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Canonical filtrations and stability of direct images by Frobenius morphisms II

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Abstract. We study the stability of direct images by Frobenius morphisms. We prove that if the cotangent vector bundle of a nonsingular projective surface X is semistable with respect to a numerically positive polarization divisor satisfying certain conditions, then the direct images of the cotangent vector bundle tensored with line bundles on X by Frobenius morphisms are semistable with respect to the polarization. Hence we see that the de Rham complex of X consists of semistable vector bundles if X has the semistable cotangent vector bundle with respect to the polarization with certain mild conditions.

1. Introduction

This is a continuation of our previous paper [9]. Let k be an algebraically closed field of characteristic $p > 0$, X a nonsingular projective variety of dimension n over k, $F = F_X$ the absolute Frobenius morphism of X and H a numerically positive divisor on X. A divisor H on X is called *numerically positive* if it is numerically effective and $H^n > 0$. We define the *slope* of a torsion free sheaf $\mathscr{E}$ on X with respect to H by

$$\mu(\mathscr{E}) = \frac{c_1(\mathscr{E})H^{n-1}}{r(\mathscr{E})},$$

where $c_1(\mathscr{E})$ is the first Chern class of $\mathscr{E}$ and $r(\mathscr{E})$ is the rank of $\mathscr{E}$. Then a torsion free sheaf $\mathscr{E}$ on X is called *semistable* (resp. *stable*) with respect to H if for all nonzero proper subsheaves $\mathscr{F}$ of $\mathscr{E}$, $\mu(\mathscr{F}) \leq \mu(\mathscr{E})$ (resp. $\mu(\mathscr{F}) < \mu(\mathscr{E})$).

As for the semistability of Frobenius pull-backs of vector bundles, a lot of useful results have been obtained (see, for examples, [3], [7], [17]). On the other hand, H. Lange and C. Pauly proved the following theorem on the stability of Frobenius direct images of line bundles.

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THEOREM (Lange-Pauly [10]). *Let X be a nonsingular projective curve over k of genus $g(X) \geq 2$. Then $F_*\mathscr{L}$ is stable for any line bundle $\mathscr{L}$ on X.*

Recently we have proved in [9] the following theorems on the semistability of Frobenius direct images, which are generalizations of Lange-Pauly's result to nonsingular projective surfaces.

THEOREM 1.1. *Let X be a nonsingular projective surface over k and let H be a numerically positive divisor on X such that $|mH|$ is base point free and it contains a nonsingular member for sufficiently large integers m. Assume that Ω_X^1 is semistable with respect to H and $K_X H > 0$, where Ω_X^1 is the cotangent vector bundle of X and K_X is the canonical divisor of X. Then $F_*\mathscr{L}$ is semistable with respect to H for any line bundle $\mathscr{L}$ on X.*

THEOREM 1.2. *Let X be a nonsingular projective surface over k and let H be a numerically positive divisor on X such that $|mH|$ is base point free and it contains a nonsingular member for sufficiently large integers m. Assume that $K_X \underset{\text{num}}{\equiv} 0$ (numerically equivalent to 0) and Ω_X^1 is semistable with respect to H. Then $F_*\mathscr{L}$ is semistable with respect to H for any line bundle $\mathscr{L}$ on X.*

As an application of these theorems, we obtained the following result on the geography of nonsingular projective minimal surfaces of general type.

THEOREM 1.3. *Let X be a nonsingular projective minimal surface of general type over k. Assume that Ω_X^1 is semistable with respect to K_X.*

(1) *(Bogomolov's inequality) If Ω_X^1 is strongly semistable, i.e., $(F^e)^*(\Omega_X^1)$ is semistable for every $e \in \mathbf{N}$ with respect to K_X, then we have*

$$c_1^2(X) \leq 4c_2(X).$$

(2) *If $(F^{e-1})^*(\Omega_X^1)$ is semistable with respect to K_X and $(F^e)^*(\Omega_X^1)$ is not semistable with respect to K_X for a positive integer e, then we have*

$$c_1^2(X) \leq \frac{4p^{2e}}{p^{2e} - (p-1)^2} c_2(X).$$

In particular, we obtain that $c_2(X) > 0$.

Meanwhile, H. Lange and C. Pauly's theorem was generalized to vector bundles as follows.

THEOREM (Mehta-Pauly [12], Sun [20], Kitadai-Sumihiro [9]). *Let X be a nonsingular projective curve over k of genus $g(X) \geq 2$ and $\mathscr{E}$ a stable (resp. semistable) vector bundle on X. Then $F_*\mathscr{E}$ is stable (resp. semistable).*

Hence it is quite natural to consider the following question:

Problem. *Let X be a nonsingular projective variety of dimension n over k such that Ω_X^1 is semistable with respect to H and $K_X H^{n-1} > 0$. Then is $F_*\mathscr{E}$ semistable (resp. stable) with respect to H for any semistable (resp. stable) vector bundle $\mathscr{E}$ with respect to H on X?*

In this paper, we shall give the following partial affirmative answer to the problem when X is a nonsingular projective surface.

Theorem 3.1. *Let X be a nonsingular projective surface over k and let H be a numerically positive divisor on X such that $|mH|$ is base point free and it contains a nonsingular member for sufficiently large integers m. Assume that Ω_X^1 is semistable with respect to H and $K_X H > 0$. Then $F_*(\mathscr{L} \otimes \Omega_X^1)$ is semistable with respect to H for any line bundle $\mathscr{L}$ on X.*

Hence we see by Theorem 1.1 and Theorem 3.1 that the de Rham complex

$$F_*\mathscr{O}_X \xrightarrow{d_0} F_*\Omega_X^1 \xrightarrow{d_1} F_*(\mathscr{O}_X(K_X))$$

of X consists of semistable vector bundles with respect to H if X and H satisfy the assumptions in Theorem 3.1. Thus we shall consider the semistability of the images and the kernels.

Problem. *Are $P_1 = \mathrm{Im}(d_0)$, $P_2 = \mathrm{Im}(d_1)$ and $Q_1 = \mathrm{Ker}(d_1)$ semistable with respect to H?*

We shall show the following.

Theorem 3.3. *Under the assumptions in Theorem 3.1, P_1 and P_2 are semistable with respect to H.*

It is well-known that the de Rham complex $(F_*(\Omega_X^\bullet), d)$ of X plays an important role in the proof of Deligne and Illusie's theorem [1]. Hence it seems that the above results might be useful in the studies of geography, Kodaira vanishing theorem etc., of nonsingular projective varieties of general type in positive characteristic in the future.

2. Canonical filtrations and canonical connections

In this section, we recall several basic results on canonical filtrations (cf. [9], [20]) and canonical connections (cf. [8]) because they play an essential role in the proofs of Theorem 3.1 and Theorem 3.3. For details, please refer to [9].

