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Partial primitives, polyprimitives and decompositions of the class of infinitely differentiable functions

Dedicated to Professor Yoshihiro Mizuta on the occasion of his sixtieth birthday

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ABSTRACT. We give two kinds of direct sum decompositions of the class of infinitely differentiabe functions. They are related to the kernel of the higher order partial derivative operator and the polynomial space.

1. Introduction

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space. The points of $\mathbf{R}^n$ are ordered n-tuples $x = (x_1, \dots, x_n)$ where each x_j is a real number. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n-tuple of nonnegative integers, then α is called a multi-index. We let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \dots \alpha_n!$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ the higher partial derivative D^α is defined by

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n},$$

where $D_j = \partial/\partial x_j$ $(j = 1, \dots, n)$. For multi-indices α and β, the notation $\alpha \geq \beta$ stands for $\alpha_j \geq \beta_j$ for $1 \leq j \leq n$. Further, $\alpha > \beta$ means that $\alpha \geq \beta$ and $\alpha_j > \beta_j$ for some j.

The following problem has been studied by several authors. For a function space H and a differential operator T on H, the kernel of T in H is denoted by

$$\mathrm{Ker}\, T|H = \{u \in H : Tu = 0\}.$$

For a subspace V of H, if a subspace W of H satisfies the condition

$$H = V \oplus W,$$

then W is called a complementary space of V in H where the symbol $\oplus$ means

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a direct sum. We write $W = V^{cp}|H$. The problem is to describe a complementary space of $\mathrm{Ker}\, T|H$ in H.

In L^2-spaces, complementary spaces of the kernels of the iterated Laplace operator and the iterated Cauchy-Riemann operator are studied in [1], [2] and [4]. In L^p-spaces, a complementary space of the kernel of the divergence operator is treated in [6]. Moreover, the articles [3] and [5] deal with complementary spaces of the kernels of the Laplace operator and the divergence operator in Sobolev spaces.

In this paper we are concerned with the case $H = C^\infty(\mathbf{R}^n)$ (the class of infinitely differentiable functions on $\mathbf{R}^n$) and $T = D^\alpha$. We investigate $\mathrm{Ker}\, D^\alpha \,|\, C^\infty(\mathbf{R}^n)$ and a complementary space of $\mathrm{Ker}\, D^\alpha \,|\, C^\infty(\mathbf{R}^n)$ in $C^\infty(\mathbf{R}^n)$. For a positive integer ℓ, it is clear that

$$\bigcap\nolimits_{|\alpha|=\ell} \mathrm{Ker}\, D^\alpha \,|\, C^\infty(\mathbf{R}^n) = \mathscr{P}^\ell(\mathbf{R}^n),$$

where $\mathscr{P}^\ell(\mathbf{R}^n)$ is the class of polynomials on $\mathbf{R}^n$ of order $\ell - 1$. We also study a complementary space of $\mathscr{P}^\ell(\mathbf{R}^n)$ in $C^\infty(\mathbf{R}^n)$. In one dimensional case we note the following fact. For a positive integer ℓ, D^ℓ denotes the derivative of order ℓ, and for $f \in C^\infty(\mathbf{R}^1)$ we put

$$K^\ell f(x) = \int_0^x \frac{(x-t)^{\ell-1}}{(\ell-1)!} f(t)dt.$$

By the integral remainder formula for Taylor's theorem we obtain

$$(\mathrm{Ker}\, D^\ell \,|\, C^\infty(\mathbf{R}^1))^{cp} \,|\, C^\infty(\mathbf{R}^1) = (\mathscr{P}^\ell(\mathbf{R}^1))^{cp} \,|\, C^\infty(\mathbf{R}^1) = \{K^\ell f : f \in C^\infty(\mathbf{R}^1)\}.$$

In section 2 for a nonzero multi-index α and a positive integer p with $1 \le p \le \#(M_\alpha)$ (see §2), we introduce quasi-polynomials of order (α, p), and give a necessary and sufficient condition that a quasi-polynomial of order $(\alpha, 1)$ vanishes (Theorem 2.3). Moreover we prove that $\mathrm{Ker}\, D^\alpha \,|\, C^\infty(\mathbf{R}^n)$ is the class $\mathscr{P}^{\alpha,1}(\mathbf{R}^n)$ of quasi-polynomials of order $(\alpha, 1)$ (Theorem 2.5). In section 3 for a nonzero multi-index α we define the class $\mathscr{K}^\alpha(\mathbf{R}^n)$ of partial primitives of order α, and give the direct sum decomposition of $C^\infty(\mathbf{R}^n)$ by $\mathscr{P}^{\alpha,1}(\mathbf{R}^n)$ and $\mathscr{K}^\alpha(\mathbf{R}^n)$ (Theorem 3.5). Hence we obtain that $(\mathrm{Ker}\, D^\alpha \,|\, C^\infty(\mathbf{R}^n))^{cp} \,|\, C^\infty(\mathbf{R}^n) = \mathscr{K}^\alpha(\mathbf{R}^n)$. In section 4 for a positive integer ℓ we introduce the class $\mathscr{K}^\ell(\mathbf{R}^n)$ of polyprimitives of order ℓ, and give the direct sum decomposition of $C^\infty(\mathbf{R}^n)$ by the class $\mathscr{P}^\ell(\mathbf{R}^n)$ and the class $\mathscr{K}^\ell(\mathbf{R}^n)$ (Theorem 4.13). Therefore we obtain that $(\mathscr{P}^\ell(\mathbf{R}^n))^{cp} \,|\, C^\infty(\mathbf{R}^n) = \mathscr{K}^\ell(\mathbf{R}^n)$.

2. Quasi-polynomials and kernels of higher order partial derivatives

First we introduce quasi-polynomials and study their properties. For a subset $\{i_1, \dots, i_p\} \subset \{1, \dots, n\}$ and $x = (x_1, \dots, x_n)$, the symbol $x(x_{i_1}, \dots, x_{i_p};$

$t_1, \dots, t_p)$ means the replacement of x_{i_j} by t_j $(j = 1, \dots, p)$, and the notation $f(\{x_{i_1}, \dots, x_{i_p}\}^c)$ stands for a function of the remaining variables of $\{x_{i_1}, \dots, x_{i_p}\}$ in the variables $x_1, \dots, x_n$. For example, $x(x_1; t_1) = (t_1, x_2, \dots, x_n)$ and $f(\{x_1\}^c) = f(x_2, \dots, x_n)$.

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, we set

$$M_\alpha = \{j \in \{1, \dots, n\} : \alpha_j \neq 0\}.$$

We denote by $\#(M_\alpha)$ the number of elements of M_α.

Let α be a nonzero multi-index and $1 \le p \le \#(M_\alpha)$. The notation $M_{\alpha,p}$ denotes the collection of subsets of M_α which have p elements. If a function $P(x)$ has the following form

$$P(x) = \sum_{\{i_1,\dots,i_p\} \in M_{\alpha,p}} \sum_{j_1=0}^{\alpha_{i_1}-1} \cdots \sum_{j_p=0}^{\alpha_{i_p}-1} v_{i_1,\dots,i_p;j_1,\dots,j_p}(\{x_{i_1}, \dots, x_{i_p}\}^c) x_{i_1}^{j_1} \dots x_{i_p}^{j_p},$$

where $v_{i_1,\dots,i_p;j_1,\dots,j_p} \in C^\infty(\mathbf{R}^{n-p})$, then we call $P(x)$ a quasi-polynomial of order (α, p). We denote by $\mathscr{P}^{\alpha,p}$ the set of all quasi-polynomials of order (α, p). We note that $\mathscr{P}^{\alpha,p} \supset \mathscr{P}^{\alpha,q}$ for $1 \le p \le q \le \#(M_\alpha)$.

We put

$$\mathbf{R}^{n,j} = \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_j = 0\}, \qquad j = 1, \dots, n.$$

For a nonzero multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ and $1 \le p \le \#(M_\alpha)$, we say that a function v satisfies condition $C_{\alpha,p}$, if for any $\{i_1, \dots, i_p\} \in M_{\alpha,p}$ and $0 \le j_1 \le \alpha_{i_1} - 1, \dots, 0 \le j_p \le \alpha_{i_p} - 1$,

$$D_{i_1}^{j_1} \dots D_{i_p}^{j_p} v(x) = 0 \qquad \text{for } x \in \textstyle\bigcap_{k=1}^{p} \mathbf{R}^{n,i_k}.$$

Lemma 2.1. *Let α be a nonzero multi-index with $\#(M_\alpha) \ge 2$ and $1 \le p \le \#(M_\alpha) - 1$. If a quasi-polynomial P of order (α, p) satisfies condition $C_{\alpha,p}$, then P is a quasi-polynomial of order $(\alpha, p+1)$.*

Proof. Let

$$P(x) = \sum_{\{i_1,\dots,i_p\} \in M_{\alpha,p}} \sum_{j_1=0}^{\alpha_{i_1}-1} \cdots \sum_{j_p=0}^{\alpha_{i_p}-1} v_{i_1,\dots,i_p;j_1,\dots,j_p}(\{x_{i_1}, \dots, x_{i_p}\}^c) x_{i_1}^{j_1} \dots x_{i_p}^{j_p}, \tag{2.1}$$

where $v_{i_1,\dots,i_p;j_1,\dots,j_p} \in C^\infty(\mathbf{R}^{n-p})$. We show that for each $\{k_1, \dots, k_p\} \in M_{\alpha,p}$ and $0 \le \ell_1 \le \alpha_{k_1} - 1, \dots, 0 \le \ell_p \le \alpha_{k_p} - 1$, the function $v_{k_1,\dots,k_p;\ell_1,\dots,\ell_p}(\{x_{k_1}, \dots, x_{k_p}\}^c)$ becomes the following form:

$$(2.2)\qquad v_{k_1,\dots,k_p;\ell_1,\dots,\ell_p}(\{x_{k_1},\dots,x_{k_p}\}^c)$$

$$= \sum_{q\in M_\alpha-\{k_1,\dots,k_p\}} \sum_{j_q=0}^{\alpha_q-1} v^{q;j_q}_{k_1,\dots,k_p;\ell_1,\dots,\ell_p}(\{x_{k_1},\dots,x_{k_p},x_q\}^c)x_q^{j_q},$$

where $v^{q;j_q}_{k_1,\dots,k_p;\ell_1,\dots,\ell_p}\in C^\infty(\mathbf{R}^{n-p-1})$. We fix $\{k_1,\dots,k_p\}\in M_{\alpha,p}$ and $0\le\ell_1\le\alpha_{k_1}-1,\dots,0\le\ell_p\le\alpha_{k_p}-1$. We divide $P(x)$ as follows:

