^{j,-} - \tilde{U}^{j,-})_{zz} J^0 \, dz + \int_0^\infty (\tilde{u}^{j,+} - \tilde{U}^{j,+})_{zz} J^0 \, dz\right] dS_y^\Gamma \tag{3.54a}$$

$$+ \int_{\mathscr{M}} \left[\int_{-\infty}^{0} (\tilde{u}^{1,-} - \tilde{U}^{1,-})_{zz} J^{j-1} \, dz \right. \tag{3.54b}$$

$$\left. + \int_0^\infty (\tilde{u}^{1,+} - \tilde{U}^{1,+})_{zz} J^{j-1} \, dz\right] dS_y^\Gamma$$

$$+ \int_{\mathscr{M}} \left[\int_{-\infty}^{0} (\tilde{u}^{0,-} - \tilde{U}^{0,-})_{zz} J^j \, dz + \int_0^\infty (\tilde{u}^{0,+} - \tilde{U}^{0,+})_{zz} J^j \, dz\right] dS_y^\Gamma \tag{3.54c}$$

$$+ \int_{\mathscr{M}} \left[\int_{-\infty}^{0} D^j (\tilde{u}^{0,-} - \tilde{U}^{0,-}) J^0 \, dz \right. \tag{3.54d}$$

$$\left. + \int_0^\infty D^j (\tilde{u}^{0,+} - \tilde{U}^{0,+}) J^0 \, dz\right] dS_y^\Gamma$$

$$+ \int_{\mathscr{M}} \left[\int_{-\infty}^{0} D^1 (\tilde{u}^{0,-} - \tilde{U}^{0,-}) J^{j-1} \, dz \right. \tag{3.54e}$$

$$\left. + \int_0^\infty D^1 (\tilde{u}^{0,+} - \tilde{U}^{0,+}) J^{j-1} \, dz\right] dS_y^\Gamma$$

$$+ \cdots,$$

where we employed the expression "$\cdots$" to denote the lower order terms depending only on $(\Gamma, R^1, \dots, R^{j-2}; v^*, v^1, \dots, v^{j-1})$. We can compute so that

$$
\begin{aligned}
[\cdot] \text{ in } (3.54\text{a}) &= \int_{-\infty}^{0} (\tilde{u}^{j,-} - \tilde{U}^{j,-})_{zz}\, dz + \int_{0}^{\infty} (\tilde{u}^{j,+} - \tilde{U}^{j,+})_{zz}\, dz \\
&= \tilde{u}_z^{j,-}|_{z=0} - \tilde{u}_z^{j,+}|_{z=0} + (\tilde{u}^{j,+} - \tilde{U}^{j,+})_z|_{z=\infty} - (\tilde{u}^{j,-} - \tilde{U}^{j,-})_z|_{z=-\infty} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
[\cdot] \text{ in } (3.54\text{b}) &= [(\tilde{u}^{1,-} - \tilde{U}^{1,-})_z(\kappa R^{j-1} + \cdots)]_{-\infty}^{0} \\
&\quad + [(\tilde{u}^{1,+} - \tilde{U}^{1,+})_z(\kappa R^{j-1} + \cdots)]_{0}^{\infty} \\
&\quad - \int_{-\infty}^{0} (\tilde{u}^{1,-} - \tilde{U}^{1,-})_z(\kappa R^{j-1} + \cdots)_z\, dz \\
&\quad - \int_{0}^{\infty} (\tilde{u}^{1,+} - \tilde{U}^{1,+})_z(\kappa R^{j-1} + \cdots)_z\, dz \\
&= -\int_{-\infty}^{0} (\tilde{u}^{1,-} - \tilde{U}^{1,-})_z(\cdots)dz - \int_{0}^{\infty} (\tilde{u}^{1,+} - \tilde{U}^{1,+})_z(\cdots)dz \\
&= \cdots,
\end{aligned}
$$

$$
\begin{aligned}
[\cdot] \text{ in } (3.54\text{c}) &= \int_{-\infty}^{\infty} Q_{zz}^* J^j\, dz = -\int_{-\infty}^{\infty} Q_z^* J_z^j\, dz = -\int_{-\infty}^{\infty} Q_z^*(2H_2 R^{j-1} + \cdots)dz \\
&= -2[h]^* H_2 R^{j-1} + \cdots,
\end{aligned}
$$

$$
\begin{aligned}
[\cdot] \text{ in } (3.54\text{d}) &= \int_{-\infty}^{\infty} \left[\left(\frac{\partial}{\partial t} - \Delta_{\mathscr{M}}^{\Gamma} - \sum_{i=1}^{N-1} \kappa_i^2 \right) R^{j-1} \right] Q_z^*\, dz \\
&\quad + \int_{-\infty}^{\infty} 2\nabla_{\mathscr{M}}^{\Gamma} R^{j-1} \cdot \nabla_{\mathscr{M}}^{\Gamma} R^1 Q_{zz}^*\, dz + \cdots \\
&= [h]^*(c'(v^*)v^j + \cdots) + 2\nabla_{\mathscr{M}}^{\Gamma} R^{j-1} \cdot \nabla_{\mathscr{M}}^{\Gamma} R^1 [Q_z^*(\infty) - Q_z^*(-\infty)] + \cdots \\
&= [h]^* c'(v^*)v^j + \cdots,
\end{aligned}
$$

$$
\begin{aligned}
[\cdot] \text{ in } (3.54\text{e}) &= \int_{-\infty}^{\infty} (\gamma_t \cdot v + \kappa) Q_z^*(\kappa R^{j-1} + \cdots)dz \\
&= (\gamma_t \cdot v + \kappa)\kappa R^{j-1} \int_{-\infty}^{\infty} Q_z^*\, dz + \cdots \\
&= [h]^*(\gamma_t \cdot v + \kappa)\kappa R^{j-1} + \cdots.
\end{aligned}
$$

The second term in (3.48) is

$$
\dot{U}^{j-1,-}|\Omega^-| + \dot{U}^{j-1,+}|\Omega^+| = \cdots,
$$

since $U^{j-1,\pm}$ depend only on $v^*, v^1, \dots, v^{j-1}$. Hence, (3.48) is recast as

$$[h]^*|\Gamma|\left(c'(v^*)v^j - \frac{1}{|\Gamma|}\int_{\mathcal{M}}(2H_2 - (\gamma_t \cdot v + \kappa)\kappa)R^{j-1}\,dS_y^\Gamma + \cdots\right) = 0.$$

By $[h]^*|\Gamma| > 0$ and changing the form of the integral kernel, we arrive at (3.50b). This completes the proof. □

3.5. Uniform approximation. In §3.3 and §3.4, we derived two series of equalities (3.33) and (3.50). These two give rise to the following series of parabolic equations for $t > 0$, $y \in \mathcal{M}$:

(3.55a) $$\gamma_t \cdot v = -\kappa + \frac{1}{|\Gamma|}\int_{\mathcal{M}} \kappa\,dS_y^\Gamma,$$

(3.55b) $$R_t^{j-1} = \left(\Delta_{\mathcal{M}}^\Gamma + \sum_{i=1}^{N-1}\kappa_i^2\right)R^{j-1} - \frac{1}{|\Gamma|}\int_{\mathcal{M}}\left(\sum_{i=1}^{N-1}\kappa_i^2 + \kappa(\gamma_t \cdot v)\right)R^{j-1}\,dS_y^\Gamma + (\rho_{j-1} + \sigma_{j-1}).$$

The first equation (3.55a) is nothing but the averaged mean curvature flow. Thus, once a smooth initial interface is given, it is guaranteed that there exists unique smooth solution Γ to (3.55a) on a time interval $[0, T]$ (cf. [14, 16, 11, 21]). The equation (3.55b) is a nonlocal nonhomogeneous linear parabolic equation of the following form

$$R_t = \left(\Delta_{\mathcal{M}}^\Gamma + \sum_{i=1}^{N-1}\kappa_i^2\right)R + \int_{\mathcal{M}} a^\Gamma R\,dy + b,$$

where a^Γ is a function defined by (cf. (3.39))

$$a^\Gamma := -\frac{1}{|\Gamma|}\left(\sum_{i=1}^{N-1}\kappa_i^2 + \kappa(\gamma_t \cdot v)\right)\sqrt{G}|_{r=0}$$

and b a nonhomogereous term. This is expressed as

$$R_t(t,y) = \mathbf{A}(t,y)R(t,y) + b(t,y), \qquad t > 0,\ y \in \mathcal{M},$$

and the generator of $\mathbf{A}$ is sectorial because the linear differential operator $\Delta_{\mathcal{M}}^\Gamma + \sum_{i=1}^{N-1}\kappa_i^2$, called the *Jacobi operator*, generates a sectorial operator while the linear nonlocal effect $\int_{\mathcal{M}} a^\Gamma R\,dy$ defines a bounded operator. Therefore, by the abstract theory for evolution equations (cf. e.g. [19]), it is ensured that there exists a unique smooth solution R to (3.55b) on $[0, T]$, provided that

Γ, b and initial data are all smooth. In this subsection, we will determine $(\Gamma, R^1, \dots; v^0, v^1, \dots)$ by employing (3.55) and construct approximate solutions.