Let k be an algebraically closed field of characteristic $p > 0$, X a nonsingular projective variety over k of dimension n, $F = F_X$ the absolute Frobenius morphism of X and let $\mathscr{E}$ be a vector bundle on X.

2.1. Canonical filtrations. Let I be the kernel of the natural surjection $F^*F_*\mathcal{O}_X \to \mathcal{O}_X$. Since $F^*F_*\mathcal{O}_X$ is an $\mathcal{O}_X$-algebra, we obtain a descending filtration

$$I^0 := F^*F_*\mathcal{O}_X \supset I^1 := I \supset I^2 \supset I^3 \supset \cdots$$

on $F^*F_*\mathcal{O}_X$. Utilizing the descending filtration, we can define a descending filtration on $F^*F_*\mathcal{E}$ as follows.

$$W^0 = F^*F_*\mathcal{E} \supset W^1 = F^*F_*\mathcal{E} \cdot I \supset \cdots \supset W^i$$
$$= F^*F_*\mathcal{E} \cdot I^i \supset \cdots \supset W^{n(p-1)+1} = (0).$$

We call this filtration $W^\bullet = \{W^i\}$ (resp. $I^\bullet = \{I^i\}$) the *canonical filtration* on $F^*F_*\mathcal{E}$ (resp. $F^*F_*\mathcal{O}_X$).

Let $U = \operatorname{Spec} A \subset X$ be a nonempty affine open subset. Then the exact sequence

$$0 \to I \to F^*F_*\mathcal{O}_X \to \mathcal{O}_X \to 0$$

is locally expressed in the following way:

$$0 \to I \to A \otimes_{A^p} A \to A \to 0$$

and $I = \langle a \otimes 1 - 1 \otimes a \mid a \in A \rangle A$. By shrinking U if necessary, every element $a \in A$ can be written as $a = \sum_{0 \le i_1, \dots, i_n \le p-1} a_{i_1,\dots,i_n}^p x_1^{i_1} \dots x_n^{i_n}$, where $\{x_1, \dots, x_n\}$ is a regular system of parameters and $a_{i_1,\dots,i_n} \in A$. Hence, locally, $I = \langle x_1^{i_1} \dots x_n^{i_n} \otimes 1 - 1 \otimes x_1^{i_1} \dots x_n^{i_n} \mid 0 \le i_1, \dots, i_n \le p-1 \rangle A$. Further putting $\omega_i = x_i \otimes 1 - 1 \otimes x_i$ $(1 \le i \le n)$, we obtain the following.

Lemma 2.1. *With the above notation, we have*

(1) *I is a free A-module with a basis $\{\omega^\alpha = \omega_1^{\alpha_1} \dots \omega_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \neq 0, 0 \le \alpha_k \le p-1, 1 \le k \le n\}$.*

$$I = \bigoplus_{\alpha \neq 0} \omega^\alpha A.$$

(2) $I^i/I^{i+1} = \bigoplus_{|\alpha|=i} \omega^\alpha A$ *for* $0 \le i \le n(p-1)$, *where* $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Hence we observe from Lemma 2.1 that $\operatorname{Gr}^i(I^\bullet) = I^i/I^{i+1}$ $(0 \le i \le n(p-1))$ is a vector bundle on X with rank $= \#\{\alpha = (\alpha_1, \dots, \alpha_n) \mid |\alpha| = i, 0 \le \alpha_k \le p-1 \ (1 \le k \le n)\}$. In particular, it is easily seen that $r(\operatorname{Gr}^i(I^\bullet)) = \binom{n+i-1}{i}$ for $0 \le i \le p-1$ and $r(I^{n(p-1)}) = 1$, i.e., $I^{n(p-1)}$ is a line bundle on X. Since $I^{n(p-1)}|_U = \omega^{(p-1,\dots,p-1)}A$ and $I^i/I^{i+1}|_U = \bigoplus_{|\alpha|=i} \omega^\alpha A$ locally, we see that $I^{n(p-1)} \simeq K_X^{\otimes(p-1)}$ and $I^i/I^{i+1} \simeq S^i(\Omega_X^1)$ on X for $0 \le i \le p-1$ by computing the transition matrices between those bases.

In addition, there exists a perfect pairing

$$I^i/I^{i+1} \otimes I^{n(p-1)-i}/I^{n(p-1)-i+1} \ni \xi \otimes \eta \mapsto \xi\eta \in I^{n(p-1)} \simeq K_X^{\otimes(p-1)}$$

for $0 \le i \le n(p-1)/2$, from which we see $I^{n(p-1)-i}/I^{n(p-1)-i+1} \simeq K_X^{\otimes(p-1)} \otimes (I^i/I^{i+1})^\vee$, where $\mathscr{E}^\vee$ is the dual vector bundle of $\mathscr{E}$. Thus we obtain the following result concerning $\mathrm{Gr}^i(I^\bullet) = I^i/I^{i+1}$.

Lemma 2.2. *Let $I^\bullet = \{I^i\}$ $(0 \le i \le n(p-1)+1)$ be the canonical filtration on $F^*F_*\mathcal{O}_X$. Then we have*
(1) $I^{n(p-1)} \simeq K_X^{\otimes(p-1)}$,
(2) $I^i/I^{i+1} \simeq S^i(\Omega_X^1)$ *for* $0 \le i \le p-1$,
(3) $I^{n(p-1)-i}/I^{n(p-1)-i+1} \simeq K_X^{\otimes(p-1)} \otimes (I^i/I^{i+1})^\vee$ *for* $0 \le i \le n(p-1)/2$.

As a corollary of Lemma 2.2, we can describe all $\mathrm{Gr}^i(I^\bullet)$ explicitly as follows when X is a curve or a surface.

Corollary 2.3. *We observe that*
(1) *If* $\dim X = 1$, *then* $I^i/I^{i+1} = K_X^{\otimes i}$ *for* $0 \le i \le p-1$.
(2) *If* $\dim X = 2$, *then*

$$I^i/I^{i+1} = \begin{cases} S^i(\Omega_X^1), & 0 \le i \le p-1, \\ K_X^{\otimes(i-p+1)} \otimes S^{2p-2-i}(\Omega_X^1), & p \le i \le 2p-2. \end{cases}$$

Assume that $\mathscr{E}|_U = \tilde{M}$ is the vector bundle on U associated to a finitely generated projective A-module M. Then we observe that

$$F^*F_*\mathscr{E}|_U = (M \otimes_{A^p} A)^\sim = (M \otimes_A (A \otimes_{A^p} A))^\sim$$
$$W^i|_U = ((M \otimes_{A^p} A) \cdot I^i)^\sim = (M \otimes_A I^i)^\sim,$$

from which it follows

$$\mathrm{Gr}^i(W^\bullet) \simeq \mathscr{E} \otimes_{\mathcal{O}_X} \mathrm{Gr}^i(I^\bullet) \qquad \text{on } X \text{ for } 0 \le i \le n(p-1).$$

Combining with Corollary 2.3, we obtain the following.