$$P(x)=\sum_{j_1=0}^{\alpha_{k_1}-1}\cdots\sum_{j_p=0}^{\alpha_{k_p}-1} v_{k_1,\dots,k_p;j_1,\dots,j_p}(\{x_{k_1},\dots,x_{k_p}\}^c)x_{k_1}^{j_1}\dots x_{k_p}^{j_p}$$

$$+\sum_{\{i_1,\dots,i_p\}\ne\{k_1,\dots,k_p\}}\sum_{j_1=0}^{\alpha_{i_1}-1}\cdots\sum_{j_p=0}^{\alpha_{i_p}-1} v_{i_1,\dots,i_p;j_1,\dots,j_p}(\{x_{i_1},\dots,x_{i_p}\}^c)x_{i_1}^{j_1}\dots x_{i_p}^{j_p}$$

$$=P_1(x)+P_2(x).$$

We denote by $u(x)|_{\bigcap_{j=1}^p\mathbf{R}^{n,k_j}}$ the restriction of a function $u(x)$ on $\mathbf{R}^n$ to $\bigcap_{j=1}^p\mathbf{R}^{n,k_j}$. Namely, $u(x)|_{\bigcap_{j=1}^p\mathbf{R}^{n,k_j}}=u(x(x_{k_1},\dots,x_{k_p};0,\dots,0))$. By condition $C_{\alpha,p}$ we have

$$(2.3)\qquad 0=D_{k_1}^{\ell_1}\dots D_{k_p}^{\ell_p}P(x)|_{\bigcap_{j=1}^p\mathbf{R}^{n,k_j}}$$

$$=D_{k_1}^{\ell_1}\dots D_{k_p}^{\ell_p}P_1(x)|_{\bigcap_{j=1}^p\mathbf{R}^{n,k_j}}+D_{k_1}^{\ell_1}\dots D_{k_p}^{\ell_p}P_2(x)|_{\bigcap_{j=1}^p\mathbf{R}^{n,k_j}}$$

$$=I+J.$$

For I we see that

$$(2.4)\qquad I=\sum_{j_1=0}^{\alpha_{k_1}-1}\cdots\sum_{j_p=0}^{\alpha_{k_p}-1} v_{k_1,\dots,k_p;j_1,\dots,j_p}(\{x_{k_1},\dots,x_{k_p}\}^c)$$

$$\times D_{k_1}^{\ell_1}\dots D_{k_p}^{\ell_p}(x_{k_1}^{j_1}\dots x_{k_p}^{j_p})|_{\bigcap_{j=1}^p\mathbf{R}^{n,k_j}}$$

$$=\sum_{j_1=\ell_1}^{\alpha_{k_1}-1}\cdots\sum_{j_p=\ell_p}^{\alpha_{k_p}-1} v_{k_1,\dots,k_p;j_1,\dots,j_p}(\{x_{k_1},\dots,x_{k_p}\}^c)$$

$$\times (j_1)_{\ell_1}\dots(j_p)_{\ell_p}x_{k_1}^{j_1-\ell_1}\dots x_{k_p}^{j_p-\ell_p}|_{\bigcap_{j=1}^p\mathbf{R}^{n,k_j}}$$

$$=\ell_1!\dots\ell_p!v_{k_1,\dots,k_p;\ell_1,\dots,\ell_p}(\{x_{k_1},\dots,x_{k_p}\}^c),$$

where

$$(j_i)_{\ell_i}=j_i(j_i-1)\dots(j_i-\ell_i+1),\qquad i=1,\dots,p.$$

For J we have

$$J = \sum_{\{i_1,\dots,i_p\}\neq\{k_1,\dots,k_p\}} \sum_{j_1=0}^{\alpha_{i_1}-1} \cdots \sum_{j_p=0}^{\alpha_{i_p}-1}$$

$$\times D_{k_1}^{\ell_1} \dots D_{k_p}^{\ell_p}(v_{i_1,\dots,i_p;j_1,\dots,j_p}(\{x_{i_1},\dots,x_{i_p}\}^c)x_{i_1}^{j_1}\dots x_{i_p}^{j_p})|_{\bigcap_{j=1}^p \mathbf{R}^{n,k_j}}$$

$$= \sum_{\{i_1,\dots,i_p\}\neq\{k_1,\dots,k_p\}} \sum_{j_1=0}^{\alpha_{i_1}-1} \cdots \sum_{j_p=0}^{\alpha_{i_p}-1} J_{i_1,\dots,i_p;j_1,\dots,j_p}^{k_1,\dots,k_p;\ell_1,\dots,\ell_p}.$$

In order to calculate $J_{i_1,\dots,i_p;j_1,\dots,j_p}^{k_1,\dots,k_p;\ell_1,\dots,\ell_p}$, we put $t = \#\{\{i_1,\dots,i_p\}\cap\{k_1,\dots,k_p\}\}$. Since $\{i_1,\dots,i_p\}\neq\{k_1,\dots,k_p\}$, we note that $0 \le t \le p-1$. We may assume that $\{i_1,\dots,i_p\}\cap\{k_1,\dots,k_p\} = \{i_1,\dots,i_t\}$ and $i_{t+1} < \cdots < i_p$. We note that since $0 \le t \le p-1$, i_{t+1} exists and is uniquely determined. Further, let $i_1 = k_{n_1},\dots,i_t = k_{n_t}$ and $\{1,\dots,p\} - \{n_1,\dots,n_t\} = \{n_{t+1},\dots,n_p\}$. Then we have

$$J_{i_1,\dots,i_p;j_1,\dots,j_p}^{k_1,\dots,k_p;\ell_1,\dots,\ell_p} = D_{k_{n_{t+1}}}^{\ell_{n_{t+1}}} \dots D_{k_{n_p}}^{\ell_{n_p}} v_{i_1,\dots,i_p;j_1,\dots,j_p}(\{x_{i_1},\dots,x_{i_p}\}^c)$$

$$\times D_{i_1}^{\ell_{n_1}} \dots D_{i_t}^{\ell_{n_t}}(x_{i_1}^{j_1}\dots x_{i_p}^{j_p})|_{\bigcap_{j=1}^p \mathbf{R}^{n,k_j}}.$$

If $(j_1,\dots,j_t)\neq(\ell_{n_1},\dots,\ell_{n_t})$, then

$$D_{i_1}^{\ell_{n_1}} \dots D_{i_t}^{\ell_{n_t}}(x_{i_1}^{j_1}\dots x_{i_p}^{j_p})|_{\bigcap_{j=1}^p \mathbf{R}^{n,k_j}} = 0$$

and hence

$$J_{i_1,\dots,i_p;j_1,\dots,j_p}^{k_1,\dots,k_p;\ell_1,\dots,\ell_p} = 0.$$

When $j_1 = \ell_{n_1},\dots,j_t = \ell_{n_t}$, we have

$$J_{i_1,\dots,i_p;\ell_{n_1},\dots,\ell_{n_t},j_{t+1},\dots,j_p}^{k_1,\dots,k_p;\ell_1,\dots,\ell_p} = D_{k_{n_{t+1}}}^{\ell_{n_{t+1}}} \dots D_{k_{n_p}}^{\ell_{n_p}} v_{i_1,\dots,i_p;\ell_{n_1},\dots,\ell_{n_t},j_{t+1},\dots,j_p}(\{x_{i_1},\dots,x_{i_p}\}^c)$$

$$\times \ell_{n_1}!\dots\ell_{n_t}! x_{i_{t+1}}^{j_{t+1}}\dots x_{i_p}^{j_p}|_{\bigcap_{j=1}^p \mathbf{R}^{n,k_j}}.$$

Since the function

$$D_{k_{n_{t+1}}}^{\ell_{n_{t+1}}} \dots D_{k_{n_p}}^{\ell_{n_p}} v_{i_1,\dots,i_p;\ell_{n_1},\dots,\ell_{n_t},j_{t+1},\dots,j_p}(\{x_{i_1},\dots,x_{i_p}\}^c)\ell_{n_1}!\dots\ell_{n_t}! x_{i_{t+2}}^{j_{t+2}}\dots x_{i_p}^{j_p}|_{\bigcap_{j=1}^p \mathbf{R}^{n,k_j}}$$

does not contain $x_{k_1},\dots,x_{k_p},x_{i_{t+1}}$ as variables, we can replace it by

$$w_{i_1,\dots,i_p;\ell_{n_1},\dots,\ell_{n_t},j_{t+1},\dots,j_p}^{k_1,\dots,k_p;\ell_1,\dots,\ell_p}(\{x_{k_1},\dots,x_{k_p},x_{i_{t+1}}\}^c).$$

Therefore we have

$$J = \sum_{\{i_1,\dots,i_p\}\neq\{k_1,\dots,k_p\}} \sum_{j_{t+1}=0}^{\alpha_{i_{t+1}}-1} \cdots \sum_{j_p=0}^{\alpha_{i_p}-1} w_{i_1,\dots,i_p;\ell_{n_1},\dots,\ell_{n_t},j_{t+1},\dots,j_p}^{k_1,\dots,k_p;\ell_1,\dots,\ell_p}(\{x_{k_1},\dots,x_{k_p},x_{i_{t+1}}\}^c)x_{i_{t+1}}^{j_{t+1}}.$$

Since $i_{t+1} \in M_\alpha - \{k_1, \dots, k_p\}$ and $0 \le j_{t+1} \le \alpha_{i_{t+1}} - 1$, J becomes the following form:

$$J = \sum_{q \in M_\alpha - \{k_1,\dots,k_p\}} \sum_{j_q=0}^{\alpha_q - 1} u^{q;j_q}_{k_1,\dots,k_p;\ell_1,\dots,\ell_p}(\{x_{k_1},\dots,x_{k_p},x_q\}^c) x_q^{j_q}, \tag{2.5}$$

where

$$u^{q;j_q}_{k_1,\dots,k_p;\ell_1,\dots,\ell_p}(\{x_{k_1},\dots,x_{k_p},x_q\}^c)$$
$$= \sum_{\{i_1,\dots,i_p\} \neq \{k_1,\dots,k_p\},\, i_{t+1}=q} \sum_{j_{t+2}=0}^{\alpha_{i_{t+2}}-1} \cdots \sum_{j_p=0}^{\alpha_{i_p}-1} w^{k_1,\dots,k_p;\ell_1,\dots,\ell_p}_{i_1,\dots,i_p;\ell_{n_1},\dots,\ell_{n_t},j_q,j_{t+2},\dots,j_p}$$
$$(\{x_{k_1},\dots,x_{k_p},x_{i_{t+1}}\}^c).$$