We give a smooth initial interface Γ_0 and define Γ as a unique smooth solution to (1.2) on a time interval $[0, T]$. We also set

$$v^0 := v^*, \qquad t \in [0, T]. \tag{3.56}$$

Then, our choice determines the functions $U^{0,\pm}$ as

$$\begin{aligned} &U^{0,-}(t) \equiv h^-(v^*) \quad \text{on } \Omega^-(t), \qquad t \in [0, T], \\ &U^{0,+}(t) \equiv h^+(v^*) \quad \text{on } \Omega^+(t), \qquad t \in [0, T] \end{aligned} \tag{3.57}$$

by (3.5a), and $\tilde{u}^{0,\pm}$ as in (3.30) for $t \in [0, T]$ and $y \in \mathscr{M}$. For fixed $k \geq 2$, we can also determine the functions $U^{j,\pm}(t)$ $(t \in [0, T])$ and $\tilde{u}^{j,\pm}(t, z, y)$ $(t \in [0, T], \pm z \in [0, \infty), y \in \mathscr{M})$ for $j = 1, \dots, k$ via (3.5b) and (3.31), by successively solving (3.55b) for $R^1, \dots, R^{k-1}$ on $[0, T]$ together with given initial data

$$R_0^1(y), \dots, R_0^{k-1}(y), \qquad y \in \mathscr{M},$$

and by setting as, for $t \in [0, T]$,

$$\begin{aligned} &v^1 := \frac{1}{c'(v^*)|\Gamma|} \int_{\mathscr{M}} \kappa \, dS_y^\Gamma, \\ &v^j := -\frac{1}{c'(v^*)} \left[\frac{1}{|\Gamma|} \int_{\mathscr{M}} \left(\sum_{i=1}^{N-1} \kappa_i^2 + \kappa(\gamma_t \cdot \nu) \right) R^{j-1} \, dS_y^\Gamma + \sigma_{j-1} \right]. \end{aligned} \tag{3.58}$$

We notice that the C^1-matching conditions (3.32) $(j = 0, 1, \dots, k)$ and the nonlocal equalities (3.4) $(j = 0, 1, \dots, k-1)$ are all fulfilled, since Γ, $R^1, \dots,$ R^{k-1} solve (3.55) while $v^0, v^1, \dots, v^k$ are defined via (3.56) and (3.58) (cf. Lemma 3.3 and Lemma 3.4).

Let us now define

$$\begin{aligned} &R_A^\varepsilon(t, y) := \sum_{j=1}^{k-1} \varepsilon^{j-1} R^j(t, y), \\ &v_A^\varepsilon(t) := \sum_{j=0}^{k} \varepsilon^j v^j(t). \end{aligned} \tag{3.59}$$

We also set

$$U_A^{\varepsilon,\pm}(t) := \sum_{j=0}^{k} \varepsilon^j U^{j,\pm}(t), \qquad \tilde{u}_A^{\varepsilon,\pm}(t,z,y) := \sum_{j=0}^{k} \varepsilon^j \tilde{u}^{j,\pm}(t,z,y),$$

$$D_A^{\varepsilon} := \sum_{j=0}^{k} \varepsilon^j D^j, \qquad J_A^{\varepsilon}(t,z,y) := \sum_{j=0}^{k-1} \varepsilon^j J^j(t,z,y),$$

$$I_A^{\varepsilon}(t) := \sum_{j=0}^{k-1} \varepsilon^j I^j(t),$$

$$U_A^{\varepsilon}(t,x) := \begin{cases} U_A^{\varepsilon,-}(t), & t \in [0,T],\ x \in \Omega^-(t), \\ U_A^{\varepsilon,+}(t), & t \in [0,T],\ x \in \Omega^+(t), \end{cases}$$

$$\tilde{U}_A^{\varepsilon}(t,z,y) := \begin{cases} U_A^{\varepsilon,-}(t), & t \in [0,T],\ z \in (-\infty,0),\ y \in \mathcal{M}, \\ U_A^{\varepsilon,+}(t), & t \in [0,T],\ z \in (0,\infty),\ y \in \mathcal{M}, \end{cases}$$

$$\tilde{u}_A^{\varepsilon}(t,z,y) := \begin{cases} \tilde{u}_A^{\varepsilon,-}(t,z,y), & t \in [0,T],\ z \in (-\infty,0],\ y \in \mathcal{M}, \\ \tilde{u}_A^{\varepsilon,+}(t,z,y), & t \in [0,T],\ z \in [0,\infty),\ y \in \mathcal{M}. \end{cases}$$

Here, the coefficients D^j, J^j and I^j are determined via (3.22), (3.47) and (3.49), respectively. We note that I_A^{ε} is represented as

$$I_A^{\varepsilon} = \int_{\mathcal{M}} \int_{-\infty}^{\infty} [(\tilde{u}_A^{\varepsilon} - \tilde{U}_A^{\varepsilon})_{zz} + D_A^{\varepsilon}(\tilde{u}_A^{\varepsilon} - \tilde{U}_A^{\varepsilon})] J_A^{\varepsilon}\, dz dS_y^{\Gamma}. \tag{3.60}$$

Let $\Theta(r)$, $0 \le \Theta \le 1$, be a smooth cut-off function satisfying

$$\Theta(r) = \begin{cases} 1, & |r| \le \delta/2, \\ 0, & |r| \ge \delta, \end{cases}$$

and define our smooth approximate solution u_A^{ε} on $\bar{\Omega}_T := [0,T] \times \bar{\Omega}$ as follows:

$$u_A^{\varepsilon}(t,x) := U_A^{\varepsilon}(t,x) + \Theta(r(t,x))[\tilde{u}_A^{\varepsilon}(t,x) - U_A^{\varepsilon}(t,x)], \qquad (t,x) \in \bar{\Omega}_T. \tag{3.61}$$

where we employed the interchangeable expressions in terms of $(t,x) \leftrightarrow (t,r,y) \leftrightarrow (t,z,y)$ via (3.7) and $z = \varepsilon^{-1}[r - \varepsilon R_A^{\varepsilon}(t,y)]$. By our way of construction, we can easily find the following.

(i) Approximation of (3.1a) by outer solutions (cf. §3.1):

$$\max_{[0,T]} \left\| \varepsilon^2 \frac{\partial U_A^{\varepsilon}}{\partial t} - \varepsilon^2 \Delta U_A^{\varepsilon} - f(U_A^{\varepsilon}) + v_A^{\varepsilon} \right\|_{L^\infty(\Omega)} = O(\varepsilon^{k+1}), \tag{3.62}$$

(ii) Approximation of (3.1a) by inner solutions (cf. §3.2):

$$\max_{[0,T]} \left\| \varepsilon^2 \frac{\partial \tilde{u}_A^{\varepsilon}}{\partial t} - \varepsilon^2 \Delta \tilde{u}_A^{\varepsilon} - f(\tilde{u}_A^{\varepsilon}) + v_A^{\varepsilon} \right\|_{L^\infty(\Gamma^{\delta/2})} = O(\varepsilon^{k+1}), \tag{3.63}$$

(iii) Approximation of (3.1b) by outer and inner solutions (cf. §3.4):

(3.64) $$\max_{[0,T]} |I_A^\varepsilon - \varepsilon(\dot{U}_A^{\varepsilon,-}|\Omega^-| + \dot{U}_A^{\varepsilon,+}|\Omega^+|)| = O(\varepsilon^k),$$

I_A^ε being (3.60).