Corollary 2.4. *Let $W^\bullet = \{W^i\}$ be the canonical filtration on $F^*F_*\mathscr{E}$. Then we have*
(1) *If* $\dim X = 1$, *then* $W^i/W^{i+1} = \mathscr{E} \otimes K_X^{\otimes i}$ *for* $0 \le i \le p-1$.
(2) *If* $\dim X = 2$, *then*

$$W^i/W^{i+1} = \begin{cases} \mathscr{E} \otimes S^i(\Omega_X^1), & 0 \le i \le p-1, \\ \mathscr{E} \otimes K_X^{\otimes(i-p+1)} \otimes S^{2p-2-i}(\Omega_X^1), & p \le i \le 2p-2. \end{cases}$$

By shrinking U if necessary, we may assume that $\mathscr{E}|_U \simeq \bigoplus^r \mathcal{O}_U$ $(r = r(\mathscr{E}))$ and choose a basis $\{e_1, \dots, e_r\}$ of $\Gamma(U, \mathscr{E})$. Then it follows from Lemma 2.1

that every element $f \in \Gamma(U, W^i|_U)$ $(0 \le i \le n(p-1))$ can be written in the following way uniquely:

$$f = \sum_{i=1}^{r} e_i \otimes \sum_{|\alpha|=i} \omega^\alpha f_\alpha^{(i)} + (\text{higher terms}),$$

where $f_\alpha^{(i)} \in A$. This is useful for computing $\nabla(f)$, where $\nabla : F^*F_*\mathscr{E} \to F^*F_*\mathscr{E} \otimes \Omega_X^1$ is the canonical connection of $F^*F_*\mathscr{E}$ defined in the next subsection.

2.2. Canonical connections. Let $\mathscr{E}$ be a quasi-coherent sheaf on a non-singular projective variety X of dimension n. Then there exists a connection $\nabla : F^*\mathscr{E} \to F^*\mathscr{E} \otimes \Omega_X^1$, which is called the *canonical connection* (cf. [8]). This is locally written as

$$\begin{array}{ccc} M \otimes_A A \to M \otimes_A A \otimes_A \Omega^1_{A/k} & \simeq & M \otimes_A \Omega^1_{A/k}, \\ \cup & & \cup \\ m \otimes f & \mapsto & m \otimes df \end{array}$$

where $A = \Gamma(U, \mathcal{O}_X)$ and $M = \Gamma(U, \mathscr{E})$ for an affine open subset U of X. Here A is considered as an A-module through Frobenius morphism. Hence the canonical connection is the positive characteristic version of the Gauss-Manin connection. In particular, we get a connection on $F^*F_*\mathscr{E}$

$$\nabla : F^*F_*\mathscr{E} \to F^*F_*\mathscr{E} \otimes \Omega_X^1.$$

Let $\{x_1, \dots, x_n\}$ be a regular system of parameters on $U = \operatorname{Spec} A$ and $\omega_i = x_i \otimes 1 - 1 \otimes x_i$ for $1 \le i \le n$. Then we have by straightforward computation,

Lemma 2.5.

$$\begin{aligned} &\nabla(e \otimes \omega_1^{\alpha_1} \dots \omega_n^{\alpha_n} f) \\ &\quad = e \otimes \sum_{k=1}^{n} \left(-\alpha_k \omega_1^{\alpha_1} \dots \omega_k^{\alpha_k - 1} \dots \omega_n^{\alpha_n} f + \omega_1^{\alpha_1} \dots \omega_n^{\alpha_n} \frac{\partial f}{\partial x_k} \right) \otimes dx_k, \end{aligned}$$

where $e \in \Gamma(U, \mathscr{E})$ *and* $f \in A$.

It turns out from Lemma 2.5 and Lemma 2.1 that the $\mathcal{O}_X$-homomorphism

$$\begin{aligned} \overline{\nabla}_i : W^i/W^{i+1} = \mathscr{E} \otimes I^i/I^{i+1} &\to \mathscr{E} \otimes I^{i-1}/I^i \otimes \Omega_X^1 \\ &= W^{i-1}/W^i \otimes \Omega_X^1 \qquad (0 \le i \le n(p-1)) \end{aligned}$$

induced from ∇ is injective. Hence we see that W^i/W^{i+1} is a subsheaf of $\mathscr{E} \otimes (\Omega_X^1)^{\otimes i}$ through the $\mathscr{O}_X$-homomorphism $\overline{\nabla}_1 \circ \cdots \circ \overline{\nabla}_{i-1} \circ \overline{\nabla}_i$ $(0 \le i \le n(p-1))$.

Let $\mathscr{E}$ be a nonzero coherent torsion free sheaf ($r = r(\mathscr{E})$) and S a nonzero torsion free subsheaf of $F_*\mathscr{E}$ and let $\nabla : F^*F_*\mathscr{E} \to F^*F_*\mathscr{E} \otimes \Omega_X^1$ be the canonical connection on $F^*F_*\mathscr{E}$. Further, let $W^\bullet = \{W^i\}$ $(0 \le i \le n(p-1))$ be the canonical filtration of $F^*F_*\mathscr{E}$. Then the filtration $W^\bullet$ induces a descending filtration $F^*S \cap W^\bullet = \{F^*S \cap W^i\}$ $(0 \le i \le n(p-1))$ of F^*S. Let $S_i = F^*S \cap W^i/F^*S \cap W^{i+1}$ and $r_i = r(S_i)$ for $0 \le i \le n(p-1)$. Then $r_0 = r(S_0)$ is always a positive integer. In fact, since the image of F^*S by the canonical surjection $F^*F_*\mathscr{E} \to \mathscr{E}$ is nonzero, F^*S is not contained in W^1. Hence we see $r(S_0) > 0$. When $n = 2$, we can observe the following fact concerning r_i $(0 \le i \le p-1)$. Let $K = k(X)$ be the function field of X over k. Then we have $r_i = \dim_K(S_i \otimes K) = \dim_K(F^*S \cap W^i/F^*S \cap W^{i+1} \otimes K)$. Let

$$f = \sum_{i=1}^{r} e_i \otimes \left(\sum_{\substack{\alpha_1+\alpha_2=2(p-1)-i,\\ 0\le\alpha_1,\alpha_2\le p-1}} \omega_1^{\alpha_1}\omega_2^{\alpha_2} f_{\alpha_1\alpha_2}^{(i)} + (\text{higher}) \right), \qquad f_{\alpha_1\alpha_2}^{(i)} \in K$$

be an element of $(F^*S \cap W^{2(p-1)-i}) \otimes K$ $(0 \le i \le p-1)$ whose residue class $\bar{f}$ in $S_{2(p-1)-i} \otimes K$ is not zero, i.e.,

$$0 \ne \sum_{i=1}^{r} e_i \otimes \sum_{\substack{\alpha_1+\alpha_2=2(p-1)-i,\\ 0\le\alpha_1,\alpha_2\le p-1}} \omega_1^{\alpha_1}\omega_2^{\alpha_2} f_{\alpha_1\alpha_2}^{(i)} \in (\mathscr{E} \otimes I^{2(p-1)-i}/I^{2(p-1)-i+1}) \otimes K.$$