By (2.3), (2.4) and (2.5) we have

$$v_{k_1,\dots,k_p;\ell_1,\dots,\ell_p}(\{x_{k_1},\dots,x_{k_p}\}^c)$$
$$= -\frac{1}{\ell_1!\dots\ell_p!} \sum_{q \in M_\alpha - \{k_1,\dots,k_p\}} \sum_{j_q=0}^{\alpha_q-1} u^{q;j_q}_{k_1,\dots,k_p;\ell_1,\dots,\ell_p}(\{x_{k_1},\dots,x_{k_p},x_q\}^c) x_q^{j_q}$$
$$= \sum_{q \in M_\alpha - \{k_1,\dots,k_p\}} \sum_{j_q=0}^{\alpha_q-1} v^{q;j_q}_{k_1,\dots,k_p;\ell_1,\dots,\ell_p}(\{x_{k_1},\dots,x_{k_p},x_q\}^c) x_q^{j_q},$$

where we put

$$v^{q;j_q}_{k_1,\dots,k_p;\ell_1,\dots,\ell_p}(\{x_{k_1},\dots,x_{k_p},x_q\}^c) = -\frac{1}{\ell_1!\dots\ell_p!} u^{q;j_q}_{k_1,\dots,k_p;\ell_1,\dots,\ell_p}(\{x_{k_1},\dots,x_{k_p},x_q\}^c).$$

Thus we obtain (2.2). We substitute (2.2) into (2.1). Then we have

$$P(x) = \sum_{\{i_1,\dots,i_p\} \in M_{\alpha,p}} \sum_{j_1=0}^{\alpha_{i_1}-1} \cdots \sum_{j_p=0}^{\alpha_{i_p}-1}$$
$$\times \left(\sum_{q \in M_\alpha - \{i_1,\dots,i_p\}} \sum_{j_q=0}^{\alpha_q-1} v^{q;j_q}_{i_1,\dots,i_p;j_1,\dots,j_p}(\{x_{i_1},\dots,x_{i_p},x_q\}^c) x_q^{j_q} \right) x_{i_1}^{j_1} \dots x_{i_p}^{j_p}$$
$$= \sum_{\{i_1,\dots,i_{p+1}\} \in M_{\alpha,p+1}} \sum_{j_1=0}^{\alpha_{i_1}-1} \cdots \sum_{j_{p+1}=0}^{\alpha_{i_{p+1}}-1} v_{i_1,\dots,i_{p+1};j_1,\dots,j_{p+1}}(\{x_{i_1},\dots,x_{i_{p+1}}\}^c) x_{i_1}^{j_1} \dots x_{i_{p+1}}^{j_{p+1}}$$

by putting

$$v_{i_1,\dots,i_{p+1};j_1,\dots,j_{p+1}}(\{x_{i_1},\dots,x_{i_{p+1}}\}^c)$$
$$= \sum_{k=1}^{p+1} v^{i_k;j_k}_{i_1,\dots,\widehat{i_k},\dots,i_{p+1};j_1,\dots,\widehat{j_k},\dots,j_{p+1}}(\{x_{i_1},\dots,\widehat{x_{i_k}},\dots,x_{i_{p+1}},x_{i_k}\}^c),$$

where the symbol $\widehat{\ }$ indicates that the variable underneath is deleted. Since $v_{i_1,\dots,i_{p+1};j_1,\dots,j_{p+1}} \in C^\infty(\mathbf{R}^{n-p-1})$, we obtain the lemma.

Lemma 2.2. *If a quasi-polynomial P of order $(\alpha, \#(M_\alpha))$ satisfies condition $C_{\alpha,\#(M_\alpha)}$, then $P = 0$.*

Proof. Let $\#(M_\alpha) = m$, $M_\alpha = \{r_1,\dots,r_m\}$ and

$$P(x) = \sum_{j_1=0}^{\alpha_{r_1}-1} \cdots \sum_{j_m=0}^{\alpha_{r_m}-1} v_{r_1,\dots,r_m;j_1,\dots,j_m}(\{x_{r_1},\dots,x_{r_m}\}^c) x_{r_1}^{j_1}\dots x_{r_m}^{j_m}.$$

For each $\ell_1,\dots,\ell_m$ with $0 \le \ell_1 \le \alpha_{r_1} - 1, \dots, 0 \le \ell_m \le \alpha_{r_m} - 1$, by condition $C_{\alpha,m}$ we have

$$0 = D_{r_1}^{\ell_1}\dots D_{r_m}^{\ell_m} P(x)|_{\bigcap_{j=1}^m \mathbf{R}^{n,r_j}}$$
$$= \sum_{j_1=0}^{\alpha_{r_1}-1} \cdots \sum_{j_m=0}^{\alpha_{r_m}-1} v_{r_1,\dots,r_m;j_1,\dots,j_m}(\{x_{r_1},\dots,x_{r_m}\}^c) D_{r_1}^{\ell_1}\dots D_{r_m}^{\ell_m}(x_{r_1}^{j_1}\dots x_{r_m}^{j_m})|_{\bigcap_{j=1}^m \mathbf{R}^{n,r_j}}$$
$$= \ell_1!\dots\ell_m! v_{r_1,\dots,r_m;\ell_1,\dots,\ell_m}(\{x_{r_1},\dots,x_{r_m}\}^c).$$

This implies that $v_{r_1,\dots,r_m;\ell_1,\dots,\ell_m} = 0$, and hence $P = 0$.

By Lemmas 2.1 and 2.2 we see that

Theorem 2.3. *Let P be a quasi-polynomial of order $(\alpha, 1)$. Then $P = 0$ if and only if P satisfies conditions $C_{\alpha,p}$ for $1 \le p \le \#(M_\alpha)$.*

Next we determine $\operatorname{Ker} D^\alpha \mid C^\infty(\mathbf{R}^n)$. The symbol e_j stands for the multi-index (or the point of $\mathbf{R}^n$) which has 1 in the jth spot and 0 everywhere else $(j = 1,\dots,n)$. For a positive integer ℓ we denote by $C^\ell(\mathbf{R}^n)$ the class of differentiable functions on $\mathbf{R}^n$ up to the order ℓ and continuous with their derivatives. For $f \in C^\infty(\mathbf{R}^n)$ we set

$$I_j f(x) = \int_0^{x_j} f(x(x_j;t))dt, \qquad j = 1,\dots,n.$$

We need the following lemma which is obtained by elementary calculations.

Lemma 2.4. *Let $f \in C^\infty(\mathbf{R}^n)$. Then for any positive integer ℓ, $I_j f \in C^\ell(\mathbf{R}^n)$ and for $|\alpha| = \ell$*

$$D^\alpha I_j f = \begin{cases} D^{\alpha-e_j} f, & \alpha_j \geq 1, \\ I_j D^\alpha f, & \alpha_j = 0. \end{cases}$$

In particular, $I_j f \in C^\infty(\mathbf{R}^n)$.

We prove

THEOREM 2.5. *Let α be a nonzero multi-index and $v \in C^\infty(\mathbf{R}^n)$. Then $D^\alpha v = 0$ if and only if v is a quasi-polynomial of order $(\alpha, 1)$.*

PROOF. ("if" part) Let $M_\alpha = \{r_1, \dots, r_m\}$ and

$$v(x) = \sum_{i=1}^{m} \sum_{\ell_i=0}^{\alpha_{r_i}-1} v_{r_i;\ell_i}(\{x_{r_i}\}^c) x_{r_i}^{\ell_i},$$

where $v_{r_i;\ell_i}(\{x_{r_i}\}^c) \in C^\infty(\mathbf{R}^{n-1})$. Then we have

$$\begin{aligned} D^\alpha v(x) &= \sum_{i=1}^{m} \sum_{\ell_i=0}^{\alpha_{r_i}-1} D^\alpha (v_{r_i;\ell_i}(\{x_{r_i}\}^c) x_{r_i}^{\ell_i}) \\ &= \sum_{i=1}^{m} \sum_{\ell_i=0}^{\alpha_{r_i}-1} (D_{r_1}^{\alpha_{r_1}} \dots \widehat{D_{r_i}^{\alpha_{r_i}}} \dots D_{r_m}^{\alpha_{r_m}} v_{r_i;\ell_i}(\{x_{r_i}\}^c)) D_{r_i}^{\alpha_{r_i}} x_{r_i}^{\ell_i} \\ &= 0 \end{aligned}$$

because $D_{r_i}^{\alpha_{r_i}} x_{r_i}^{\ell_i} = 0$.

("only if" part) We must prove the assertion that if $D^\alpha v = 0$, then v is a quasi-polynomial of order $(\alpha, 1)$. We show the assertion by induction with respect to α. We consider the case $\alpha = e_j$ $(j = 1, \dots, n)$. Let $D_j v = 0$. Then

$$\begin{aligned} v(x) &= \int_0^{x_j} D_j v(x(x_j; t)) dt + v(x(x_j; 0)) \\ &= v(x(x_j; 0)) = v_{j;0}(\{x_j\}^c). \end{aligned}$$

Since $v_{j;0}(\{x_j\}^c) \in C^\infty(\mathbf{R}^{n-1})$, the assertion holds for $\alpha = e_j$. We assume that the assertion holds for α. We show that the assertion holds for $\alpha + e_j$ $(j = 1, \dots, n)$. Let $M_\alpha = \{r_1, \dots, r_m\}$. First we consider the case $j \in M_\alpha$. Let $j = r_s$ and $D^{\alpha + e_{r_s}} v = 0$. We put $u = D^{e_{r_s}} v$. Then $D^\alpha u = 0$. By the assumption of induction we have

$$u(x) = \sum_{i=1}^{m} \sum_{\ell_i=0}^{\alpha_{r_i-1}} u_{r_i;\ell_i}(\{x_{r_i}\}^c) x_{r_i}^{\ell_i},$$

where $u_{r_i;\ell_i}(\{x_{r_i}\}^c) \in C^\infty(\mathbf{R}^{n-1})$. Hence we have

$$\begin{aligned}
v(x) &= \int_0^{x_{r_s}} D_{r_s} v(x(x_{r_s};t))dt + v(x(x_{r_s};0)) \\
&= \int_0^{x_{r_s}} u(x(x_{r_s};t))dt + v(x(x_{r_s};0)) \\
&= \sum_{i=1,\dots,m,\, i\neq s} \sum_{\ell_i=0}^{\alpha_{r_i}-1} \int_0^{x_{r_s}} u_{r_i;\ell_i}(\{x_{r_i}\}^c(x_{r_s};t)) x_{r_i}^{\ell_i}\, dt \\
&\quad + \sum_{\ell_s=0}^{\alpha_{r_s}-1} u_{r_s;\ell_s}(\{x_{r_s}\}^c) \int_0^{x_{r_s}} t^{\ell_s}\, dt + v(x(x_{r_s};0)).
\end{aligned}$$