Using these results, we have the following

Lemma 3.5. *Let* u_A^ε, v_A^ε *be the functions defined by* (3.61) *and* (3.59), *respectively. Then*

(3.65a) $$\max_{[0,T]} \left\| \varepsilon^2 \frac{\partial u_A^\varepsilon}{\partial t} - \varepsilon^2 \Delta u_A^\varepsilon - f(u_A^\varepsilon) + v_A^\varepsilon \right\|_{L^\infty(\Omega)} = O(\varepsilon^{k+1}),$$

(3.65b) $$\max_{[0,T]} |v_A^\varepsilon - \langle f(u_A^\varepsilon)\rangle| = O(\varepsilon^{k+1}).$$

Proof. Let us first prove (3.65a). Since $u_A^\varepsilon(t,x) = U_A^\varepsilon(t,x)$ on $\Omega\backslash\Gamma(t)^\delta$, the estimate (3.62) immediately yields that

(3.66) $$\max_{[0,T]} \left\| \varepsilon^2 \frac{\partial u_A^\varepsilon}{\partial t} - \varepsilon^2 \Delta u_A^\varepsilon - f(u_A^\varepsilon) + v_A^\varepsilon \right\|_{L^\infty(\Omega\backslash\Gamma^\delta)} = O(\varepsilon^{k+1}).$$

In $\Gamma(t)^\delta\backslash\Gamma(t)^{\delta/2}$, putting $p_A^\varepsilon := \tilde{u}_A^\varepsilon - U_A^\varepsilon$, we can compute as

$$\begin{aligned}
&\varepsilon^2 \Delta u_A^\varepsilon + f(u_A^\varepsilon) - v_A^\varepsilon - \varepsilon^2 \frac{\partial u_A^\varepsilon}{\partial t} \\
&\quad = \varepsilon^2 \Delta U_A^\varepsilon + f(U_A^\varepsilon) - v_A^\varepsilon - \varepsilon^2 \frac{\partial U_A^\varepsilon}{\partial t} \\
&\qquad + \varepsilon^2 \Delta(p_A^\varepsilon \Theta(r)) + f(U_A^\varepsilon + p_A^\varepsilon \Theta(r)) - f(U_A^\varepsilon) - \varepsilon^2 \frac{\partial(p_A^\varepsilon \Theta(r))}{\partial t} \\
&\quad = \varepsilon^2 \Delta U_A^\varepsilon + f(U_A^\varepsilon) - v_A^\varepsilon - \varepsilon^2 \frac{\partial U_A^\varepsilon}{\partial t} \\
&\qquad + \varepsilon^2 \Theta(r) \Delta p_A^\varepsilon + 2\varepsilon^2 \Theta'(r) \nabla p_A^\varepsilon \cdot \nu + \varepsilon^2 p_A^\varepsilon (\Theta''(r) + \Theta'(r)\Delta r) \\
&\qquad + p_A^\varepsilon \Theta(r) \int_0^1 f'(U_A^\varepsilon + s p_A^\varepsilon \Theta(r)) ds + \varepsilon^2 \Theta'(r) p_A^\varepsilon \mathbf{v} - \varepsilon^2 \Theta(r) \frac{\partial p_A^\varepsilon}{\partial t},
\end{aligned}$$

where the following identities were employed:

$$r_t(t,x) = -\mathbf{v}(x;\Gamma(t)), \qquad \nabla r(t,x) = \nu(x;\Gamma(t)).$$

By the estimate (3.62) and the fact that p_A^ε and its derivatives with respect to t, x of any order are $O(e^{-\eta/\varepsilon})$ for some $\eta > 0$, we obtain

(3.67) $$\max_{[0,T]} \left\| \varepsilon^2 \frac{\partial u_A^\varepsilon}{\partial t} - \varepsilon^2 \Delta u_A^\varepsilon - f(u_A^\varepsilon) + v_A^\varepsilon \right\|_{L^\infty(\Gamma^\delta\backslash\Gamma^{\delta/2})} = O(\varepsilon^{k+1}).$$

Combining (3.63), (3.66) and (3.67) together, we obtain (3.65a).

Let us next prove (3.65b). The terms in the left-hand side of (3.65b) are recast as

$$\langle f(u_A^\varepsilon)\rangle - v_A^\varepsilon = \frac{1}{|\Omega|}\int_\Omega [f(u_A^\varepsilon) - v_A^\varepsilon]dx$$

$$= \frac{1}{|\Omega|}\int_\Omega [f(U_A^\varepsilon) - v_A^\varepsilon]dx + \frac{1}{|\Omega|}\int_\Omega [f(u_A^\varepsilon) - f(U_A^\varepsilon)]dx$$

$$= \frac{1}{|\Omega|}\int_\Omega [f(U_A^\varepsilon) - v_A^\varepsilon]dx + \frac{1}{|\Omega|}\int_{\Gamma^{\delta/2}} [f(u_A^\varepsilon) - f(U_A^\varepsilon)]dx \tag{3.68a}$$

$$+ \frac{1}{|\Omega|}\int_{\Gamma^\delta\backslash\Gamma^{\delta/2}} [f(u_A^\varepsilon) - f(U_A^\varepsilon)]dx \tag{3.68b}$$

$$+ \frac{1}{|\Omega|}\int_{\Omega\backslash\Gamma^\delta} [f(u_A^\varepsilon) - f(U_A^\varepsilon)]dx. \tag{3.68c}$$

Using (3.62), (3.64) and (3.65a), we treat (3.68a) as follows.

$$\begin{aligned}
(3.68\text{a}) &= \frac{1}{|\Omega|}\int_{\Omega^-} \varepsilon^2[(U_A^{\varepsilon,-})_t - \Delta U_A^{\varepsilon,-}]dx + \frac{1}{|\Omega|}\int_{\Omega^+} \varepsilon^2[(U_A^{\varepsilon,+})_t - \Delta U_A^{\varepsilon,+}]dx \\
&\quad + \frac{1}{|\Omega|}\int_{\Gamma^{\delta/2}} \varepsilon^2[(\tilde{u}_A^\varepsilon - U_A^\varepsilon)_t - \Delta(\tilde{u}_A^\varepsilon - U_A^\varepsilon)]dx + O(\varepsilon^{k+1}) \\
&= \frac{\varepsilon^2}{|\Omega|}(\dot{U}_A^{\varepsilon,-}|\Omega^-| + \dot{U}_A^{\varepsilon,+}|\Omega^+|) \\
&\quad - \frac{\varepsilon}{|\Omega|}\int_{\mathscr{M}}\int_{-\delta/2\varepsilon - R_A^\varepsilon}^{\delta/2\varepsilon - R_A^\varepsilon} [(\tilde{u}_A^\varepsilon - \tilde{U}_A^\varepsilon)_{zz} + D_A^\varepsilon(\tilde{u}_A^\varepsilon - \tilde{U}_A^\varepsilon)]J_A^\varepsilon\, dzdS_y^\Gamma + O(\varepsilon^{k+1}) \\
&= \frac{\varepsilon}{|\Omega|}[\varepsilon(\dot{U}_A^{\varepsilon,-}|\Omega^-| + \dot{U}_A^{\varepsilon,+}|\Omega^+|) - I_A^\varepsilon + O(e^{-\eta/\varepsilon})] + O(\varepsilon^{k+1}) \\
&= \varepsilon O(\varepsilon^k) + \varepsilon O(e^{-\eta/\varepsilon}) + O(\varepsilon^{k+1}) \\
&= O(\varepsilon^{k+1}).
\end{aligned}$$

On the other hand, (3.68b) is computed as

$$\begin{aligned}
(3.68\text{b}) &= \frac{1}{|\Omega|}\int_{\Gamma^\delta\backslash\Gamma^{\delta/2}} [f(U_A^\varepsilon + p_A^\varepsilon\Theta(r)) - f(U_A^\varepsilon)]dx \\
&= \frac{1}{|\Omega|}\int_{\Gamma^\delta\backslash\Gamma^{\delta/2}} p_A^\varepsilon\Theta(r)\left(\int_0^1 f'(U_A^\varepsilon + sp_A^\varepsilon\Theta(r))ds\right)dx \\
&= O(e^{-\eta/\varepsilon}),
\end{aligned}$$

and

$$(3.68c) \equiv 0$$

since $u_A^\varepsilon(t,x) = U_A^\varepsilon(t,x)$ on $\Omega \backslash \Gamma(t)^\delta$. Therefore, we obtain (3.65b). □

Our approximate solution u_A^ε in (3.61) is obviously smooth. From Lemma 3.5, we immediately obtain (2.1a). It also turns out that u_A^ε satisfies the boundary conditions (2.1b) since $u_A^\varepsilon(t,x) \equiv U_A^{\varepsilon,+}(t)$, spatially homogeneous, on $\partial\Omega$. Furthermore, we can verify that (2.1c) and (2.1d) are fulfilled because of (3.56), (3.57) and the fact that $\Gamma(t)$ is the solution to (1.2) for $[0,T]$. Therefore, our u_A^ε defined as in (3.61) is the desired approximate solution. This completes the proof of Proposition 2.1.