Then we see by Lemma 2.5 that

$$\begin{aligned} \nabla(f) = \sum_{i=1}^{r} e_i \otimes \Big(&\sum (-\alpha_1\omega_1^{\alpha_1-1}\omega_2^{\alpha_2} f_{\alpha_1\alpha_2}^{(i)} + (\text{higher})) \otimes dx_1 \\ &+ \sum (-\alpha_2\omega_1^{\alpha_1}\omega_2^{\alpha_2-1} f_{\alpha_1\alpha_2}^{(i)} + (\text{higher})) \otimes dx_2 \Big). \end{aligned}$$

Since ∇ induces a connection on F^*S, if we put

$$\nabla_{x_1}(f) = \sum_{i=1}^{r} e_i \otimes \sum (-\alpha_1\omega_1^{\alpha_1-1}\omega_2^{\alpha_2} f_{\alpha_1\alpha_2}^{(i)} + (\text{higher})),$$

$$\nabla_{x_2}(f) = \sum_{i=1}^{r} e_i \otimes \sum (-\alpha_2\omega_1^{\alpha_1}\omega_2^{\alpha_2-1} f_{\alpha_1\alpha_2}^{(i)} + (\text{higher})),$$

then $\nabla_{x_1}(f)$ and $\nabla_{x_2}(f)$ are elements of $(F^*S \cap W^{2p-3-i}) \otimes K$. Let us put

$$\nabla_{x_1}^{p-1-i}\nabla_{x_2}^{p-1-i} = \underbrace{\nabla_{x_1} \circ \cdots \circ \nabla_{x_1}}_{(p-1-i)\text{-times}} \circ \underbrace{\nabla_{x_2} \circ \cdots \circ \nabla_{x_2}}_{(p-1-i)\text{-times}}.$$

Then it turns out that

$$\nabla_{x_1}^{p-1-i}\nabla_{x_2}^{p-1-i}(f) = \sum_{i=1}^{r} e_i \otimes \sum_{\substack{\alpha_1+\alpha_2=2(p-1)-i,\\ 0\le\alpha_1,\alpha_2\le p-1}} \frac{\alpha_1!}{(\alpha_1-(p-1-i))!} \times \frac{\alpha_2!}{(\alpha_2-(p-1-i))!} \cdot \omega_1^{\alpha_1-(p-1-i)}\omega_2^{\alpha_2-(p-1-i)} f_{\alpha_1\alpha_2}^{(i)} + (\text{higher})$$

and $\nabla_{x_1}^{p-1-i}\nabla_{x_2}^{p-1-i}(f)$ is an element of $(F^*S \cap W^i) \otimes K$ whose residue class corresponds to

$$\sum_{i=1}^{r} e_i \otimes \sum_{\substack{\alpha_1+\alpha_2=2(p-1)-i,\\ 0\le\alpha_1,\alpha_2\le p-1}} \frac{\alpha_1!}{(\alpha_1-(p-1-i))!}\,\frac{\alpha_2!}{(\alpha_2-(p-1-i))!} \cdot \omega_1^{\alpha_1-(p-1-i)}\omega_2^{\alpha_2-(p-1-i)} f_{\alpha_1\alpha_2}^{(i)} \in (\mathscr{E} \otimes I^i/I^{i+1}) \otimes K.$$

Hence we have the following commutative diagram

$$\begin{array}{ccc} (F^*S \cap W^{2(p-1)-i}/F^*S \cap W^{2(p-1)-i+1}) \otimes K & \longrightarrow & (W^{2(p-1)-i}/W^{2(p-1)-i+1}) \otimes K \\ \downarrow & & \wr\downarrow\varphi \\ (F^*S \cap W^i/F^*S \cap W^{i+1}) \otimes K & \longrightarrow & (W^i/W^{i+1}) \otimes K, \end{array}$$

where φ is the isomorphism obtained by multiplications by nonzero elements $\frac{\alpha_1!}{(\alpha_1-(p-1-i))!}\frac{\alpha_2!}{(\alpha_2-(p-1-i))!}$ for $\alpha_1+\alpha_2 = 2(p-1)-i$ $(0 \le \alpha_1, \alpha_2 \le p-1)$. Therefore we have proved the following.

LEMMA 2.6. *With the above notation, we have* $r_i \ge r_{2(p-1)-i}$ *for* $0 \le i \le p-1$.

In particular, assume $r_{2p-2} = r$. Then we can take a basis $\{e_i\}$ $(1 \le i \le r)$ of $\mathscr{E} \otimes K$ such that $\{e_i \otimes \omega_1^{p-1}\omega_2^{p-1}\}$ $(1 \le i \le r)$ is a basis of $(F^*S \cap W^{2p-2}) \otimes K$. Hence taking Corollary 2.4 into consideration, we obtain the following by arguments similar to the above.

LEMMA 2.7. *If* $r_{2p-2} = r$, *then we have*

$$r_i = \begin{cases} r(i+1), & 0 \le i \le p-1, \\ r(2p-1-i), & p \le i \le 2p-2. \end{cases}$$

3. Main results

Using the canonical filtrations and canonical connections, we prove the following theorem, which is a partial affirmative answer to the problem when X is an algebraic surface.