We put

$$v_{r_i;\ell_i}(\{x_{r_i}\}^c) = \int_0^{x_{r_s}} u_{r_i;\ell_i}(\{x_{r_i}\}^c(x_{r_s};t))dt, \qquad i \neq s,\ \ell_i = 0,\dots,\alpha_{r_i}-1,$$

$$v_{r_s;\ell_s}(\{x_{r_s}\}^c) = \frac{u_{r_s;\ell_s-1}(\{x_{r_s}\}^c)}{\ell_s}, \qquad \ell_s = 1,\dots,\alpha_{r_s}$$

and

$$v_{r_s;0}(\{x_{r_s}\}^c) = v(x(x_{r_s};0)).$$

Then we have

$$v(x) = \sum_{i=1,\dots,m,\, i\neq s} \sum_{\ell_i=0}^{\alpha_{r_i}-1} v_{r_i;\ell_i}(\{x_{r_i}\}^c) x_{r_i}^{\ell_i} + \sum_{\ell_s=0}^{\alpha_{r_s}} v_{r_s;\ell_s}(\{x_{r_s}\}^c) x_{r_s}^{\ell_s}.$$

Since $v_{r_i;\ell_i} \in C^\infty(\mathbf{R}^{n-1})$ by Lemma 2.4, the assertion holds for $\alpha + e_{r_s}$ $(s = 1,\dots,m)$. Next we consider the case $j \notin M_\alpha$. Let $D^{\alpha+e_j} v = 0$. We put $u = D^{e_j} v$. Then $D^\alpha u = 0$. By the assumption of induction we have

$$u(x) = \sum_{i=1}^{m} \sum_{\ell_i=0}^{\alpha_{r_i}-1} u_{r_i;\ell_i}(\{x_{r_i}\}^c) x_{r_i}^{\ell_i},$$

where $u_{r_i;\ell_i}(\{x_{r_i}\}^c) \in C^\infty(\mathbf{R}^{n-1})$. Since $j \notin M_\alpha$, we see that

$$\begin{aligned}
v(x) &= \int_0^{x_j} D_j v(x(x_j;t))dt + v(x(x_j;0)) \\
&= \int_0^{x_j} u(x(x_j;t))dt + v(x(x_j;0)) \\
&= \sum_{i=1}^{m} \sum_{\ell_i=0}^{\alpha_{r_i}-1} \int_0^{x_j} u_{r_i;\ell_i}(\{x_{r_i}\}^c(x_j;t)) x_{r_i}^{\ell_i}\, dt + v(x(x_j;0)).
\end{aligned}$$

We put

$$v_{r_i;\ell_i}(\{x_{r_i}\}^c) = \int_0^{x_j} u_{r_i;\ell_i}(\{x_{r_i}\}^c(x_j;t))dt, \qquad i = 1,\dots,m,\ \ell_i = 0,\dots,\alpha_{r_i}-1$$

and

$$v_{j;0}(\{x_j\}^c) = v(x(x_j;0)).$$

Then

$$v(x) = \sum_{i=1}^{m}\sum_{\ell_i=0}^{\alpha_{r_i}-1} v_{r_i;\ell_i}(\{x_{r_i}\}^c)x_{r_i}^{\ell_i} + v_{j;0}(\{x_j\}^c).$$

Since $v_{r_i;\ell_i}(\{x_{r_i}\}^c) \in C^\infty(\mathbf{R}^{n-1})$ by Lemma 2.4, the assertion holds for $\alpha + e_j$ with $j \notin M_\alpha$. We complete the proof of the theorem.

3. Partial primitives and the first decomposition of $C^\infty(\mathbf{R}^n)$

In this section we introduce partial primitives of order α, and give the first direct sum decomposition of $C^\infty(\mathbf{R}^n)$.

For a multi-index α with $M_\alpha = \{r_1,\dots,r_m\}$ and $f \in C^\infty(\mathbf{R}^n)$ we put

$$K^\alpha f(x) = \int_0^{x_{r_1}} \cdots \int_0^{x_{r_m}} \frac{(x_{r_1}-t_1)^{\alpha_{r_1}-1}\dots(x_{r_m}-t_m)^{\alpha_{r_m}-1}}{(\alpha_{r_1}-1)!\cdots(\alpha_{r_m}-1)!} \times f(x(x_{r_1},\dots,x_{r_m};t_1,\dots,t_m))dt_1\dots dt_m.$$

In order to investigate properties of $K^\alpha f$, we introduce the following operators: For $f \in C^\infty(\mathbf{R}^n)$, we set $I_j^0 = f$, $I_j^1 f = I_j f$ and $I_j^\ell f = I_j(I_j^{\ell-1}f)$, $\ell = 2,3,\dots,$ and for a multi-index $\alpha = (\alpha_1,\dots,\alpha_n)$ we put

$$I^\alpha f = I_1^{\alpha_1}\dots I_n^{\alpha_n} f.$$

The order of $I_1^{\alpha_1}\dots I_n^{\alpha_n}$ is irrelevant since $I_k I_j f = I_j I_k f$ by Fubini's theorem.

By Lemma 2.4 we see that

LEMMA 3.1. *Let α, β be multi-indices with $\alpha \geq \beta$ and $f \in C^\infty(\mathbf{R}^n)$. Then $I^\alpha f \in C^\infty(\mathbf{R}^n)$ and $D^\beta I^\alpha f = I^{\alpha-\beta} f$.*

By Fubini's theorem we have

LEMMA 3.2. *Let $f \in C^\infty(\mathbf{R}^n)$. Then*

$$K^\alpha f = I^\alpha f.$$

The properties of $K^\alpha f$ are given by the following proposition.

Proposition 3.3. *Let* $f \in C^\infty(\mathbf{R}^n)$. *Then*

(i) $K^\alpha f \in C^\infty(\mathbf{R}^n)$.

(ii) *For multi-indices* α *and* β *with* $\alpha \geq \beta$, $D^\beta K^\alpha f = K^{\alpha-\beta} f$. *In particular*, $D^\alpha K^\alpha f = f$.

(iii) *For multi-indices* α *and* β *with* $\alpha > \beta$, $D^\beta K^\alpha f(x) = 0$ *for* $x \in \bigcup_{j \in M_{\alpha-\beta}} \mathbf{R}^{n,j}$.

Proof. The assertions (i) and (ii) follow from Lemmas 3.1 and 3.2. Further, the definition of K^α and (ii) give (iii).

Noting that $K^\alpha f$ satisfies the property (iii) of Proposition 3.3, we introduce the following definition: For a nonzero multi-index α we say that a function v satisfies condition B_α if for any multi-index β with $\beta < \alpha$, $D^\beta v(x) = 0$ for $x \in \bigcup_{j \in M_{\alpha-\beta}} \mathbf{R}^{n,j}$. Moreover we denote by $\mathscr{K}^\alpha(R^n)$ the image of K^α. Namely

$$\mathscr{K}^\alpha(\mathbf{R}^n) = \{K^\alpha f : f \in C^\infty(\mathbf{R}^n)\}.$$

We state a relation among condition B_α, conditions $C_{\alpha,p}$ for $1 \leq p \leq \#(M_\alpha)$ and $\mathscr{K}^\alpha(\mathbf{R}^n)$.

Proposition 3.4. *Let* α *be a nonzero multi-index. For* $v \in C^\infty(\mathbf{R}^n)$, *the following three conditions are equivalent.*

(i) v *satisfies condition* B_α.

(ii) v *satisfies conditions* $C_{\alpha,p}$ *for* $1 \leq p \leq \#(M_\alpha)$.

(iii) $v \in \mathscr{K}^\alpha(\mathbf{R}^n)$.

Proof. (i) $\Rightarrow$ (ii). Let $1 \leq p \leq \#(M_\alpha)$, $\{i_1, \dots, i_p\} \in M_{\alpha,p}$ and $0 \leq j_1 \leq \alpha_{i_1} - 1, \dots, 0 \leq j_p \leq \alpha_{i_p} - 1$. Let β be the multi-index which has j_k in the i_kth spot ($k = 1, \dots, p$) and 0 everywhere else. Since $\beta < \alpha$ and $M_{\alpha-\beta} \supset \{i_1, \dots, i_p\}$, the condition (i) implies $D^\beta v(x) = D_{i_1}^{j_1} \dots D_{i_p}^{j_p} v(x) = 0$ for $x \in \bigcup_{k=1}^p \mathbf{R}^{n,i_k}$. Since $\bigcup_{k=1}^p \mathbf{R}^{n,i_k} \supset \bigcap_{k=1}^p \mathbf{R}^{n,i_k}$, we obtain (ii).

(ii) $\Rightarrow$ (iii). We put $f = D^\alpha v$. Then, by Proposition 3.3 (ii) we have

$$D^\alpha(v - K^\alpha f) = f - f = 0.$$

Hence Theorem 2.5 implies that $P = v - K^\alpha f \in \mathscr{P}^{\alpha,1}$. By Proposition 3.3 (iii) and the above proof (i) $\Rightarrow$ (ii), $K^\alpha f$ satisfies the conditions $C_{\alpha,p}$ for $1 \leq p \leq \#(M_\alpha)$. Moreover, since v satisfies the condition $C_{\alpha,p}$ for $1 \leq p \leq \#(M_\alpha)$ by the assumption (ii), P satisfies the condition $C_{\alpha,p}$ for $1 \leq p \leq \#(M_\alpha)$. Therefore, by Theorem 2.3 we see that $P = 0$. Consequently $v = K^\alpha f \in \mathscr{K}^\alpha(\mathbf{R}^n)$.

(iii) $\Rightarrow$ (i). This follows from Proposition 3.3 (iii).

Now we give the first direct sum decomposition of $C^\infty(\mathbf{R}^n)$.

Theorem 3.5. $C^\infty(\mathbf{R}^n) = \mathscr{K}^\alpha(\mathbf{R}^n) \oplus \mathscr{P}^{\alpha,1}(\mathbf{R}^n)$.