Remark. (i) The linear part in the equation (3.55b)

$$R_t = \left(\Delta_{\mathscr{M}}^{\Gamma} + \sum_{i=1}^{N-1} \kappa_i^2\right) R - \frac{1}{|\Gamma|}\int_{\Gamma} \left(\sum_{i=1}^{N-1} \kappa_i^2 + \kappa(\gamma_t \cdot \nu)\right) R \, dS_y^{\Gamma}$$

is characterized as the *linearization* of the averaged mean curvature flow (3.55a) in the direction of

$$\{\gamma(t,y) + R(t,y)\nu(t,y) \mid y \in \mathscr{M}\}.$$

It is also verified that the function R satisfies

$$\frac{d}{dt}\int_{\Gamma} R(t,y) dS_y^{\Gamma} \equiv 0, \qquad \frac{d}{dt}\bar{R} = \frac{d}{dt}\left(\frac{1}{|\Gamma|}\int_{\mathscr{M}} R(t,y) dS_y^{\Gamma}\right) \equiv 0.$$

(ii) The level-set interface $\Gamma^\varepsilon(t)$ (cf. (3.12), (3.13)) is approximated by

$$\begin{aligned}\Gamma_A^\varepsilon(t) &:= \{x \in \Omega \mid u_A^\varepsilon(t,x) = 0\} \\ &= \{x \in \Omega \mid x = \gamma(t,y) + \varepsilon R_A^\varepsilon(t,y)\nu(t,y),\ y \in \mathscr{M}\}.\end{aligned}$$

(iii) By virtue of the homogeneous Neumann boundary conditions and the existence of nonlocal term, the true solution u^ε to (1.1) preserves its spatial average:

$$\langle u^\varepsilon(t,\cdot)\rangle \equiv \langle u^\varepsilon(0,\cdot)\rangle, \qquad t \in [0,T],$$

while the approximate solution u_A^ε does *not*. However, it does *approximately* in the sense that

$$\langle u_A^\varepsilon(t,\cdot)\rangle = \langle u_A^\varepsilon(0,\cdot)\rangle + O(\varepsilon^{k-1}), \qquad t \in [0,T].$$

4. Proof of Proposition 2.3

In this section, we prove Proposition 2.3 by means of several lemmas. The first lemma shows that the operator $\mathscr{A}^{\varepsilon}(\tau)$ is sectorial for all $\tau \in [0, T/\varepsilon^s]$.

Lemma 4.1. *There exist some constants $\lambda_* > 0$, $\theta_* \in (0, \pi/2)$ and $M_* > 0$ such that*

$$\rho(\mathscr{A}^{\varepsilon}(\tau)) \supset S_* := \{\lambda \in \mathbf{C} \mid \lambda \neq \varepsilon^s \lambda_*, |\arg(\lambda - \varepsilon^s \lambda_*)| < \pi/2 + \theta_*\}$$

and the following resolvent estimate is valid for all $\tau \in [0, T/\varepsilon^s]$:

$$\|(\lambda - \mathscr{A}^{\varepsilon}(\tau))^{-1}\|_{0,0} \leq \frac{M_*}{|\lambda - \varepsilon^s \lambda_*|}, \qquad \lambda \in S_*. \tag{4.1}$$

Proof. We first treat the case where $p = 2$. It is easy to verify that $\mathscr{L}^{\varepsilon}(t)$ under the Neumann boundary condition is formally self-adjoint in $L^2(\Omega) \cap \mathbf{M}$, and therefore eigenvalues are real. We also obtain by the variational characterization for the principal eigenvalue λ^{ε} of $\mathscr{L}^{\varepsilon}(t)$ that

$$\begin{aligned}\lambda^{\varepsilon} &= \sup_{\substack{\varphi \in H^1(\Omega) \\ \varphi \neq 0, \langle \varphi \rangle = 0}} \frac{\int_{\Omega} -|\nabla \varphi|^2 + \varepsilon^{-2} f'(u_A^{\varepsilon})|\varphi|^2 dx}{\|\varphi\|_{L^2(\Omega)}^2} \\ &\leq \sup_{\substack{\varphi \in H^1(\Omega) \\ \varphi \neq 0}} \frac{\int_{\Omega} -|\nabla \varphi|^2 + \varepsilon^{-2} f'(u_A^{\varepsilon})|\varphi|^2 dx}{\|\varphi\|_{L^2(\Omega)}^2}.\end{aligned}$$

This says that λ^{ε} is estimated from above by the principal eigenvalue of the linearized Allen-Cahn operator $\Delta + \varepsilon^{-2} f'(u_A^{\varepsilon})$. On the other hand, according to the results established by Alikakos et al. [6] and Chen [8], the principal eigenvalue of $\Delta + \varepsilon^{-2} f'(u_A^{\varepsilon})$ is bounded above for $\varepsilon > 0$ and $t \in [0, T]$. Thus we have $\lambda^{\varepsilon} \leq \lambda_*$ for some $\lambda_* > 0$.

For $\lambda \in \mathbf{C}$ and a complex-valued function v with zero average, let us now consider the resolvent equation

$$\lambda u - \mathscr{L}^{\varepsilon}(t) u = v, \qquad \frac{\partial u}{\partial \mathbf{n}} = 0. \tag{4.2}$$

Multiplying the equation in (4.2) by the complex conjugate $\bar{u}$ of u and integrating over Ω, we have

$$\lambda \|u\|_{L^2(\Omega)}^2 = (\mathscr{L}^{\varepsilon}(t) u, u)_{L^2(\Omega)} + (v, u)_{L^2(\Omega)}, \tag{4.3}$$

where the symbol $(\cdot, \cdot)_{L^2(\Omega)}$ stands for the usual L^2-inner product. We decompose $\lambda \in \mathbf{C}$, $u : \Omega \to \mathbf{C}$ and $v : \Omega \to \mathbf{C}$ so that

$$\lambda = \lambda^R + i\lambda^I, \qquad u = u^R + iu^I, \qquad v = v^R + iv^I. \tag{4.4}$$

We note that the real-valued functions $u^R : \Omega \to \mathbf{R}$ and $u^I : \Omega \to \mathbf{R}$ also have zero average and satisfy the Neumann boundary conditions. Associated with the decomposition in (4.4), the real part of (4.3) is computed as

$$\begin{aligned}
\lambda^R \|u\|^2_{L^2(\Omega)} &= (\mathscr{L}^\varepsilon(t)u^R, u^R)_{L^2(\Omega)} + (\mathscr{L}^\varepsilon(t)u^I, u^I)_{L^2(\Omega)} \\
&\quad + (u^R, v^R)_{L^2(\Omega)} + (u^I, v^I)_{L^2(\Omega)} \\
&\le \lambda_*(\|u^R\|^2_{L^2(\Omega)} + \|u^I\|^2_{L^2(\Omega)}) \\
&\quad + \|u^R\|_{L^2(\Omega)}\|v^R\|_{L^2(\Omega)} + \|u^I\|_{L^2(\Omega)}\|v^I\|_{L^2(\Omega)} \\
&\le \lambda_*(\|u^R\|^2_{L^2(\Omega)} + \|u^I\|^2_{L^2(\Omega)}) \\
&\quad + (\|u^R\|^2_{L^2(\Omega)} + \|u^I\|^2_{L^2(\Omega)})^{1/2}(\|v^R\|^2_{L^2(\Omega)} + \|v^I\|^2_{L^2(\Omega)})^{1/2} \\
&\le \lambda_*\|u\|^2_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}
\end{aligned}$$

to obtain

$$(\lambda^R - \lambda_*)\|u\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)}. \tag{4.5}$$

On the other hand, the imaginary part of (4.3) becomes

$$\begin{aligned}
\lambda^I \|u\|^2_{L^2(\Omega)} &= -(\mathscr{L}^\varepsilon(t)u^R, u^I)_{L^2(\Omega)} + (\mathscr{L}^\varepsilon(t)u^I, u^R)_{L^2(\Omega)} \\
&\quad + (u^R, v^I)_{L^2(\Omega)} - (u^I, v^R)_{L^2(\Omega)} \\
&= (u^R, v^I)_{L^2(\Omega)} - (u^I, v^R)_{L^2(\Omega)},
\end{aligned}$$

where integration by parts and the Neumann boundary conditions are used. Thus we have

$$|\lambda^I|\,\|u\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)}. \tag{4.6}$$

From (4.5) and (4.6), we obtain the estimate

$$[(\lambda^R - \lambda_*) + |\lambda^I|]\|u\|_{L^2(\Omega)} \le 2\|v\|_{L^2(\Omega)},$$

which implies that

$$\|u\|_{L^2(\Omega)} \le \frac{M_*}{|\lambda - \lambda_*|}\|v\|_{L^2(\Omega)} \tag{4.7}$$

is valid for $\lambda \in \{\lambda \in \mathbf{C} \mid \lambda \ne \lambda_*, |\arg(\lambda - \lambda_*)| < \pi/2 + \theta_*\} \subset \rho(\mathscr{L}^\varepsilon(t))$ with $\theta_* \in (0, \pi/4)$ and $M_* := \sqrt{2}/\cos(\theta_* + \pi/4)$.