THEOREM 3.1. *Let X be a nonsingular projective surface over k and let H be a numerically positive divisor on X such that $|mH|$ is base point free and it contains a nonsingular member for sufficiently large integers m. Assume that Ω_X^1 is semistable with respect to H and $K_X H > 0$. Then $F_*(\mathscr{L} \otimes \Omega_X^1)$ is semistable with respect to H for any line bundle $\mathscr{L}$ on X.*

PROOF. Assuming that $F_*(\mathscr{L} \otimes \Omega_X^1)$ is not semistable with respect to H, we shall derive a contradiction. Let S be the maximal destabilizing subsheaf of $F_*(\mathscr{L} \otimes \Omega_X^1)$. Then we have by Theorem 2.1 [9]

$$\mu(S) > \mu(F_*(\mathscr{L} \otimes \Omega_X^1)) = \frac{1}{2} K_X H + \frac{1}{p} c_1(\mathscr{L}) H.$$

Let $W^\bullet = \{W^i\}$ $(0 \le i \le 2p-1)$ be the canonical filtration of $F^* F_*(\mathscr{L} \otimes \Omega_X^1)$. Then we see by Corollary 2.4 that

$$F^* F_*(\mathscr{L} \otimes \Omega_X^1) = W^0 \supset W^1 \supset \cdots \supset W^{2p-1} = (0)$$

and

$$W^i / W^{i+1} = \begin{cases} \mathscr{L} \otimes \Omega_X^1 \otimes S^i(\Omega_X^1), & 0 \le i \le p-1, \\ \mathscr{L} \otimes \Omega_X^1 \otimes K_X^{\otimes(i-p+1)} \otimes S^{2p-2-i}(\Omega_X^1), & p \le i \le 2p-2. \end{cases}$$

It follows from Ilangovan-Mehta-Parameswaran's Theorem ([5, 16]) and the restriction theorem (cf. [11, Corollary 5.4]) that W^i / W^{i+1} is semistable with respect to H for $0 \le i \le p-2$ and $p \le i \le 2p-2$ since Ω_X^1 is semistable with respect to H. In addition, $F^* S \cap W^\bullet$ is a filtration of $F^* S$. If we put

$$S_i = F^* S \cap W^i / F^* S \cap W^{i+1} \qquad (0 \le i \le 2p-2),$$

then we have that

$$\mu(S_i) \le \mu(W^i / W^{i+1}) \qquad \text{except for } i = p-1.$$

Thus the following inequalities hold for $i \ne p-1$

$$c_1(F^* S \cap W^i) H - c_1(F^* S \cap W^{i+1}) H \le \left(\frac{1}{2} + \frac{i}{2}\right) r(S_i) K_X H + r(S_i) c_1(\mathscr{L}) H.$$

Summing up the above inequalities, we see that

$$c_1(F^*S)H - c_1(S_{p-1})H \le \left(\frac{1}{2}\sum{}' r(S_i) + \frac{1}{2}\sum{}' ir(S_i)\right)K_X H + \sum{}' r(S_i)c_1(\mathscr{L})H \tag{1}$$

where $\sum' = \sum_{i \ne p-1} = \sum_{i=0}^{p-2} + \sum_{i=p}^{2p-2}$.

1) Assume that $\Omega_X^1 \otimes S^{p-1}(\Omega_X^1)$ is semistable with respect to H. Then we have that

$$c_1(S_{p-1})H \le \left(\frac{1}{2}r(S_{p-1}) + \frac{p-1}{2}r(S_{p-1})\right)K_X H + r(S_{p-1})c_1(\mathscr{L})H$$

because W^{p-1}/W^p is semistable with respect to H. Hence combining the above inequality with the inequality (1), we have

$$c_1(F^*S)H \le \left(\frac{1}{2}\sum_{i=0}^{2p-2} r(S_i) + \frac{1}{2}\sum_{i=0}^{2p-2} ir(S_i)\right)K_X H + \sum_{i=0}^{2p-2} r(S_i)c_1(\mathscr{L})H,$$

$$\mu(F^*S) \le \left(\frac{1}{2} + \frac{1}{2}\frac{\sum ir(S_i)}{\sum r(S_i)}\right)K_X H + c_1(\mathscr{L})H.$$

On the other hand,

$$\mu(F^*S) = p\mu(S) > p\left(\frac{1}{2}K_X H + \frac{1}{p}c_1(\mathscr{L})H\right) = \frac{p}{2}K_X H + c_1(\mathscr{L})H.$$

Therefore it follows that

$$0 < \sum_{i=0}^{2p-2}(i - (p-1))r(S_i) = \sum_{i=0}^{p-2}(i - (p-1))(r(S_i) - r(S_{2p-2-i})).$$

However $r(S_i) \ge r(S_{2p-2-i})$ for $0 \le i \le p-1$ by Lemma 2.6 and we have a contradiction.

2) Assume that $r(S_{2p-2}) \ne 0$.

Take a sufficiently small affine open subset $U = \operatorname{Spec} A$ of X such that $W^{2p-2} = \omega^{p-1}\eta^{p-1}A$ on U, where $\omega = x \otimes 1 - 1 \otimes x$, $\eta = y \otimes 1 - 1 \otimes y$ ($\{x, y\}$ being a regular system of parameters of A) and $\mathscr{L} \otimes \Omega_X^1$ is free on U. Let $f = \sum_{i=1}^{2} e_i \otimes \omega^{p-1}\eta^{p-1}f_i$ be a nonzero element of $\Gamma(U, F^*S \cap W^{2p-2})$, where $\{e_1, e_2\}$ is a basis of $\Gamma(U, \mathscr{L} \otimes \Omega_X^1)$ and $f_i \in \Gamma(U, \mathscr{O}_X)$ $(i = 1, 2)$, and let $\nabla : F^*F_*(\mathscr{L} \otimes \Omega_X^1) \to F^*F_*(\mathscr{L} \otimes \Omega_X^1) \otimes \Omega_X^1$ be the canonical connection of $F^*F_*(\mathscr{L} \otimes \Omega_X^1)$. Then it holds from Lemma 2.5 that

$$\begin{aligned}\nabla(f) &= \sum_i e_i \otimes \left(-(p-1)\omega^{p-2}\eta^{p-1}f_i + \omega^{p-1}\eta^{p-1}\frac{\partial f_i}{\partial x}\right) \otimes dx \\ &\quad + \sum_i e_i \otimes \left(-(p-1)\omega^{p-1}\eta^{p-2}f_i + \omega^{p-1}\eta^{p-1}\frac{\partial f_i}{\partial y}\right) \otimes dy.\end{aligned}$$

Hence if we write

$$\nabla_x(f) = \sum_i e_i \otimes \left(-(p-1)\omega^{p-2}\eta^{p-1}f_i + \omega^{p-1}\eta^{p-1}\frac{\partial f_i}{\partial x}\right),$$

$$\nabla_y(f) = \sum_i e_i \otimes \left(-(p-1)\omega^{p-1}\eta^{p-2}f_i + \omega^{p-1}\eta^{p-1}\frac{\partial f_i}{\partial y}\right),$$

then both $\nabla_x(f)$ and $\nabla_y(f)$ are contained in $\Gamma(U, F^*S \cap W^{2p-3})$ because ∇ induces a connection on F^*S. For every α and β $(0 \leq \alpha \leq p-1, 0 \leq \beta \leq p-1)$, let us put

$$\nabla_{x^\alpha y^\beta} = \underbrace{\nabla_x \circ \cdots \circ \nabla_x}_{\alpha\text{-times}} \circ \underbrace{\nabla_y \circ \cdots \circ \nabla_y}_{\beta\text{-times}}.$$