PROOF. For $v \in C^{\infty}(\mathbf{R}^n)$, we put $f = D^{\alpha}v$. Then by the same argument as in (ii) $\Rightarrow$ (iii) part of Proposition 3.4, we see that $P = v - K^{\alpha}f \in \mathscr{P}^{\alpha,1}$. Hence $v = K^{\alpha}f + P \in \mathscr{K}^{\alpha}(\mathbf{R}^n) + \mathscr{P}^{\alpha,1}(\mathbf{R}^n)$. Moreover, let $v \in \mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \mathscr{P}^{\alpha,1}(\mathbf{R}^n)$. Since $v \in \mathscr{K}^{\alpha}(\mathbf{R}^n)$, by Proposition 3.4 v satisfies the conditions $C_{\alpha,p}$ for $1 \le p \le \#(M_{\alpha})$. Further, since $v \in \mathscr{P}^{\alpha,1}$, Theorem 2.3 implies that $v = 0$. Thus we obtain the theorem.

4. Polyprimitives and the second decomposition of $C^{\infty}(\mathbf{R}^n)$

By Theorem 2.5 we see that

$$\bigcap_{|\alpha|=\ell} \mathscr{P}^{\alpha,1}(\mathbf{R}^n) = \bigcap_{|\alpha|=\ell} \{u \in C^{\infty}(\mathbf{R}^n) : D^{\alpha}u = 0\} = \mathscr{P}^{\ell}(\mathbf{R}^n). \tag{4.1}$$

In section 3, we gave the direct sum decomposition of $C^{\infty}(R^n)$ by $\mathscr{P}^{\alpha,1}(\mathbf{R}^n)$ and $\mathscr{K}^{\alpha}(\mathbf{R}^n)$. In this section we study a direct sum decomposition of $C^{\infty}(\mathbf{R}^n)$ by $\mathscr{P}^{\ell}(\mathbf{R}^n)$ and its complementary space.

LEMMA 4.1. *Let $u \in C^{\infty}(R^n)$ and $j \in \{1, \dots, n\}$. If $u(x) = 0$ on $\mathbf{R}^{n,j}$, then $D^{\beta}u(x) = 0$ and $I^{\beta}u(x) = 0$ on $R^{n,j}$ for β with $j \notin M_{\beta}$.*

PROOF. Let $x \in \mathbf{R}^{n,j}$. Then for $k \ne j$ we have $x + he_k \in \mathbf{R}^{n,j}$ and $x(x_k, t) \in \mathbf{R}^{n,j}$. Hence the condition $u = 0$ on $\mathbf{R}^{n,j}$ implies that $D_k u(x) = I_k u(x) = 0$. Next let $M_{\beta} = \{j_1, \dots, j_t\}$. Since $D^{\beta} = D_{j_1}^{\beta_{j_1}} \dots D_{j_t}^{\beta_{j_t}}$, $I^{\beta} = I_{j_1}^{\beta_{j_1}} \dots I_{j_t}^{\beta_{j_t}}$ and $j \ne j_i$ $(i = 1, \dots, t)$, by repeating the above argument we obtain the lemma.

For an n-tuple $\delta = (\delta_1, \dots, \delta_n)$ of integers, we define

$$\delta^+ = (\max(\delta_1, 0), \dots, \max(\delta_n, 0)).$$

Then δ^+ is a multi-index.

LEMMA 4.2. *Let $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ be multi-indices and $u \in C^{\infty}(\mathbf{R}^n)$. Then*

(i) $D^{\alpha}I^{\beta}u = I^{(\beta-\alpha)^+}D^{(\alpha-\beta)^+}u$.

(ii) $D^{\alpha}I^{\beta}D^{\beta}u = I^{(\beta-\alpha)^+}D^{(\beta-\alpha)^+}D^{\alpha}u$.

PROOF. We put

$$\{j : \beta_j > \alpha_j\} = \{j_1, \dots, j_t\}, \qquad \{j : \beta_j \le \alpha_j\} = \{j_{t+1}, \dots, j_n\}.$$

Since $D_j I_k = I_k D_j$ for $k \ne j$ by Lemma 2.4, we have

$$D^{\alpha}I^{\beta}u = (D_{j_1}^{\alpha_{j_1}} I_{j_1}^{\beta_{j_1}}) \dots (D_{j_t}^{\alpha_{j_t}} I_{j_t}^{\beta_{j_t}})(D_{j_{t+1}}^{\alpha_{j_{t+1}}} I_{j_{t+1}}^{\beta_{j_{t+1}}}) \dots (D_{j_n}^{\alpha_{j_n}} I_{j_n}^{\beta_{j_n}})u$$

and

$$D^{\alpha}I^{\beta}D^{\beta}u = (D_{j_1}^{\alpha_{j_1}} I_{j_1}^{\beta_{j_1}}) \dots (D_{j_t}^{\alpha_{j_t}} I_{j_t}^{\beta_{j_t}})(D_{j_{t+1}}^{\alpha_{j_{t+1}}} I_{j_{t+1}}^{\beta_{j_{t+1}}}) \dots (D_{j_n}^{\alpha_{j_n}} I_{j_n}^{\beta_{j_n}}) \\ D_{j_1}^{\beta_{j_1}} \dots D_{j_t}^{\beta_{j_t}} D_{j_{t+1}}^{\beta_{j_{t+1}}} \dots D_{j_n}^{\beta_{j_n}} u.$$

Since

$$D_j^{\alpha_j} I_j^{\beta_j} = \begin{cases} D_j^{\alpha_j - \beta_j}, & \alpha_j \geq \beta_j, \\ I_j^{\beta_j - \alpha_j}, & \alpha_j < \beta_j, \end{cases}$$

by Lemma 2.4, we obtain that

$$\begin{aligned} D^{\alpha}I^{\beta}u &= I_{j_1}^{\beta_{j_1}-\alpha_{j_1}} \dots I_{j_t}^{\beta_{j_t}-\alpha_{j_t}} D_{j_{t+1}}^{\alpha_{j_{t+1}}-\beta_{j_{t+1}}} \dots D_{j_n}^{\alpha_{j_n}-\beta_{j_n}} u \\ &= I^{(\beta-\alpha)^+} D^{(\alpha-\beta)^+} u \end{aligned}$$

and

$$\begin{aligned} D^{\alpha}I^{\beta}D^{\beta}u &= I_{j_1}^{\beta_{j_1}-\alpha_{j_1}} \dots I_{j_t}^{\beta_{j_t}-\alpha_{j_t}} D_{j_{t+1}}^{\alpha_{j_{t+1}}-\beta_{j_{t+1}}} \dots D_{j_n}^{\alpha_{j_n}-\beta_{j_n}} D_{j_1}^{\beta_{j_1}} \dots D_{j_t}^{\beta_{j_t}} D_{j_{t+1}}^{\beta_{j_{t+1}}} \dots D_{j_n}^{\beta_{j_n}} u \\ &= I_{j_1}^{\beta_{j_1}-\alpha_{j_1}} \dots I_{j_t}^{\beta_{j_t}-\alpha_{j_t}} D_{j_{t+1}}^{\alpha_{j_{t+1}}} \dots D_{j_n}^{\alpha_{j_n}} D_{j_1}^{\beta_{j_1}} \dots D_{j_t}^{\beta_{j_t}} u \\ &= I_{j_1}^{\beta_{j_1}-\alpha_{j_1}} \dots I_{j_t}^{\beta_{j_t}-\alpha_{j_t}} D_{j_1}^{\beta_{j_1}-\alpha_{j_1}} \dots D_{j_t}^{\beta_{j_t}-\alpha_{j_t}} D_{j_1}^{\alpha_{j_1}} \dots D_{j_t}^{\alpha_{j_t}} D_{j_{t+1}}^{\alpha_{j_{t+1}}} \dots D_{j_n}^{\alpha_{j_n}} u \\ &= I^{(\beta-\alpha)^+} D^{(\beta-\alpha)^+} D^{\alpha} u. \end{aligned}$$

Thus we obtain (i) and (ii).

Lemma 4.3. *Let α and β be nonzero multi-indices.*
(i) *If $P \in \mathscr{P}^{\alpha,1}(\mathbf{R}^n)$, then $I^{\beta}D^{\beta}P \in \mathscr{P}^{\alpha,1}(\mathbf{R}^n)$.*
(ii) *If $v \in \mathscr{K}^{\alpha}(\mathbf{R}^n)$, then $I^{\beta}D^{\beta}v \in \mathscr{K}^{\alpha}(\mathbf{R}^n)$.*

Proof. (i) Let $P \in \mathscr{P}^{\alpha,1}(\mathbf{R}^n)$. Since $D^{\alpha}P = 0$ by Theorem 2.5, Lemma 4.2(ii) implies that

$$D^{\alpha}I^{\beta}D^{\beta}P = I^{(\beta-\alpha)^+} D^{(\beta-\alpha)^+} D^{\alpha}P = 0.$$

Hence $I^{\beta}D^{\beta}P \in \mathscr{P}^{\alpha,1}(\mathbf{R}^n)$.

(ii) Suppose that $v \in \mathscr{K}^{\alpha}(\mathbf{R}^n)$. By Proposition 3.4 it suffices to show that for $\gamma < \alpha$, $D^{\gamma}I^{\beta}D^{\beta}v = 0$ on $\bigcup_{j \in M_{\alpha-\gamma}} R^{n,j}$. Let $\gamma < \alpha$ and $j \in M_{\alpha-\gamma}$. By Lemma 4.2(ii) we have

$$D^{\gamma}I^{\beta}D^{\beta}v(x) = I^{(\beta-\gamma)^+} D^{(\beta-\gamma)^+} D^{\gamma}v(x).$$

When $j \in M_{(\beta-\gamma)^+}$, Proposition 3.3(iii) implies that

$$I^{(\beta-\gamma)^+} D^{(\beta-\gamma)^+} D^{\gamma}v(x) = 0 \qquad \text{on } \mathbf{R}^{n,j}.$$

We consider the case $j \in M_{\alpha-\gamma}$ and $j \notin M_{(\beta-\gamma)^+}$. From the assumption $v \in \mathscr{K}^{\alpha}(\mathbf{R}^n)$, $j \in M_{\alpha-\gamma}$ and Proposition 3.3(iii), it follows that $D^{\gamma}v(x) = 0$ on $\mathbf{R}^{n,j}$. Since $j \notin M_{(\beta-\gamma)^+}$, Lemma 4.1 implies that $D^{(\beta-\gamma)^+}D^{\gamma}v(x) = 0$ on $\mathbf{R}^{n,j}$ and $I^{(\beta-\gamma)^+}D^{(\beta-\gamma)^+}D^{\gamma}v(x) = 0$ on $\mathbf{R}^{n,j}$. Thus we obtain that $D^{\gamma}I^{\beta}D^{\beta}v(x) = 0$ on $\bigcup_{j \in M_{\alpha-\gamma}} \mathbf{R}^{n,j}$. The lemma is proved.