Once this is established, we find, along the line of arguments in Tanabe [34], that the following L^p-version $(p > 2)$ of (4.7)

$$\|u\|_{L^p(\Omega)} \le \frac{M_*}{|\lambda - \lambda_*|}\|v\|_{L^p(\Omega)} \tag{4.8}$$

holds for all $\lambda \in \{\lambda \in \mathbf{C} \mid \lambda \neq \lambda_*, |\arg(\lambda - \lambda_*)| < \pi/2 + \theta_*\} \subset \rho(\mathscr{L}^\varepsilon(t))$ with the same $\lambda_* > 0$ in (4.7), replacing θ_* and M_* by other constants. The estimate (4.1) then follows from (4.8) and the time rescale in (2.3), which completes the proof of Lemma 4.1. □

On the other hand, one can easily find that the operator $\mathscr{A}^\varepsilon(\tau) - \mathscr{A}^\varepsilon(\sigma)$ consists of a multiplication operator and an integral operator. In particular, it does not involve any differential operator. Thanks to this fact, the operator norm of $\mathscr{A}^\varepsilon(\tau) - \mathscr{A}^\varepsilon(\sigma)$ has the following characterization for $s \ge 4$.

LEMMA 4.2. *Let* $\alpha_0 \in [0, 1/2)$. *Then, there exists a constant* $M_0 = M(\alpha_0) > 0$ *such that the following estimate holds for sufficiently small* $\varepsilon > 0$, $s \ge 4$ *and* $0 \le \sigma \le \tau \le T/\varepsilon^s$*:*

$$\|\mathscr{A}^\varepsilon(\tau) - \mathscr{A}^\varepsilon(\sigma)\|_{1,\alpha_0} \le M_0 \varepsilon^s(\tau - \sigma). \tag{4.9}$$

PROOF. In the proof, we simply write α and M instead of α_0 and M_0, respectively.

Let $\varphi \in X_1^\varepsilon$ and define the linear operator $\mathscr{E}^\varepsilon_{\tau,\sigma}$ by

$$\mathscr{E}^\varepsilon_{\tau,\sigma}\varphi := (\mathscr{A}^\varepsilon(\tau) - \mathscr{A}^\varepsilon(\sigma))\varphi. \tag{4.10}$$

Then, by (2.6a), an elementary calculation gives

$$\mathscr{E}^\varepsilon_{\tau,\sigma}\varphi(x) = \varepsilon^{s-2}[F^\varepsilon_{\tau,\sigma}(x)G^\varepsilon_{\tau,\sigma}(x)\varphi(x) - \langle F^\varepsilon_{\tau,\sigma}G^\varepsilon_{\tau,\sigma}\varphi\rangle]\varepsilon^s(\tau - \sigma), \tag{4.11}$$

where $F^\varepsilon_{\tau,\sigma}$ and $G^\varepsilon_{\tau,\sigma}$ are

$$\begin{aligned} F^\varepsilon_{\tau,\sigma}(x) &:= \int_0^1 f''(u_A^\varepsilon(\varepsilon^s\sigma, x) + \theta(u_A^\varepsilon(\varepsilon^s\tau, x) - u_A^\varepsilon(\varepsilon^s\sigma, x)))d\theta, \\ G^\varepsilon_{\tau,\sigma}(x) &:= \int_0^1 \frac{\partial u_A^\varepsilon}{\partial t}(\varepsilon^s(\sigma + \theta(\tau - \sigma)), x)d\theta. \end{aligned} \tag{4.12}$$

We divide the proof into two cases; (i) $\alpha = 0$, and (ii) $\alpha \in (0, 1/2)$.

Case (i): $\alpha = 0$. We notice that $F^\varepsilon_{\tau,\sigma}$ and $G^\varepsilon_{\tau,\sigma}$ $(0 \le \sigma \le \tau \le T/\varepsilon^s)$ in (4.12) satisfy the following estimates:

$$\begin{aligned} \|F^\varepsilon_{\tau,\sigma}\|_{L^\infty(\Omega)} &= O(1), \\ \|G^\varepsilon_{\tau,\sigma}\|_{L^\infty(\Omega)} &= O(\varepsilon^{-2}). \end{aligned} \tag{4.13}$$

Thus, by (4.11), (4.13) and the embedding $X_1^\varepsilon \hookrightarrow X_0^\varepsilon \subset L^p(\Omega)$, we obtain

$$\begin{aligned}\|\mathscr{E}_{\tau,\sigma}^\varepsilon\varphi\|_0 &\le 2\varepsilon^{s-2}\|F_{\tau,\sigma}^\varepsilon\|_{L^\infty(\Omega)}\|G_{\tau,\sigma}^\varepsilon\|_{L^\infty(\Omega)}\|\varphi\|_{L^p(\Omega)}\varepsilon^s(\tau-\sigma)\\ &\le M\varepsilon^{s-4}\cdot\varepsilon^s(\tau-\sigma)\|\varphi\|_1.\end{aligned}$$

For $s \ge 4$, we have

$$\|\mathscr{E}_{\tau,\sigma}^\varepsilon\varphi\|_0 \le M\varepsilon^s(\tau-\sigma)\|\varphi\|_1 \tag{4.14}$$

This together with (4.10) implies

$$\|\mathscr{A}^\varepsilon(\tau)-\mathscr{A}^\varepsilon(\sigma)\|_{1,0} \le M\varepsilon^s(\tau-\sigma),$$

which establishes (4.9) with $\alpha = 0$.

Case (ii): $\alpha \in (0,1/2)$. We note, by virtue of the relation between Besov and Sobolev-Slobodeckii spaces [1, 35], that the interpolation spaces X_α^ε in this situation are characterized as

$$X_\alpha^\varepsilon = W_\varepsilon^{2\alpha,p}(\Omega)\cap\mathbf{M}.$$

Here, $W_\varepsilon^{2\alpha,p}(\Omega) = W^{2\alpha,p}(\Omega)$ as a set, equipped with the weighted norm

$$\|u\|_{W_\varepsilon^{2\alpha,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \varepsilon^{s\alpha}[u]_{W^{2\alpha,p}(\Omega)}, \tag{4.15}$$

in which $[u]_{W^{2\alpha,p}(\Omega)}$ is the seminorm defined by

$$[u]_{W^{2\alpha,p}(\Omega)} := \left(\int\int_{\Omega\times\Omega}\frac{|u(x)-u(x')|^p}{|x-x'|^{N+2\alpha p}}\,dxdx'\right)^{1/p}. \tag{4.16}$$

Let $\varphi \in X_1^\varepsilon$, and we will estimate $\mathscr{E}_{\tau,\sigma}^\varepsilon\varphi$ by the norm $\|\cdot\|_{W_\varepsilon^{2\alpha,p}(\Omega)}$. In (4.15), the estimate

$$\|\mathscr{E}_{\tau,\sigma}^\varepsilon\varphi\|_{L^p(\Omega)} \le M\varepsilon^s(\tau-\sigma)\|\varphi\|_1 \tag{4.17}$$

immediately follows from (4.14). Hence, it suffices to examine the seminorm part.