By direct calculations, we observe that

$$\begin{aligned}&\nabla_{x^\alpha y^\beta}(f) \\ &\quad = \sum_i e_i \otimes \left((-1)^{\alpha+\beta}\frac{(p-1)!}{(p-1-\alpha)!}\frac{(p-1)!}{(p-1-\beta)!}\omega^{p-1-\alpha}\eta^{p-1-\beta}f_i + (\text{higher})\right) \\ &\quad \in \Gamma(U, F^*S \cap W^{2p-2-(\alpha+\beta)}).\end{aligned}$$

Hence for every $\{\alpha, \beta\}$ such that $\alpha + \beta = p-1$, we see that

$$\begin{array}{ccc} S_{p-1} = F^*S \cap W^{p-1}/F^*S \cap W^p & \hookrightarrow & W^{p-1}/W^p = \mathscr{L} \otimes \Omega_X^1 \otimes S^{p-1}(\Omega_X^1) \\ \Psi & & \Psi \\ \overline{\nabla_{x^\alpha y^\beta}(f)} & \longmapsto & \sum_i \frac{(p-1)!}{(p-1-\alpha)!}\frac{(p-1)!}{(p-1-\beta)!} f_i e_i \otimes dx^{p-1-\alpha} dy^{p-1-\beta} \end{array}$$

on U, where $\overline{\nabla_{x^\alpha y^\beta}(f)}$ is the residue class of $\nabla_{x^\alpha y^\beta}(f)$ in $F^*S \cap W^{p-1}/F^*S \cap W^p$. Therefore, there exists a rank 1 torsion free subsheaf M of $\mathscr{L} \otimes \Omega_X^1$ satisfying the conditions

(a) $r(S_{p-1} \cap M \otimes S^{p-1}(\Omega_X^1)) \geq p$,

(b) $\mathscr{L} \otimes \Omega_X^1/M$ is a rank 1 torsion free sheaf.

Indeed, the saturated rank 1 torsion free subsheaf M of $\mathscr{L} \otimes \Omega_X^1$ which is an extension of the line bundle on U associated to the rank 1 free A-module $\sum_{i=1}^2 f_i e_i A$ to X satisfies the above conditions (a) and (b).

Consider the following commutative diagram with exact columns:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
S_{p-1} \cap (M \otimes S^{p-1}(\Omega_X^1)) & \longrightarrow & M \otimes S^{p-1}(\Omega_X^1) \\
\downarrow & & \downarrow \\
S_{p-1} & \longrightarrow & \mathscr{L} \otimes \Omega_X^1 \otimes S^{p-1}(\Omega_X^1) \\
\downarrow & & \downarrow \\
S_{p-1}/S_{p-1} \cap (M \otimes S^{p-1}(\Omega_X^1)) & \longrightarrow & (\mathscr{L} \otimes \Omega_X^1/M) \otimes S^{p-1}(\Omega_X^1) \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}$$

Since $M \otimes S^{p-1}(\Omega_X^1)$ and $(\mathscr{L} \otimes \Omega_X^1/M) \otimes S^{p-1}(\Omega_X^1)$ are semistable with respect to H, it follows that

$$c_1(S_{p-1} \cap (M \otimes S^{p-1}(\Omega_X^1)))H \leq \left(c_1(M)H + \frac{p-1}{2}K_X H\right)t_1,$$

$$c_1(S_{p-1}/S_{p-1} \cap (M \otimes S^{p-1}(\Omega_X^1)))H$$
$$\leq \left(2c_1(\mathscr{L})H + K_X H - c_1(M)H + \frac{p-1}{2}K_X H\right)t_2,$$

where $t_1 = r(S_{p-1} \cap (M \otimes S^{p-1}(\Omega_X^1)))$ and $t_2 = r(S_{p-1}/S_{p-1} \cap (M \otimes S^{p-1}(\Omega_X^1)))$. Thus combining the above inequalities, we obtain that

$$c_1(S_{p-1})H \leq \left(\frac{p-1}{2}(t_1+t_2) + t_2\right)K_X H + (t_1 - t_2)c_1(M)H + 2t_2 c_1(\mathscr{L})H.$$

Further, we have that $c_1(M)H \leq c_1(\mathscr{L})H + \frac{1}{2}K_X H$ and $t_1 - t_2 \geq 0$ because $r(S_{p-1}) = t_1 + t_2 \leq 2p$ and $t_1 \geq p$. Hence it follows that

$$c_1(S_{p-1})H \leq \left(\frac{1}{2}r(S_{p-1}) + \frac{p-1}{2}r(S_{p-1})\right)K_X H + r(S_{p-1})c_1(\mathscr{L})H,$$

which leads to a contradiction similar to the case 1).

3) Assume that $\Omega_X^1 \otimes S^{p-1}(\Omega_X^1)$ is not semistable with respect to H and $r(S_{2p-2}) = 0$.

3.1) There are the following canonical exact sequences for any rank 2 vector bundle $\mathscr{E}$ in positive characteristic $p > 0$.

$$0 \to S^{p-2}(\mathscr{E}) \otimes \det \mathscr{E} \to \mathscr{E} \otimes S^{p-1}(\mathscr{E}) \to S^{p}(\mathscr{E}) \to 0,$$

$$0 \to \mathscr{E}^{(p)} \to S^{p}(\mathscr{E}) \to S^{p-2}(\mathscr{E}) \otimes \det \mathscr{E} \to 0,$$

where $\mathscr{E}^{(p)} = F^*\mathscr{E}$ is the Frobenius pull-back of $\mathscr{E}$. Hence in particular, we have the exact sequences:

$$0 \to S^{p-2}(\Omega_X^1) \otimes \mathcal{O}_X(K_X) \to \Omega_X^1 \otimes S^{p-1}(\Omega_X^1) \to S^{p}(\Omega_X^1) \to 0, \tag{2}$$

$$0 \to (\Omega_X^1)^{(p)} \to S^{p}(\Omega_X^1) \to S^{p-2}(\Omega_X^1) \otimes \mathcal{O}_X(K_X) \to 0. \tag{3}$$