We put

$$A_{\ell} = \{\alpha; |\alpha| = \ell\} = \{\alpha^1, \alpha^2, \dots, \alpha^{d_{\ell}}\},$$

where $d_{\ell} = (n + \ell - 1)!/(\ell!(n-1)!)$.

LEMMA 4.4. *Let k be a positive integer with $k \leq d_{\ell} - 1$. If $\mathscr{H}$ is a subspace of $C^{\infty}(\mathbf{R}^n)$ with the following form*

$$\mathscr{H} = \mathscr{H}_1 \cap \mathscr{H}_2 \cap \cdots \cap \mathscr{H}_k,$$

where each $\mathscr{H}_i$ is either $\mathscr{K}^{\alpha^i}(\mathbf{R}^n)$ or $\mathscr{P}^{\alpha^i,1}(\mathbf{R}^n)$ $(i = 1, \dots, k)$, then

$$\mathscr{H} = \mathscr{H} \cap \mathscr{K}^{\alpha^{k+1}}(\mathbf{R}^n) \oplus \mathscr{H} \cap \mathscr{P}^{\alpha^{k+1},1}(\mathbf{R}^n).$$

PROOF. Since $\mathscr{H} \cap \mathscr{K}^{\alpha^{k+1}}(\mathbf{R}^n)$, $\mathscr{H} \cap \mathscr{P}^{\alpha^{k+1},1}(\mathbf{R}^n) \subset \mathscr{H}$ and $\mathscr{H}$ is a subspace, it is clear that $\mathscr{H} \cap \mathscr{K}^{\alpha^{k+1}}(\mathbf{R}^n) + \mathscr{H} \cap \mathscr{P}^{\alpha^{k+1},1}(\mathbf{R}^n) \subset \mathscr{H}$. Let $u \in \mathscr{H}$. We put $v = I^{\alpha^{k+1}}D^{\alpha^{k+1}}u$ and $P = u - v$. Then Lemma 4.3 implies that $v \in \mathscr{H}$. Moreover, since $v \in \mathscr{K}^{\alpha^{k+1}}(\mathbf{R}^n)$ by the definition, we have $v \in \mathscr{H} \cap \mathscr{K}^{\alpha^{k+1}}(\mathbf{R}^n)$. Since $u, v \in \mathscr{H}$, we see that $P = u - v \in \mathscr{H}$. Further, by Proposition 3.3(ii) we have

$$\begin{aligned} D^{\alpha^{k+1}}P &= D^{\alpha^{k+1}}u - D^{\alpha^{k+1}}I^{\alpha^{k+1}}D^{\alpha^{k+1}}u \\ &= D^{\alpha^{k+1}}u - D^{\alpha^{k+1}}u = 0. \end{aligned}$$

Therefore Theorem 2.5 implies that $P \in \mathscr{P}^{\alpha^{k+1},1}(\mathbf{R}^n)$, and hence $P \in \mathscr{H} \cap \mathscr{P}^{\alpha^{k+1},1}(\mathbf{R}^n)$. Since $u = v + P$, $u \in \mathscr{H} \cap \mathscr{K}^{\alpha^{k+1}}(\mathbf{R}^n) + \mathscr{H} \cap \mathscr{P}^{\alpha^{k+1},1}(\mathbf{R}^n)$. Thus $\mathscr{H} = \mathscr{H} \cap \mathscr{K}^{\alpha^{k+1}}(\mathbf{R}^n) + \mathscr{H} \cap \mathscr{P}^{\alpha^{k+1},1}(\mathbf{R}^n)$. Since $\mathscr{K}^{\alpha^{k+1}}(\mathbf{R}^n) \cap \mathscr{P}^{\alpha^{k+1},1}(\mathbf{R}^n) = \{0\}$ by Theorem 3.5, the sum is a direct sum. Thus we obtain the lemma.

Since $C^{\infty}(\mathbf{R}^n) = \mathscr{K}^{\alpha^1}(\mathbf{R}^n) \oplus \mathscr{P}^{\alpha^1,1}(\mathbf{R}^n)$ by Theorem 3.5, by using Lemma 4.4 repeatedly we obtain

LEMMA 4.5. *The space $C^{\infty}(\mathbf{R}^n)$ is the direct sum of the $2^{d_{\ell}}$ subspaces of the following form*

$$\mathscr{H}_1 \cap \mathscr{H}_2 \cap \cdots \cap \mathscr{H}_{d_{\ell}},$$

where each $\mathscr{H}_i$ is either $\mathscr{K}^{\alpha^i}(\mathbf{R}^n)$ or $\mathscr{P}^{\alpha^i,1}(\mathbf{R}^n)$ $(i = 1, 2, \dots, d_{\ell})$.

By Lemma 4.5 it is possible to write

$$C^{\infty}(\mathbf{R}^n) = \bigoplus_{\mathscr{H}_i = \mathscr{K}^{\alpha^i}(\mathbf{R}^n) \text{ or } \mathscr{P}^{\alpha^i,1}(\mathbf{R}^n)} \mathscr{H}_1 \cap \mathscr{H}_2 \cap \cdots \cap \mathscr{H}_{d_\ell}.$$

As we saw in (4.1),

$$\bigcap_{i=1}^{d_\ell} \mathscr{P}^{\alpha^i,1}(\mathbf{R}^n) = \bigcap_{|\alpha|=\ell} \mathscr{P}^{\alpha}(\mathbf{R}^n) = \mathscr{P}^{\ell}(\mathbf{R}^n).$$

Namely, if $\mathscr{H}_i = \mathscr{P}^{\alpha^i,1}(\mathbf{R}^n)$ for $i = 1, 2, \ldots, d_\ell$, then $\mathscr{H}_1 \cap \mathscr{H}_2 \cap \cdots \cap \mathscr{H}_{d_\ell} = \mathscr{P}^{\ell}(\mathbf{R}^n)$. We give a characterization of functions which belong to the subspace $\mathscr{H}_1 \cap \mathscr{H}_2 \cap \cdots \cap \mathscr{H}_{d_\ell}$, where each $\mathscr{H}_i$ is either $\mathscr{K}^{\alpha^i}(\mathbf{R}^n)$ or $\mathscr{P}^{\alpha^i,1}(\mathbf{R}^n)$ and $\mathscr{H}_j \neq \mathscr{P}^{\alpha^j,1}(\mathbf{R}^n)$ for some j.

Lemma 4.6. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be nonzero multi-indices. Then*

$$\mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \mathscr{K}^{\beta}(\mathbf{R}^n) = \mathscr{K}^{\max(\alpha,\beta)}(\mathbf{R}^n),$$

where

$$\max(\alpha, \beta) = (\max(\alpha_1, \beta_1), \ldots, \max(\alpha_n, \beta_n)).$$

Proof. Since $\alpha, \beta \leq \max(\alpha, \beta)$, we see that $\mathscr{K}^{\max(\alpha,\beta)}(\mathbf{R}^n) \subset \mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \mathscr{K}^{\beta}(\mathbf{R}^n)$. We show the converse. Let $u \in \mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \mathscr{K}^{\beta}(\mathbf{R}^n)$. In order to prove that $u \in \mathscr{K}^{\max(\alpha,\beta)}(\mathbf{R}^n)$, by Proposition 3.4 it suffices to show that for $\gamma < \max(\alpha, \beta)$, $D^{\gamma}u(x) = 0$ on $\bigcup_{j \in M_{\max(\alpha,\beta)-\gamma}} \mathbf{R}^{n,j}$. Let $\gamma < \max(\alpha, \beta)$ and $j \in M_{\max(\alpha,\beta)-\gamma}$. Since $j \in M_{\max(\alpha,\beta)-\gamma}$, we have $\max(\alpha_j, \beta_j) > \gamma_j$. Hence $\gamma_j < \alpha_j$ or $\gamma_j < \beta_j$. We put $\eta = \gamma_j e_j$. If $\gamma_j < \alpha_j$ (resp. $\gamma_j < \beta_j$), then $D^{\eta}u(x) = 0$ on $R^{n,j}$ by Proposition 3.3(iii) because $u \in \mathscr{K}^{\alpha}(\mathbf{R}^n)$ (resp. $u \in \mathscr{K}^{\beta}(\mathbf{R}^n)$), $\eta < \alpha$ (resp. $\eta < \beta$) and $j \in M_{\alpha-\eta}$ (resp. $j \in M_{\beta-\eta}$). Therefore $D_j^{\gamma_j}u(x) = D^{\eta}u(x) = 0$ on $\mathbf{R}^{n,j}$. Hence by Lemma 4.1

$$D^{\gamma}u(x) = D_1^{\gamma_1} \ldots \widehat{D_j^{\gamma_j}} \ldots D_n^{\gamma_n} D_j^{\gamma_j} u(x) = 0$$

on $\mathbf{R}^{n,j}$. Thus we obtain the lemma.

Corollary 4.7. *Let $\gamma^1, \ldots, \gamma^m$ be nonzero multi-indices. Then*

$$\bigcap_{i=1}^{m} \mathscr{K}^{\gamma^i}(\mathbf{R}^n) = \mathscr{K}^{\max(\gamma^1,\ldots,\gamma^m)}(\mathbf{R}^n),$$

where $\max(\gamma^1, \ldots, \gamma^m) = (\max(\gamma_1^1, \ldots, \gamma_1^m), \ldots, \max(\gamma_n^1, \ldots, \gamma_n^m))$.

Corollary 4.8.

$$\bigcap_{i=1}^{d_\ell} \mathscr{K}^{\alpha^i}(\mathbf{R}^n) = \bigcap_{|\alpha|=\ell} \mathscr{K}^{\alpha}(\mathbf{R}^n) = \mathscr{K}^{\ell e}(\mathbf{R}^n),$$

where $e = (1, \ldots, 1)$.