Let us compute $\varepsilon^{s\alpha}[\mathscr{E}_{\tau,\sigma}^\varepsilon\varphi]_{W^{2\alpha,p}(\Omega)}$. From (4.11), we can calculate as

$$\begin{aligned}\mathscr{E}_{\tau,\sigma}^\varepsilon\varphi(x)-\mathscr{E}_{\tau,\sigma}^\varepsilon\varphi(x') = \varepsilon^{s-2}[&(F_{\tau,\sigma}^\varepsilon(x)-F_{\tau,\sigma}^\varepsilon(x'))G_{\tau,\sigma}^\varepsilon(x)\varphi(x)\\ &+F_{\tau,\sigma}^\varepsilon(x')(G_{\tau,\sigma}^\varepsilon(x)-G_{\tau,\sigma}^\varepsilon(x'))\varphi(x)\\ &+F_{\tau,\sigma}^\varepsilon(x')G_{\tau,\sigma}^\varepsilon(x')(\varphi(x)-\varphi(x'))]\varepsilon^s(\tau-\sigma).\end{aligned}$$

By (4.16), it is easily verified that

$$\varepsilon^{s\alpha}[\mathscr{E}_{\tau,\sigma}^\varepsilon\varphi]_{W^{2\alpha,p}(\Omega)} \le M\big(I_1^\varepsilon+I_2^\varepsilon+I_3^\varepsilon\big)\varepsilon^s(\tau-\sigma), \tag{4.18}$$

where

$$I_1^\varepsilon := \varepsilon^{s-2}\varepsilon^{s\alpha}\left(\iint_{\Omega\times\Omega} \frac{|F_{\tau,\sigma}^\varepsilon(x) - F_{\tau,\sigma}^\varepsilon(x')|^p |G_{\tau,\sigma}^\varepsilon(x)|^p |\varphi(x)|^p}{|x-x'|^{N+2\alpha p}}\,dxdx'\right)^{1/p},$$

$$I_2^\varepsilon := \varepsilon^{s-2}\varepsilon^{s\alpha}\left(\iint_{\Omega\times\Omega} \frac{|F_{\tau,\sigma}^\varepsilon(x')|^p |G_{\tau,\sigma}^\varepsilon(x) - G_{\tau,\sigma}^\varepsilon(x')|^p |\varphi(x)|^p}{|x-x'|^{N+2\alpha p}}\,dxdx'\right)^{1/p},$$

$$I_3^\varepsilon := \varepsilon^{s-2}\varepsilon^{s\alpha}\left(\iint_{\Omega\times\Omega} \frac{|F_{\tau,\sigma}^\varepsilon(x')|^p |G_{\tau,\sigma}^\varepsilon(x')|^p |\varphi(x) - \varphi(x')|^p}{|x-x'|^{N+2\alpha p}}\,dxdx'\right)^{1/p}.$$

We first examine I_1^ε. Let $D^\delta := (\Omega\times\Omega)\backslash(\Gamma^{\delta/2}\times\Gamma^{\delta/2})$, namely,

$$D^\delta = [(\Omega\backslash\Gamma^{\delta/2})\times(\Omega\backslash\Gamma^{\delta/2})] \cup [(\Omega\backslash\Gamma^{\delta/2})\times\Gamma^{\delta/2}] \cup [\Gamma^{\delta/2}\times(\Omega\backslash\Gamma^{\delta/2})].$$

We also define $S^\delta \subset D^\delta$ by

$$S^\delta := \{(x,x') \in D^\delta; |x-x'| < \delta/4\}$$

and introduce

$$[u]_{\mathrm{Lip}(S^\delta)} := \sup_{\substack{x,x'\in S^\delta\\ x\neq x'}} \frac{|u(x)-u(x')|}{|x-x'|}.$$

Note that $F_{\tau,\sigma}^\varepsilon$ for $0 \le \sigma \le \tau \le T/\varepsilon^s$ enjoys the following properties

$$[F_{\tau,\sigma}^\varepsilon]_{\mathrm{Lip}(S^\delta)} = O(1), \tag{4.19a}$$

$$[F_{\tau,\sigma}^\varepsilon]_{C^\beta(\bar\Gamma^{\delta/2})} = O(\varepsilon^{-\beta}) \tag{4.19b}$$

for $\beta \in (0,1)$.

We now fix β so that $2\alpha < \beta < \min\{4\alpha, 1\}$. Using (4.13), (4.19) together with spherical coodinates, we can compute I_1^ε as

$$\begin{aligned} I_1^\varepsilon \le{}& M\varepsilon^{s-2}\varepsilon^{s\alpha}\left(\iint_{S^\delta} \frac{|F_{\tau,\sigma}^\varepsilon(x) - F_{\tau,\sigma}^\varepsilon(x')|^p |G_{\tau,\sigma}^\varepsilon(x)|^p |\varphi(x)|^p}{|x-x'|^{N+2\alpha p}}\,dxdx'\right)^{1/p} \\ &+ M\varepsilon^{s-2}\varepsilon^{s\alpha}\left(\iint_{D^\delta\backslash S^\delta} \frac{|F_{\tau,\sigma}^\varepsilon(x) - F_{\tau,\sigma}^\varepsilon(x')|^p |G_{\tau,\sigma}^\varepsilon(x)|^p |\varphi(x)|^p}{|x-x'|^{N+2\alpha p}}\,dxdx'\right)^{1/p} \\ &+ M\varepsilon^{s-2}\varepsilon^{s\alpha}\left(\iint_{\Gamma^{\delta/2}\times\Gamma^{\delta/2}} \frac{|F_{\tau,\sigma}^\varepsilon(x) - F_{\tau,\sigma}^\varepsilon(x')|^p |G_{\tau,\sigma}^\varepsilon(x)|^p |\varphi(x)|^p}{|x-x'|^{N+2\alpha p}}\,dxdx'\right)^{1/p} \end{aligned}$$

$$\leq M\varepsilon^{s\alpha+s-2}[F^{\varepsilon}_{\tau,\sigma}]_{\mathrm{Lip}(S^{\delta})}\|G^{\varepsilon}_{\tau,\sigma}\|_{L^{\infty}(\Omega)}\left(\int\int_{S^{\delta}}\frac{|\varphi(x)|^{p}dxdx'}{|x-x'|^{N+(2\alpha-1)p}}\right)^{1/p}$$

$$+ M\varepsilon^{s\alpha+s-2}\|F^{\varepsilon}_{\tau,\sigma}\|_{L^{\infty}(\Omega)}\|G^{\varepsilon}_{\tau,\sigma}\|_{L^{\infty}(\Omega)}\left(\int\int_{D^{\delta}\setminus S^{\delta}}\frac{|\varphi(x)|^{p}dxdx'}{|x-x'|^{N+2\alpha p}}\right)^{1/p}$$

$$+ M\varepsilon^{s\alpha+s-2}[F^{\varepsilon}_{\tau,\sigma}]_{C^{\beta}(\bar{\Gamma}^{\delta/2})}\|G^{\varepsilon}_{\tau,\sigma}\|_{L^{\infty}(\Omega)}\left(\int\int_{\Gamma^{\delta/2}\times\Gamma^{\delta/2}}\frac{|\varphi(x)|^{p}dxdx'}{|x-x'|^{N+(2\alpha-\beta)p}}\right)^{1/p}$$

$$\leq M\varepsilon^{s\alpha+(s-4)}\left(\int_{\Omega}|\varphi(x)|^{p}\left(\int_{\Omega}\frac{dx'}{|x-x'|^{N+(2\alpha-1)p}}\right)dx\right)^{1/p}$$

$$+ M\varepsilon^{s\alpha+(s-4)}\left(\int_{\Omega}|\varphi(x)|^{p}\left(\int_{\Omega\cap\{|x-x'|\geq\delta/4\}}\frac{dx'}{|x-x'|^{N+2\alpha p}}\right)dx\right)^{1/p}$$

$$+ M\varepsilon^{s\alpha-\beta+(s-4)}\left(\int_{\Omega}|\varphi(x)|^{p}\left(\int_{\Omega}\frac{dx'}{|x-x'|^{N+(2\alpha-\beta)p}}\right)dx\right)^{1/p}$$

$$\leq M\varepsilon^{s\alpha-\beta+(s-4)}\|\varphi\|_{L^{p}(\Omega)},$$

and by the embedding $X_1^{\varepsilon} \hookrightarrow X_0^{\varepsilon} \subset L^p(\Omega)$, we have

$$I_1^{\varepsilon} \leq M\varepsilon^{s\alpha-\beta+(s-4)}\|\varphi\|_1. \tag{4.20}$$