Thus if $(\Omega_X^1)^{(p)}$ is semistable with respect to H, then it turns out from the exact sequences (2) and (3) that $\Omega_X^1 \otimes S^{p-1}(\Omega_X^1)$ is also semistable with respect to H because $\mu((\Omega_X^1)^{(p)}) = \mu(S^{p-2}(\Omega_X^1) \otimes K_X) = \mu(S^{p}(\Omega_X^1)) = (p/2)K_X H$. Hence $(\Omega_X^1)^{(p)}$ is not semistable with respect to H. Let

$$0 \to A \to (\Omega_X^1)^{(p)} \to B \to 0 \tag{4}$$

be the Harder-Narasimhan filtration of $(\Omega_X^1)^{(p)}$ and let

$$\nabla : (\Omega_X^1)^{(p)} = F^*\Omega_X^1 \to F^*\Omega_X^1 \otimes \Omega_X^1$$

be the canonical connection of $F^*\Omega_X^1$. Then ∇ induces a nonzero $\mathcal{O}_X$-homomorphism $A \to (\Omega_X^1)^{(p)}/A \otimes \Omega_X^1$ (cf. [19]), from which we obtain

$$\frac{p}{2} K_X H < c_1(A)H \le \frac{1}{4}(1+2p)K_X H. \tag{5}$$

3.2) Consider the following commutative diagram with exact columns:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
S_{p-1} \cap (\mathscr{L} \otimes S^{p-2}(\Omega_X^1) \otimes \mathcal{O}_X(K_X)) & \longrightarrow & \mathscr{L} \otimes S^{p-2}(\Omega_X^1) \otimes \mathcal{O}_X(K_X) \\
\downarrow & & \downarrow \\
S_{p-1} & \longrightarrow & \mathscr{L} \otimes \Omega_X^1 \otimes S^{p-1}(\Omega_X^1) \\
\downarrow & & \downarrow \varphi \\
\varphi(S_{p-1}) & \longrightarrow & \mathscr{L} \otimes S^{p}(\Omega_X^1) \\
\downarrow & & \downarrow \\
0 & & 0,
\end{array} \tag{6}$$

where $\varphi : \mathscr{L} \otimes \Omega_X^1 \otimes S^{p-1}(\Omega_X^1) \to \mathscr{L} \otimes S^{p}(\Omega_X^1)$ is the canonical surjection associated to the exact sequence (2) and put $r_1 = r(S_{p-1} \cap (\mathscr{L} \otimes S^{p-2}(\Omega_X^1) \otimes \mathcal{O}_X(K_X)))$, $r_2 = r(\varphi(S_{p-1}))$. Since $\mathscr{L} \otimes S^{p-2}(\Omega_X^1) \otimes \mathcal{O}_X(K_X)$ is semistable with respect to H, it follows from (6) that

$$c_1(S_{p-1} \cap (\mathscr{L} \otimes S^{p-2}(\Omega_X^1) \otimes \mathcal{O}_X(K_X)))H \le r_1\left(\frac{p}{2}K_XH + c_1(\mathscr{L})H\right). \qquad (7)$$

Further, consider the following commutative diagram with exact columns:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\varphi(S_{p-1}) \cap \mathscr{L} \otimes (\Omega_X^1)^{(p)} & \longrightarrow & \mathscr{L} \otimes (\Omega_X^1)^{(p)} \\
\downarrow & & \downarrow \\
\varphi(S_{p-1}) & \longrightarrow & \mathscr{L} \otimes S^p(\Omega_X^1) \\
\downarrow & & \downarrow \psi \\
\psi(\varphi(S_{p-1})) & \longrightarrow & \mathscr{L} \otimes S^{p-2}(\Omega_X^1) \otimes \mathcal{O}_X(K_X) \\
\downarrow & & \downarrow \\
0 & & 0,
\end{array} \qquad (8)$$

where $\psi : \mathscr{L} \otimes S^p(\Omega_X^1) \to \mathscr{L} \otimes S^{p-2}(\Omega_X^1) \otimes \mathcal{O}_X(K_X)$ is the canonical surjection associated to the exact sequence (3) and put $s_1 = r(\varphi(S_{p-1}) \cap \mathscr{L} \otimes (\Omega_X^1)^{(p)})$, $s_2 = r(\psi(\varphi(S_{p-1})))$. Then we obtain from (8) similarly to the above argument that

$$c_1(\psi(\varphi(S_{p-1})))H \le s_2\left(\frac{p}{2}K_XH + c_1(\mathscr{L})H\right). \qquad (9)$$

3.3) Finally let us consider the following commutative diagram with exact columns:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\varphi(S_{p-1}) \cap \mathscr{L} \otimes A & \longrightarrow & \mathscr{L} \otimes A \\
\downarrow & & \downarrow \\
\varphi(S_{p-1}) \cap \mathscr{L} \otimes (\Omega_X^1)^{(p)} & \longrightarrow & \mathscr{L} \otimes (\Omega_X^1)^{(p)} \\
\downarrow & & \downarrow \xi \\
\xi(\varphi(S_{p-1}) \cap \mathscr{L} \otimes (\Omega_X^1)^{(p)}) & \longrightarrow & \mathscr{L} \otimes B \\
\downarrow & & \downarrow \\
0 & & 0,
\end{array} \qquad (10)$$

where $\xi : \mathscr{L} \otimes (\Omega_X^1)^{(p)} \to \mathscr{L} \otimes B$ is the canonical surjection associated to the Harder-Narasimhan filtration (4) and put $t_1 = r(\varphi(S_{p-1}) \cap \mathscr{L} \otimes A)$, $t_2 = r(\xi(\varphi(S_{p-1}) \cap \mathscr{L} \otimes (\Omega_X^1)^{(p)}))$. Then we have from (10) that

$$c_1(\varphi(S_{p-1}) \cap \mathscr{L} \otimes A)H \leq t_1(c_1(A)H + c_1(\mathscr{L})H),$$
$$c_1(\xi(\varphi(S_{p-1}) \cap \mathscr{L} \otimes (\Omega_X^1)^{(p)}))H \leq t_2(c_1(B)H + c_1(\mathscr{L})H).$$

Hence it holds that

$$\begin{aligned} c_1(\varphi(S_{p-1}) \cap \mathscr{L} \otimes (\Omega_X^1)^{(p)})H &\leq t_1 c_1(A)H + t_2 c_1(B)H + s_1 c_1(\mathscr{L})H \\ &= pt_2 K_X H + (t_1 - t_2)c_1(A)H + s_1 c_1(\mathscr{L})H. \end{aligned}$$

Combining the above inequality with (9) and (7), we obtain

$$c_1(\varphi(S_{p-1}))H \leq \left(t_2 + \frac{1}{2}s_2\right)pK_X H + (t_1 - t_2)c_1(A)H + r_2 c_1(\mathscr{L})H,$$
$$c_1(S_{p-1})H \leq \left(t_2 + \frac{1}{2}(r_1 + s_2)\right)pK_X H + (t_1 - t_2)c_1(A)H + r(S_{p-1})c_1(\mathscr{L})H.$$

Therefore we get the following in combination with (5) according to the values of t_i $(i = 1, 2)$:

3.3.1) $t_1 = 0$, $t_2 = 1$ or $t_1 = t_2 = 1$.

$$c_1(S_{p-1})H \leq \frac{p}{2}r(S_{p-1})K_X H + r(S_{p-1})c_1(\mathscr{L})H,$$

whence a contradiction similar to the case 1) is derived.