LEMMA 4.9. *Let α and β be nonzero multi-indices. Then*

$$\mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \mathscr{P}^{\beta,1}(\mathbf{R}^n) = \begin{cases} \{0\}, & \alpha \ge \beta, \\ \{K^{\alpha}f : f \in \mathscr{P}^{(\beta-\alpha)^+,1}(\mathbf{R}^n)\}, & \alpha \not\ge \beta. \end{cases}$$

PROOF. Let $\alpha \ge \beta$. By Theorem 3.5, $\mathscr{K}^{\beta}(\mathbf{R}^n) \cap \mathscr{P}^{\beta,1}(\mathbf{R}^n) = \{0\}$. Since $\alpha \ge \beta$ implies that $\mathscr{K}^{\alpha}(\mathbf{R}^n) \subset \mathscr{K}^{\beta}(\mathbf{R}^n)$, we obtain $\mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \mathscr{P}^{\beta,1}(\mathbf{R}^n) = \{0\}$. We consider the case $\alpha \not\ge \beta$. First we prove that $\mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \mathscr{P}^{\beta,1}(\mathbf{R}^n) \subset \{K^{\alpha}f : f \in \mathscr{P}^{(\beta-\alpha)^+,1}(\mathbf{R}^n)\}$. Let $u \in \mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \mathscr{P}^{\beta,1}(\mathbf{R}^n)$. Since $u \in \mathscr{K}^{\alpha}(\mathbf{R}^n)$, there exists $f \in C^{\infty}(\mathbf{R}^n)$ such that $u = K^{\alpha}f$. Moreover, since $u \in \mathscr{P}^{\beta,1}(\mathbf{R}^n)$, by Theorem 2.5 and Lemma 4.2(i) we obtain

$$0 = D^{\beta}K^{\alpha}f = D^{\beta}I^{\alpha}f = I^{(\alpha-\beta)^+}D^{(\beta-\alpha)^+}f.$$

Since $\alpha \not\ge \beta$, we have two cases $\beta > \alpha$ or $\beta \not\ge \alpha$. In case of $\beta > \alpha$, we have $D^{(\beta-\alpha)^+}f = 0$ because $(\alpha-\beta)^+ = 0$. Hence $f \in \mathscr{P}^{(\beta-\alpha)^+,1}(\mathbf{R}^n)$. In case of $\beta \not\ge \alpha$, we have $(\alpha-\beta)^+ \ne 0$. Hence $I^{(\alpha-\beta)^+}D^{(\beta-\alpha)^+}f = 0$ implies $D^{(\beta-\alpha)^+}f = 0$. Therefore $f \in \mathscr{P}^{(\beta-\alpha)^+,1}(\mathbf{R}^n)$. Next, let $u = K^{\alpha}f$ with $f \in \mathscr{P}^{(\beta-\alpha)^+,1}(\mathbf{R}^n)$. It is clear that $u \in \mathscr{K}^{\alpha}(\mathbf{R}^n)$. Further, Lemma 4.2(i), the condition $f \in \mathscr{P}^{(\beta-\alpha)^+,1}(\mathbf{R}^n)$ and Theorem 2.5 give

$$D^{\beta}u = D^{\beta}K^{\alpha}f = D^{\beta}I^{\alpha}f = I^{(\alpha-\beta)^+}D^{(\beta-\alpha)^+}f = 0.$$

Thus $u \in \mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \mathscr{P}^{\beta,1}(\mathbf{R}^n)$. The lemma is proved.

COROLLARY 4.10. *Let $\gamma^1, \dots, \gamma^m$ and α be nonzero multi-indices. Then*

$$\mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \left(\textstyle\bigcap_{i=1}^m \mathscr{P}^{\gamma^i,1}(\mathbf{R}^n)\right)$$
$$= \begin{cases} \{0\}, & \text{if there exists } i \text{ such that } \gamma^i \le \alpha, \\ \{K^{\alpha}f : f \in (\bigcap_{i=1}^m \mathscr{P}^{(\gamma^i-\alpha)^+,1}(R^n))\}, & \text{if } \gamma^i \not\le \alpha \text{ for all } i. \end{cases}$$

PROOF. If there exists i_0 such that $\gamma^{i_0} \le \alpha$, then by Lemma 4.9

$$\mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \left(\textstyle\bigcap_{i=1}^m \mathscr{P}^{\gamma^i,1}(\mathbf{R}^n)\right) \subset \mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \mathscr{P}^{\gamma^{i_0},1}(\mathbf{R}^n) = \{0\}.$$

Let $\gamma^i \not\le \alpha$ for $i = 1, \dots, m$. Then $(\gamma^i - \alpha)^+ \ne 0$ for $i = 1, \dots, m$. Let $u = K^{\alpha}f$ with $f \in \bigcap_{i=1}^m \mathscr{P}^{(\gamma^i-\alpha)^+,1}(R^n)$. Then by Lemma 4.9 we see that $u \in \mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \mathscr{P}^{\gamma^i,1}(\mathbf{R}^n)$ for $i = 1, \dots, m$. Hence

$$u \in \textstyle\bigcap_{i=1}^m (\mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \mathscr{P}^{\gamma^i,1}(\mathbf{R}^n)) = \mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \left(\bigcap_{i=1}^m \mathscr{P}^{\gamma^i,1}(\mathbf{R}^n)\right).$$

Conversely, let $u \in \mathscr{K}^{\alpha}(\mathbf{R}^n) \cap (\bigcap_{i=1}^m \mathscr{P}^{\gamma^i,1}(\mathbf{R}^n))$. Since $u \in \mathscr{K}^{\alpha}(\mathbf{R}^n) \cap \mathscr{P}^{\gamma^i,1}(\mathbf{R}^n)$ $(i = 1, \dots, m)$, by Lemma 4.9 there exists $f_i \in \mathscr{P}^{(\gamma^i-\alpha)^+,1}(\mathbf{R}^n)$ $(i = 1, \dots, m)$ such that $u = K^{\alpha}f_i$. Since $f_i = D^{\alpha}K^{\alpha}f_i = D^{\alpha}u$, we see that $f_1 = \cdots = f_m$. Hence, if we put $f = f_1 = \cdots = f_m$, then $u = K^{\alpha}f$ and $f \in \bigcap_{i=1}^m \mathscr{P}^{(\gamma^i-\alpha)^+,1}(\mathbf{R}^n)$.

For a subset $S = \{\alpha^{j_1}, \dots, \alpha^{j_m}\} \subset A_\ell$ we set

$$\alpha^S = \max(\alpha^{j_1}, \dots, \alpha^{j_m}).$$

COROLLARY 4.11. *Let* $\mathscr{H}_i = \mathscr{K}^{\alpha^i}(\mathbf{R}^n)$ *or* $\mathscr{P}^{\alpha^i,1}(\mathbf{R}^n)$ $(i = 1, \dots, d_\ell)$ *and* $(\mathscr{H}_1, \dots, \mathscr{H}_{d_\ell}) \neq (\mathscr{P}^{\alpha^1,1}(\mathbf{R}^n), \dots, \mathscr{P}^{\alpha^{d_\ell},1}(\mathbf{R}^n))$. *Then*

$$\mathscr{H}_1 \cap \cdots \cap \mathscr{H}_{d_\ell} = \begin{cases} \{0\}, & \text{if there exists } \gamma \in S^c \text{ such that } \gamma \leq \alpha^S, \\ \{K^{\alpha^S} f : f \in \bigcap_{\gamma \in S^c} \mathscr{P}^{(\gamma - \alpha^S)^+,1}(\mathbf{R}^n)\}, & \text{if } \gamma \nleq \alpha^S \text{ for all } \gamma \in S^c, \end{cases}$$

where $S = \{\alpha^i : \mathscr{H}_i = \mathscr{K}^{\alpha^i}(\mathbf{R}^n)\}$ *and* $S^c = A_\ell - S$.

PROOF. By Lemma 4.7 we have

$$\begin{aligned} \mathscr{H}_1 \cap \cdots \cap \mathscr{H}_{d_\ell} &= (\textstyle\bigcap_{\gamma \in S} \mathscr{K}^{\gamma}(\mathbf{R}^n)) \cap (\bigcap_{\gamma \in S^c} \mathscr{P}^{\gamma,1}(\mathbf{R}^n)) \\ &= \mathscr{K}^{\alpha^S}(\mathbf{R}^n) \cap (\textstyle\bigcap_{\gamma \in S^c} \mathscr{P}^{\gamma,1}(\mathbf{R}^n)). \end{aligned}$$

Hence the corollary follows from Corollary 4.10.

Taking Corollary 4.11 into account, it is convenient to define $\mathscr{P}^{0,1}(\mathbf{R}^n) = \{0\}$. Under the definition by Corollary 4.11 we have

PROPOSITION 4.12.

$$u \in \bigoplus_{\mathscr{H}_i = \mathscr{K}^{\alpha^i}(\mathbf{R}^n) \text{ or } \mathscr{P}^{\alpha^i,1}(\mathbf{R}^n), (\mathscr{H}_1, \dots, \mathscr{H}_{d_\ell}) \neq (\mathscr{P}^{\alpha^1,1}(\mathbf{R}^n), \dots, \mathscr{P}^{\alpha^{d_\ell},1}(\mathbf{R}^n))} \mathscr{H}_1 \cap \cdots \cap \mathscr{H}_{d_\ell}$$

if and only if

$$u = \sum_{S \subset A_\ell, S \neq \phi} K^{\alpha^S} f_S, \qquad f_S \in \textstyle\bigcap_{\gamma \in S^c} \mathscr{P}^{(\gamma - \alpha^S)^+,1}(\mathbf{R}^n). \tag{4.2}$$

If a function u has the form (4.2), then we call u a polyprimitive of order ℓ, and denote by $\mathscr{K}^\ell(\mathbf{R}^n)$ the set of all polyprimitives of order ℓ. By Lemma 4.5 and Proposition 4.12 we have

THEOREM 4.13. *Let* ℓ *be a positive integer. Then*

$$C^\infty(\mathbf{R}^n) = \mathscr{K}^\ell(\mathbf{R}^n) \oplus \mathscr{P}^\ell(\mathbf{R}^n).$$

Finally we give other characterizations of the spaces $\mathscr{K}^\ell(\mathbf{R}^n)$ and $\mathscr{P}^\ell(\mathbf{R}^n)$. For a polynomial $P(x) = \sum_{|\alpha| \leq \ell - 1} a_\alpha x^\alpha$ of order $\ell - 1$, it is clear that

$$a_\alpha = \frac{D^\alpha P(0)}{\alpha!}, \qquad |\alpha| \leq \ell - 1. \tag{4.3}$$

For a nonempty set $T \subset A_\ell$, we put

$$F^T = \sum_{T \subset S \subset A_\ell} (-1)^{\#(S-T)} K^{\alpha^S - \alpha^T} D^{\alpha^S}.$$