As for I_2^{ε}, we notice that the following properties for $G^{\varepsilon}_{\tau,\sigma}$ are valid:

$$[G^{\varepsilon}_{\tau,\sigma}]_{\mathrm{Lip}(S^{\delta})} = O(\varepsilon^{-2}), \tag{4.21a}$$

$$[G^{\varepsilon}_{\tau,\sigma}]_{C^{\beta}(\bar{\Gamma}^{\delta/2})} = O(\varepsilon^{-2-\beta}). \tag{4.21b}$$

Then the same computation as that for I_1^{ε} implies

$$I_2^{\varepsilon} \leq M\varepsilon^{s-2}\varepsilon^{s\alpha}\left(\int\int_{S^{\delta}}\frac{|F^{\varepsilon}_{\tau,\sigma}(x')|^{p}|G^{\varepsilon}_{\tau,\sigma}(x)-G^{\varepsilon}_{\tau,\sigma}(x')|^{p}|\varphi(x)|^{p}}{|x-x'|^{N+2\alpha p}}\,dxdx'\right)^{1/p}$$

$$+ M\varepsilon^{s-2}\varepsilon^{s\alpha}\left(\int\int_{D^{\delta}\setminus S^{\delta}}\frac{|F^{\varepsilon}_{\tau,\sigma}(x')|^{p}|G^{\varepsilon}_{\tau,\sigma}(x)-G^{\varepsilon}_{\tau,\sigma}(x')|^{p}|\varphi(x)|^{p}}{|x-x'|^{N+2\alpha p}}\,dxdx'\right)^{1/p}$$

$$+ M\varepsilon^{s-2}\varepsilon^{s\alpha}\left(\int\int_{\Gamma^{\delta/2}\times\Gamma^{\delta/2}}\frac{|F^{\varepsilon}_{\tau,\sigma}(x')|^{p}|G^{\varepsilon}_{\tau,\sigma}(x)-G^{\varepsilon}_{\tau,\sigma}(x')|^{p}|\varphi(x)|^{p}}{|x-x'|^{N+2\alpha p}}\,dxdx'\right)^{1/p}$$

$$\leq M\varepsilon^{s\alpha+s-2}\|F^{\varepsilon}_{\tau,\sigma}\|_{L^\infty(\Omega)}[G^{\varepsilon}_{\tau,\sigma}]_{\mathrm{Lip}(S^\delta)}\left(\iint_{S^\delta}\frac{|\varphi(x)|^p dxdx'}{|x-x'|^{N+(2\alpha-1)p}}\right)^{1/p}$$

$$+ M\varepsilon^{s\alpha+s-2}\|F^{\varepsilon}_{\tau,\sigma}\|_{L^\infty(\Omega)}\|G^{\varepsilon}_{\tau,\sigma}\|_{L^\infty(\Omega)}\left(\iint_{D^\delta\setminus S^\delta}\frac{|\varphi(x)|^p dxdx'}{|x-x'|^{N+2\alpha p}}\right)^{1/p}$$

$$+ M\varepsilon^{s\alpha+s-2}\|F^{\varepsilon}_{\tau,\sigma}\|_{L^\infty(\Omega)}[G^{\varepsilon}_{\tau,\sigma}]_{C^\beta(\bar{\Gamma}^{\delta/2})}\left(\iint_{\Gamma^{\delta/2}\times\Gamma^{\delta/2}}\frac{|\varphi(x)|^p dxdx'}{|x-x'|^{N+(2\alpha-\beta)p}}\right)^{1/p}$$

$$\leq M\varepsilon^{s\alpha+(s-4)}\left(\int_\Omega|\varphi(x)|^p\left(\int_\Omega\frac{dx'}{|x-x'|^{N+(2\alpha-1)p}}\right)dx\right)^{1/p}$$

$$+ M\varepsilon^{s\alpha+(s-4)}\left(\int_\Omega|\varphi(x)|^p\left(\int_{\Omega\cap\{|x-x'|\geq\delta/4\}}\frac{dx'}{|x-x'|^{N+2\alpha p}}\right)dx\right)^{1/p}$$

$$+ M\varepsilon^{s\alpha-\beta+(s-4)}\left(\int_\Omega|\varphi(x)|^p\left(\int_\Omega\frac{dx'}{|x-x'|^{N+(2\alpha-\beta)p}}\right)dx\right)^{1/p}$$

$$\leq M\varepsilon^{s\alpha-\beta+(s-4)}\|\varphi\|_{L^p(\Omega)},$$

and thus we obtain

$$I^{\varepsilon}_2 \leq M\varepsilon^{s\alpha-\beta+(s-4)}\|\varphi\|_1. \tag{4.22}$$

In (4.20) and (4.22), we find that $s\alpha-\beta+(s-4)\geq 4\alpha-\beta>0$ by virtue of $s\geq 4$ and our way of choice of β, and therefore we have

$$I^{\varepsilon}_i \leq M\|\varphi\|_1 \qquad i=1,2. \tag{4.23}$$

For I^{ε}_3, we can estimate, by (4.13), as

$$\begin{aligned} I^{\varepsilon}_3 &\leq \varepsilon^{s-2}\|F^{\varepsilon}_{\tau,\sigma}\|_{L^\infty(\Omega)}\|G^{\varepsilon}_{\tau,\sigma}\|_{L^\infty(\Omega)}\cdot\varepsilon^{s\alpha}[\varphi]_{W^{2\alpha,p}_{\varepsilon}(\Omega)} \\ &\leq M\varepsilon^{s-4}\cdot\varepsilon^{s\alpha}[\varphi]_{W^{2\alpha,p}_{\varepsilon}(\Omega)} \\ &\leq M\varepsilon^{s-4}\|\varphi\|_{W^{2\alpha,p}_{\varepsilon}(\Omega)}, \end{aligned}$$

which, together with $s\geq 4$ and the embedding $X^{\varepsilon}_1 \hookrightarrow X^{\varepsilon}_{\alpha}\subset W^{2\alpha,p}_{\varepsilon}(\Omega)$, implies

$$I^{\varepsilon}_3 \leq M\|\varphi\|_1. \tag{4.24}$$

By substituting (4.23) and (4.24) into (4.18), and combining with (4.17), we have

$$\|\mathscr{E}^{\varepsilon}_{\tau,\sigma}\varphi\|_{\alpha}\leq M\varepsilon^s(\tau-\sigma)\|\varphi\|_1,$$

that is,

$$\|\mathscr{A}^\varepsilon(\tau) - \mathscr{A}^\varepsilon(\sigma)\|_{1,\alpha} \le M\varepsilon^s(\tau - \sigma).$$

This completes the proof of Lemma 4.2. □

By using Lemma 4.1 and Lemma 4.2, let us now prove Proposition 2.3. Let $\alpha_0 \in [0, 1/2)$. By Lemma 4.2, it follows that

$$\|\mathscr{A}^\varepsilon(\tau) - \mathscr{A}^\varepsilon(\sigma)\|_{1,\alpha_0} \le M_0\,\varepsilon^s(\tau - \sigma). \tag{4.25}$$

Moreover, from Lemma 4.1 above and Proposition 2.3.1 in Lunardi [19], we find that for $0 \le \alpha \le \beta \le 1$, there exists a constant $M = M(\alpha, \beta) > 0$ such that the estimate

$$\|e^{(\tau-\sigma)\mathscr{A}^\varepsilon(\sigma)}\|_{\alpha,\beta} \le M(\tau - \sigma)^{\alpha-\beta} e^{\varepsilon^s\lambda_*(\tau-\sigma)} \tag{4.26}$$

is valid. We emphasize that the constant $M > 0$ can be chosen *independent* of $\varepsilon > 0$ thanks to the weighted norm (2.12). We now define the operator $k_1^\varepsilon(\tau, \sigma)$ by

$$k_1^\varepsilon(\tau, \sigma) := (\mathscr{A}^\varepsilon(\tau) - \mathscr{A}^\varepsilon(\sigma))e^{(\tau-\sigma)\mathscr{A}^\varepsilon(\sigma)}.$$

Then we can estimate $k_1^\varepsilon(\tau, \sigma)$ by employing (4.25) and (4.26) as

$$\begin{aligned}\|k_1^\varepsilon(\tau, \sigma)\|_{0,\alpha_0} &\le \|\mathscr{A}^\varepsilon(\tau) - \mathscr{A}^\varepsilon(\sigma)\|_{1,\alpha_0}\|e^{(\tau-\sigma)\mathscr{A}^\varepsilon(\sigma)}\|_{0,1}\\ &\le M_0 M\varepsilon^s\, e^{\varepsilon^s\lambda_*(\tau-\sigma)}.\end{aligned}$$