3.3.2) $t_1 = 1$, $t_2 = 0$.

$$c_1(S_{p-1})H \leq \frac{p}{2}r(S_{p-1})K_X H + \frac{1}{4}K_X H + r(S_{p-1})K_X H.$$

In the case 3.3.2), it follows that

$$\sum_{i=0}^{2p-2}(i - (p-1))r(S_i) + \frac{1}{2} > 0. \tag{11}$$

However, since $r(S_0) > r(S_{2p-2}) = 0$ from our assumption, the inequality (11) can not hold. □

As a corollary of Theorem 3.1, we have the following.

Corollary 3.2. *Under the assumptions in Theorem* 3.1, $F_*(\mathscr{L} \otimes \mathscr{T}_X)$ *is semistable with respect to* H *for any line bundle* $\mathscr{L}$ *on* X, *where* $\mathscr{T}_X$ *is the tangent vector bundle of* X.

Proof. It is known that $(F_*(\mathscr{L} \otimes \mathscr{T}_X))^\vee \simeq F_*(\Omega_X^1 \otimes \mathscr{L}^{-1} \otimes K_X^{1-p})$, where $\mathscr{E}^\vee$ is the dual vector bundle of $\mathscr{E}$ ([14]). Hence, $F_*(\mathscr{L} \otimes \mathscr{T}_X)$ is semistable with respect to H by Theorem 3.1. □

Let us consider the de Rham complex of X:

$$F_*\mathcal{O}_X \xrightarrow{d_0} F_*\Omega_X^1 \xrightarrow{d_1} F_*(\mathcal{O}_X(K_X))$$

and put $P_1 = \mathrm{Im}(d_0)$, $P_2 = \mathrm{Im}(d_1)$ and $Q_1 = \mathrm{Ker}(d_1)$. Then we observe from Theorem 1.1 and Theorem 3.1 that the de Rham complex of X consists of semistable vector bundles with respect to H under the assumptions in Theorem 3.1.

On the other hand, there exist the following exact sequences by the Cartier's isomorphism theorem (cf. [6]):

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to P_1 \to 0$$

$$0 \to P_1 \to Q_1 \to \Omega_X^1 \to 0$$

$$0 \to Q_1 \to F_*\Omega_X^1 \to P_2 \to 0$$

$$0 \to P_2 \to F_*(\mathcal{O}(K_X)) \to \mathcal{O}(K_X) \to 0.$$

Hence it follows that

$$r(P_1) = p^2 - 1, \qquad c_1(P_1) = \frac{p^2 - p}{2} K_X, \qquad \mu(P_1) = \frac{p}{2(p+1)} K_X H,$$

$$r(P_2) = p^2 - 1, \qquad c_1(P_2) = \frac{p^2 + p - 2}{2} K_X, \qquad \mu(P_2) = \frac{p+2}{2(p+1)} K_X H,$$

$$r(Q_1) = p^2 + 1, \qquad c_1(Q_1) = \frac{p^2 - p + 2}{2} K_X, \qquad \mu(Q_1) = \frac{p^2 - p + 2}{2(p^2+1)} K_X H.$$

Concerning the semistability of those vector bundles, we shall show the following.

Theorem 3.3. *Let* X *be a nonsingular projective surface over* k *and let* H *be a numerically positive divisor on* X *such that* $|mH|$ *is base point free and it contains a nonsingular member for large integers* m. *Assume that* Ω_X^1 *is semistable with respect to* H *and* $K_X H > 0$. *Then* P_1 *and* P_2 *are semistable with respect to* H.

Proof. 1) Assuming that P_1 is not semistable with respect to H, we shall derive a contradiction. Let S be the maximal destabilizing subsheaf of P_1. Then we have $c_1(S)H > (p/2(p+1))rK_XH$ where $r = r(S)$. Let $\tilde{S} = d_0^{-1}(S) \subset F_*\mathcal{O}_X$ and let $W^\bullet = \{W^i\}$ $(0 \le i \le 2p-1)$ be the canonical filtration on $F^*F_*\mathcal{O}_X$. Put $\tilde{S}_i = F^*\tilde{S} \cap W^i / F^*\tilde{S} \cap W^{i+1} \hookrightarrow W^i/W^{i+1}$ and $r_i = r(\tilde{S}_i)$ $(0 \le i \le 2p-2)$. Then it follows from Corollary 2.4 that

$$c_1(F^*\tilde{S} \cap W^i)H - c_1(F^*\tilde{S} \cap W^{i+1})H \le \frac{i}{2} r_i K_X H \qquad \text{for } 0 \le i \le 2p-2.$$

Summing up the above, we have $c_1(F^*\tilde{S})H \le ((1/2)\sum_{i=0}^{2p-2} ir_i)K_XH$. Since $c_1(\tilde{S}) = c_1(S)$ and $\sum_{i=0}^{2p-2} r_i = r+1$, the inequality

$$\sum_{i=0}^{2p-2} ((p+1)i - p^2)r_i + p^2 > 0 \tag{12}$$

holds. If $r_{2p-2} \ne 0$, i.e., $r_{2p-2} = 1$, then it follows from Lemma 2.7 that $r_i = i+1$ for $0 \le i \le p-1$ and $r_i = 2p-1-i$ for $p \le i \le 2p-2$. Hence we have $r(S) = r(\tilde{S}) - 1 = p^2 - 1$, which contradicts to $r(S) \le p^2 - 2$. Thus we see $r_{2p-2} = 0$. We shall check the inequality (12).

$$\begin{aligned}
\text{The left hand side} &= -p^2 r_0 + \sum_{i=1}^{p-2}((p+1)i - p^2)r_i - r_{p-1} \\
&\quad + \sum_{i=p}^{2p-3}((p+1)i - p^2)r_i + p^2 \\
&= \sum_{i=1}^{p-2}((p+1)i - p^2)(r_i - r_{2p-2-i}) - p^2 r_0 - r_{p-1} \\
&\quad - 2\sum_{i=1}^{p-2} r_{2p-2-i} + p^2.
\end{aligned}$$

However, since $r_0 = 1$ and $r_i \ge r_{2p-2-i}$ for $0 \le i \le p-1$ by Lemma 2.6, it follows that the left hand side ≤ 0, which is a contradiction.

2) The semistability of P_2 is proved by arguments similar to P_1 and hence we shall omit the proof. □

Remark 3.4. (1) Unfortunately, it is left open whether Q_1 is semistable with respect to H or not.

(2) Let X be a nonsingular projective curve with genus $g(X) \ge 2$ and $F_*\mathcal{O}_X \xrightarrow{d_0} F_*(\mathcal{O}_X(K_X))$ the de Rham complex and let $P_1 = \mathrm{Im}(d_0)$. Then it

is easily proved by arguments similar to 1) in Theorem 3.3 that P_1 is stable.

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