LEMMA 4.14. *If* $u = \sum_{S \subset A_\ell, S \neq \phi} K^{\alpha^S} f_S$ $(f_S \in \bigcap_{\gamma \in S^c} \mathscr{P}^{(\gamma - \alpha^S)^+, 1}(\mathbf{R}^n))$ *is a polyprimitive of order* ℓ, *then*

$$f_T = F^T u, \qquad T \subset A_\ell,\ T \neq \emptyset. \tag{4.4}$$

PROOF. First we prove that if $U \not\subset S$, then

$$D^{\alpha^U} K^{\alpha^S} f_S = 0. \tag{4.5}$$

By Lemma 4.2(i) we have

$$D^{\alpha^U} K^{\alpha^S} f_S = K^{(\alpha^S - \alpha^U)^+} D^{(\alpha^U - \alpha^S)^+} f_S. \tag{4.6}$$

Since $U \not\subset S$, there exists $\gamma_0 \notin S$ and $\gamma_0 \in U$. The condition $\gamma_0 \in U$ implies that

$$(\alpha^U - \alpha^S)^+ \geq (\gamma_0 - \alpha^S)^+. \tag{4.7}$$

Since $f_S \in \bigcap_{\gamma \in S^c} \mathscr{P}^{(\gamma - \alpha^S)^+, 1}(\mathbf{R}^n)$ and $\gamma_0 \notin S$, we see that $f_S \in \mathscr{P}^{(\gamma_0 - \alpha^S)^+, 1}(\mathbf{R}^n)$. Hence

$$D^{(\gamma_0 - \alpha^S)^+} f_S = 0. \tag{4.8}$$

The three formulae (4.6), (4.7) and (4.8) give that $D^{(\alpha^U - \alpha^S)^+} f_S = 0$ and $D^{\alpha^U} K^{\alpha^S} f_S = 0$. Thus we obtain (4.5). We put $p = \#(T)$. Then $1 \leq p \leq d_\ell$. We prove (4.4) by downward induction with respect to p. If $p = d_\ell$, then $T = A_\ell$. Hence

$$\begin{aligned} F^{A_\ell} u &= F^{A_\ell}\left(\sum_{S \subset A_\ell, S \neq \emptyset} K^{\alpha^S} f_S\right) \\ &= \sum_{S \subset A_\ell, S \neq \emptyset} F^{A_\ell} K^{\alpha^S} f_S \\ &= \sum_{S \subset A_\ell, S \neq \emptyset} \sum_{A_\ell \subset U \subset A_\ell} (-1)^{\#(U - A_\ell)} K^{\alpha^U - \alpha^{A_\ell}} D^{\alpha^U} K^{\alpha^S} f_S \\ &= \sum_{S \subset A_\ell, S \neq \emptyset} D^{\alpha^{A_\ell}} K^{\alpha^S} f_S. \end{aligned}$$

Since $A_\ell \not\subset S$ except $S = A_\ell$, by (4.5) and Proposition 3.3 (ii) we have

$$F^{A_\ell} u = D^{\alpha^{A_\ell}} K^{\alpha^{A_\ell}} f_{A_\ell} = f_{A_\ell}.$$

Thus we obtain (4.4) for $p = d_\ell$. Next let $1 \le p \le d_\ell - 1$ and we assume that (4.4) holds for T with $\#(T) = p+1, p+2, \dots, d_\ell$. We consider the case $\#(T) = p$. By (4.5) and Proposition 3.3(ii) we have

$$D^{\alpha^T} u = \sum_{S \subset A_\ell, S \neq \varnothing} D^{\alpha^T} K^{\alpha^S} f_S = \sum_{S \supset T} D^{\alpha^T} K^{\alpha^S} f_S$$

$$= f_T + \sum_{S \supset T, S \neq T} K^{\alpha^S - \alpha^T} f_S.$$

Hence

$$f_T = D^{\alpha^T} u - \sum_{S \supset T, S \neq T} K^{\alpha^S - \alpha^T} f_S. \tag{4.9}$$

Let $S \supset T$ and $S \neq T$. Since $\#(S) \ge p+1$, by the assumption of induction we have

$$f_S = F^S u = \sum_{S \subset U \subset A_\ell} (-1)^{\#(U-S)} K^{\alpha^U - \alpha^S} D^{\alpha^U} u. \tag{4.10}$$

By substituting (4.10) for (4.9) we obtain

$$f_T = D^{\alpha^T} u - \sum_{S \supset T, S \neq T} \sum_{U \supset S} (-1)^{\#(U-S)} K^{\alpha^S - \alpha^T} K^{\alpha^U - \alpha^S} D^{\alpha^U} u$$

$$= D^{\alpha^T} u - \sum_{S \supset T, S \neq T} \sum_{U \supset S} (-1)^{\#(U-S)} K^{\alpha^U - \alpha^T} D^{\alpha^U} u$$

$$= D^{\alpha^T} u - \sum_{U \supset T, U \neq T} \sum_{U \supset S \supset T, S \neq T} (-1)^{\#(U-S)} K^{\alpha^U - \alpha^T} D^{\alpha^U} u$$

$$= D^{\alpha^T} u - \sum_{U \supset T, U \neq T} K^{\alpha^U - \alpha^T} D^{\alpha^U} u \sum_{U \supset S \supset T, S \neq T} (-1)^{\#(U-S)}.$$

Here we note that

$$\sum_{U \supset S \supset T} (-1)^{\#(U-S)} = 0.$$

Hence

$$f_T = D^{\alpha^T} u + \sum_{U \supset T, U \neq T} (-1)^{\#(U-T)} K^{\alpha^U - \alpha^T} D^{\alpha^U} u$$

$$= \sum_{U \supset T} (-1)^{\#(U-T)} K^{\alpha^U - \alpha^T} D^{\alpha^U} u = F^T u.$$

Thus we obtain (4.4) for $\#(T) = p$. The lemma is proved.

Now we prove

Proposition 4.15. (i) $\mathscr{P}^{\ell}(\mathbf{R}^n) = \{u \in C^{\infty}(\mathbf{R}^n) : F^T u = 0$ *for* $T \subset A_\ell$ *and* $T \neq \varnothing\}$.

(ii) $\mathscr{K}^{\ell}(\mathbf{R}^n) = \{u \in C^{\infty}(\mathbf{R}^n) : D^{\alpha}u(0) = 0$ *for* $|\alpha| \leq \ell - 1\}$.

Proof. (i) First let $P(x) = \sum_{|\alpha| \leq \ell - 1} a_\alpha x^\alpha$ and $T \subset A_\ell$, $T \neq \varnothing$. We note that $D^\beta x^\alpha = 0$ for $|\alpha| \leq \ell - 1$ and $|\beta| \geq \ell$. Since $|\alpha^U| \geq \ell$ for $U \subset A_\ell$, $U \neq \varnothing$, we have $D^{\alpha^U} x^\alpha = 0$ for $U \subset A_\ell$, $U \neq \varnothing$ and $|\alpha| \leq \ell - 1$. Hence

$$F^T P = \sum_{T \subset U \subset A_\ell} (-1)^{\#(U-T)} K^{\alpha^U - \alpha^T} D^{\alpha^U} P = 0. \tag{4.11}$$

Conversely let $F^T u = 0$ for $T \subset A_\ell$, $T \neq \varnothing$. By Theorem 4.13 $u = v + P$, where v is a polyprimitive of order ℓ and P is a polynomial of order $\ell - 1$. Let $v = \sum_{S \subset A_\ell, S \neq \varnothing} K^{\alpha^S} f_S$. By Lemma 4.14 and (4.11), for $T \subset A_\ell$ and $T \neq \varnothing$, we have

$$0 = F^T u = F^T v + F^T P = f_T.$$

Therefore $v = 0$ and hence $u = P \in \mathscr{P}^{\ell}(\mathbf{R}^n)$.

(ii) First let $u = \sum_{S \subset A_\ell, S \neq \phi} K^{\alpha^S} f_S$ and $|\alpha| \leq \ell - 1$. Then

$$D^\alpha u(x) = \sum_{S \subset A_\ell, S \neq \phi} D^\alpha K^{\alpha^S} f_S(x).$$

Since $|\alpha^S| \geq \ell$ and $|\alpha| \leq \ell - 1$, there exists i such that $\alpha_i^S - \alpha_i > 0$. Since

$$D^\alpha K^{\alpha^S} f_S(x) = K_i^{\alpha_i^S - \alpha_i} D_1^{\alpha_1} \dots \widehat{D_i^{\alpha_i}} \dots D_n^{\alpha_n} K_1^{\alpha_1^S} \dots \widehat{K_i^{\alpha_i^S}} \dots K_n^{\alpha_n^S} f_S(x),$$

we see that $D^\alpha K^{\alpha^S} f_S(0) = 0$, and hence

$$D^\alpha u(0) = 0. \tag{4.12}$$

Conversely, let $D^\alpha u(0) = 0$ for $|\alpha| \leq \ell - 1$. By Theorem 4.13 $u = v + P$, where v is a polyprimitive of order ℓ and P is a polynomial of order $\ell - 1$. Let $P(x) = \sum_{|\beta| \leq \ell - 1} a_\beta x^\beta$. By (4.3) and (4.12), for $|\alpha| \leq \ell - 1$

$$0 = D^\alpha u(0) = D^\alpha v(0) + D^\alpha P(0) = \alpha! a_\alpha.$$

Therefore $P = 0$ and hence $u = v \in \mathscr{K}^{\ell}(\mathbf{R}^n)$.

Corollary 4.16. *Let* $u \in C^\infty(\mathbf{R}^n)$ *and* ℓ *be a positive integer. Then*

$$u(x) = \sum_{|\alpha| \leq \ell - 1} \frac{D^\alpha u(0)}{\alpha!} x^\alpha + \sum_{T \subset A_\ell, T \neq \varnothing} K^{\alpha^T} F^T u(x).$$

Proof. By Theorem 4.13 $u = P + v$ where $P \in \mathscr{P}^{\ell}(\mathbf{R}^n)$ and $v \in \mathscr{K}^{\ell}(\mathbf{R}^n)$. Let $P(x) = \sum_{|\alpha| \le \ell - 1} a_\alpha x^\alpha$ and $v = \sum_{T \subset A_\ell, T \neq \emptyset} K^{\alpha^T} f_T$. By Proposition 4.15 (ii), for $|\alpha| \le \ell - 1$ we have

$$D^\alpha u(0) = D^\alpha P(0) + D^\alpha v(0) = a_\alpha \alpha!.$$

By Lemma 4.14 and Proposition 4.15 (i), for $T \subset A_\ell$, $T \neq \emptyset$ we see that

$$F^T u = F^T P + F^T v = f_T.$$

Thus we obtain the corollary.

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