For this $k_1^\varepsilon(\tau, \sigma)$, it is known [9] that the evolution operator $\Phi^\varepsilon(\tau, \sigma)$ is the unique solution of the integral equation

$$\Phi^\varepsilon(\tau, \sigma) = e^{(\tau-\sigma)\mathscr{A}^\varepsilon(\sigma)} + \int_\sigma^\tau \Phi^\varepsilon(\tau, \sigma')k_1^\varepsilon(\sigma', \sigma)d\sigma',$$

and that the solution $\Phi^\varepsilon(\tau, \sigma)$ has the unique representation

$$\Phi^\varepsilon(\tau, \sigma) = e^{(\tau-\sigma)\mathscr{A}^\varepsilon(\sigma)} + \int_\sigma^\tau e^{(\tau-\sigma')\mathscr{A}^\varepsilon(\sigma')}k^\varepsilon(\sigma', \sigma)d\sigma' \tag{4.27}$$

with resolvent kernel $k^\varepsilon(\tau, \sigma)$. This kernel can be successively constructed starting from $k_1^\varepsilon(\tau, \sigma)$. We inductively define $k_m^\varepsilon(\tau, \sigma)$ $(m \ge 2)$ by

$$k_m^\varepsilon(\tau, \sigma) := \int_\sigma^\tau k_{m-1}^\varepsilon(\tau, \sigma')k_1^\varepsilon(\sigma', \sigma)d\sigma'.$$

By the repeated application of the following estimates

$$\|k_m^\varepsilon(\tau, \sigma)\|_{0,\alpha_0} \le \int_\sigma^\tau \|k_{m-1}^\varepsilon(\tau, \sigma')\|_{0,\alpha_0}\|k_1^\varepsilon(\sigma', \sigma)\|_{0,0}d\sigma',$$

we find, by induction, that

$$\|k_m^{\varepsilon}(\tau,\sigma)\|_{0,\alpha_0} \le \frac{(M_0 M \varepsilon^s)^m}{(m-1)!}(\tau-\sigma)^{m-1} e^{\varepsilon^s \lambda_*(\tau-\sigma)}, \qquad m \ge 1.$$

This immediately implies that the series

$$k^{\varepsilon}(\tau,\sigma) := \sum_{m=1}^{\infty} k_m^{\varepsilon}(\tau,\sigma) \tag{4.28}$$

converges and that it can be estimated as

$$\begin{aligned}
\|k^{\varepsilon}(\tau,\sigma)\|_{0,\alpha_0} &\le M_0 M \varepsilon^s e^{\varepsilon^s \lambda_*(\tau-\sigma)} \left[\sum_{m=1}^{\infty} \frac{(M_0 M \varepsilon^s(\tau-\sigma))^{m-1}}{(m-1)!}\right] \\
&= M_0 M \varepsilon^s e^{\varepsilon^s(\lambda_* + M_0 M)(\tau-\sigma)}.
\end{aligned}$$

Therefore, there exist some constants $M = M(\alpha,\beta;\alpha_0) > 0$, $K = K(\alpha,\beta;\alpha_0) > 0$ such that the resolvent kernel k^{ε} defined in (4.28) satisfies the estimate

$$\|k^{\varepsilon}(\tau,\sigma)\|_{0,\alpha_0} \le M \varepsilon^s e^{\varepsilon^s(\lambda_*+K)(\tau-\sigma)}. \tag{4.29}$$

Let us now examine the norm $\|\Phi^{\varepsilon}(\tau,\sigma)\|_{\alpha,\beta}$ by using the estimates (4.26) and (4.29) in (4.27). Suppose that $0 \le \alpha \le \beta < 1$. Then, by using (4.29) with $\alpha_0 = 0$, we have

$$\begin{aligned}
\|\Phi^{\varepsilon}(\tau,\sigma)\|_{\alpha,\beta} &\le \|e^{(\tau-\sigma)\mathscr{A}^{\varepsilon}(\sigma)}\|_{\alpha,\beta} + \int_{\sigma}^{\tau} \|e^{(\tau-\sigma')\mathscr{A}^{\varepsilon}(\sigma')}\|_{0,\beta} \|k^{\varepsilon}(\sigma',\sigma)\|_{\alpha,0} d\sigma' \\
&\le M(\tau-\sigma)^{\alpha-\beta} e^{\varepsilon^s(\lambda_*+K)(\tau-\sigma)} \\
&\quad + M \int_{\sigma}^{\tau} (\tau-\sigma')^{-\beta} e^{\varepsilon^s(\lambda_*+K)(\tau-\sigma')} \|k^{\varepsilon}(\sigma',\sigma)\|_{0,0} d\sigma' \\
&\le M(\tau-\sigma)^{\alpha-\beta} e^{\varepsilon^s(\lambda_*+K)(\tau-\sigma)} \\
&\quad + M \varepsilon^s e^{\varepsilon^s(\lambda_*+K)(\tau-\sigma)} \frac{(\tau-\sigma)^{1-\beta}}{1-\beta} \\
&\le M(\tau-\sigma)^{\alpha-\beta} e^{\varepsilon^s(\lambda_*+K)(\tau-\sigma)} \\
&\quad + \frac{M T^{1-\alpha}}{1-\beta} \varepsilon^{s\alpha} (\tau-\sigma)^{\alpha-\beta} e^{\varepsilon^s(\lambda_*+K)(\tau-\sigma)} \\
&\le M(\tau-\sigma)^{\alpha-\beta} e^{\varepsilon^s(\lambda_*+K)(\tau-\sigma)}.
\end{aligned}$$

In the case where $0 < \alpha \le \beta = 1$, we choose $\alpha_0 > 0$ so small that $\alpha > \alpha_0$. Then we have

$$\begin{aligned}
\|\Phi^{\varepsilon}(\tau,\sigma)\|_{\alpha,1} &\le \|e^{(\tau-\sigma)\mathscr{A}^{\varepsilon}(\sigma)}\|_{\alpha,1} + \int_{\sigma}^{\tau} \|e^{(\tau-\sigma')\mathscr{A}^{\varepsilon}(\sigma')}\|_{\alpha_0,1}\|k^{\varepsilon}(\sigma',\sigma)\|_{\alpha,\alpha_0}d\sigma' \\
&\le M(\tau-\sigma)^{\alpha-1}e^{\varepsilon^s(\lambda_*+K)(\tau-\sigma)} \\
&\quad + M\int_{\sigma}^{\tau}(\tau-\sigma')^{\alpha_0-1}e^{\varepsilon^s(\lambda_*+K)(\tau-\sigma')}\|k^{\varepsilon}(\sigma',\sigma)\|_{0,\alpha_0}d\sigma' \\
&\le M(\tau-\sigma)^{\alpha-1}e^{\varepsilon^s(\lambda_*+K)(\tau-\sigma)} \\
&\quad + M\varepsilon^s e^{\varepsilon^s(\lambda_*+K)(\tau-\sigma)}\frac{(\tau-\sigma)^{\alpha_0}}{\alpha_0} \\
&\le M(\tau-\sigma)^{\alpha-1}e^{\varepsilon^s(\lambda_*+K)(\tau-\sigma)} \\
&\quad + \frac{MT^{(1-\alpha)+\alpha_0}}{\alpha_0}\varepsilon^{s(\alpha-\alpha_0)}(\tau-\sigma)^{\alpha-1}e^{\varepsilon^s(\lambda_*+K)(\tau-\sigma)} \\
&\le M(\tau-\sigma)^{\alpha-1}e^{\varepsilon^s(\lambda_*+K)(\tau-\sigma)}.
\end{aligned}$$

Thus (2.17) is obtained, which completes the proof of Proposition 2.3.

REMARK. (i) *Only* the estimates $\|\Phi^{\varepsilon}(\tau,\sigma)\|_{0,\alpha}$ and $\|\Phi^{\varepsilon}(\tau,\sigma)\|_{\alpha,\alpha}$ in Proposition 2.3 are employed in the proof of Theorem 1.1.

(ii) The estimate for the case where $(\alpha,\beta)=(0,1)$ is not given. However, even if $(\alpha,\beta)=(0,1)$, the norm $\|\Phi^{\varepsilon}(\tau,\sigma)\|_{0,1}$ can be estimated, by employing (4.26) and (4.29) with $\alpha_0>0$, so that

$$\|\Phi^{\varepsilon}(\tau,\sigma)\|_{0,1} \le Me^{\varepsilon^s(\lambda_*+K)(\tau-\sigma)}[1+(\tau-\sigma)^{-1}].$$

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