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John disks are local bilipschitz images of quasidisks

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Abstract. It is proved that all bounded John disks are local bilipschitz images of quasidisks. This makes it easy to prove many necessary conditions for John disks.

It is generally acknowledged that John disks are "one sided quasidisks" [12], [17]. One common interpretation of this is that John disks have a lot of the same geometric properties as quasidisks, but with Euclidean distances replaced by internal distances. The *internal (length) distance* between z and w in some domain D is defined as

$$\lambda_D(z, w) = \inf\{\ell(\alpha)\colon \alpha \text{ rectifiable arc from } z \text{ to } w \text{ in } D\}, \tag{1}$$

where $\ell(\alpha)$ is the length of α. See e.g. Näkki and Väisälä [17], Kim and Langmeyer [16], Broch [5, 6, 7]. This distance is not able to tell the difference between Pacman (a quasidisk) and a disk minus a slit (the prototype John domain). Here we prove the following result.

Theorem 1. *A bounded, simply connected domain D in $\mathbf{R}^2$ is a c-John disk if and only if there exists a K-quasidisk Ω and an L-bilipschitz homeomorphism $F : (\Omega, \lambda_\Omega) \to (D, \lambda_D)$. The constants involved depend only on each other.*

Equivalently, F is locally L-bilipschitz in the Euclidean metrics.

Let us recall that a domain D is a *c-John domain* if any two points z and w in it may be joined by a rectifiable arc β in D such that

$$\min\{\ell(\beta(z, \zeta)), \ell(\beta(\zeta, w))\} \le c \operatorname{dist}(\zeta, \partial D) \qquad \text{for every } \zeta \in \beta.$$

Here $\beta(x, y)$ denotes the subarc of β from x to y. A simply connected planar John domain is called a *John disk*. Also remember that a *K-quasidisk* is the image of the unit disk $\mathbf{D}$ under some global K-quasiconformal map $f : \mathbf{R}^2 \to \mathbf{R}^2$. In this paper we will assume all quasidisks to be bounded.

In addition to Theorem 1 we will prove a result about locally Euclildean metrics in John disks (Theorem 2) and present some applications of Theorem 1 (Corollaries 1–3).

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Euclidean disks are denoted by $B(z,r)$. Denote by (D,d) the metric space consisting of the domain (set) D with the metric d. Let $\eta : [0,\infty) \to [0,\infty)$ be a homeomorphism with $\eta(0) = 0$. Then a homeomorphism $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is *η-quasisymmetric* if

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \le \eta\left(\frac{d_X(x,y)}{d_X(x,z)}\right) \qquad \text{for all } x, y, z \in X.$$

If L is a fixed constant, the homeomorphism f is said to be *L-bilipschitz* if

$$L^{-1} d_X(x,y) \le d_Y(f(x), f(y)) \le L d_X(x,y) \qquad \text{for all } x, y \in X.$$

A map f is *locally L-bilipschitz* if every $x \in X$ has a neighbourhood in which f is L-bilipschitz.

We first collect some results necessary for the proof of Theorem 1. The following is a local version of an observation made by S. Rohde in [19]. The proof is essentially the same as in [19], Lemma 2.1. Let J_f denote the Jacobian of a quasiconformal map f, defined almost everywhere.

Lemma 1 ([19], Lemma 2.1). *Suppose that D and D' are simply connected domains in $\mathbf{R}^2$, and that we have orientation preserving quasiconformal maps $g : \mathbf{D} \to D$ and $f : \mathbf{D} \to D'$ satisfying $L^{-1} J_f(z) \le J_g(z) \le L J_f(z)$ a.e. in $\mathbf{D}$. Then $F = f \circ g^{-1} : D \to D'$ is locally L'-bilipschitz for some constant L' or, equivalently, L'-bilipschitz with respect to internal distances.*

With every positive continuous function ϱ on $\mathbf{D}$, called a *density* [3], we can associate a metric space $\mathbf{D}_\varrho = (\mathbf{D}, d_\varrho)$, where the metric d_ϱ is defined by

$$d_\varrho(z,w) = \inf_\alpha \int_\alpha \varrho(\zeta)|d\zeta|. \tag{2}$$

The infimum is taken over all rectifiable curves α joining z and w in $\mathbf{D}$. When f is conformal in $\mathbf{D}$, we write $\varrho_f(z) = |f'(z)|$. The following elementary observation is useful.

Lemma 2. *If $f : \mathbf{D} \to D$ is a conformal map, then $d_{\varrho_f}(z,w) = \lambda_D(f(z), f(w))$.*

Write $B_\varrho(a,r) = \{z \in \mathbf{D} : d_\varrho(a,z) < r\}$ for a ball in the metric d_ϱ. Define a Borel measure associated with ϱ by $\mu_\varrho(E) = \int_{E\cap\mathbf{D}} |\varrho(z)|^2 dm$ (where m is two dimensional Lebesgue measure) for every Borel set $E \subset \mathbf{R}^2$. Then following [4] and [3] we say that ϱ is a *doubling conformal density* if there are constants A_1, A_2 and A_3 such that the following conditions hold:

(i) $A_1^{-1}\varrho(w) \le \varrho(z) \le A_1 \varrho(w)$ for all $z, w \in B(\zeta, (1-|\zeta|)/2)$, $\zeta \in \mathbf{D}$;

(ii) $\mu_\varrho(B_\varrho(a,r)) \le A_2 r^2$, and

(iii) $\mu_\varrho(B(z,2r)) \le A_3 \mu_\varrho(B(z,r))$, with $z \in \mathbf{D}$, $r > 0$.

We will use the following version of Theorem 1.15 in [3], which relates doubling conformal densities to globally defined quasiconformal maps.

Lemma 3 ([3], Theorem 1.15). *A positive continuous function ϱ on $\mathbf{D}$ is a doubling conformal density if and only if there exists a quasiconformal map $g : \mathbf{R}^2 \to \mathbf{R}^2$ and a constant $A \geq 1$ such that $A^{-1}\varrho(z) \leq J_g(z)^{1/2} \leq A\varrho(z)$ for almost every $z \in \mathbf{D}$.*

Lemma 4. *Suppose that $D \subset \mathbf{R}^2$ is a bounded, simply connected domain, and let $f : \mathbf{D} \to D$ be a conformal map. Then $\varrho_f(z) = |f'(z)|$ is a doubling conformal density on $\mathbf{D}$ if and only if D is a John disk.*

Proof. By Lemma 2 the map $f : (\mathbf{D}, d_{\varrho_f}) \to (D, \lambda_D)$ is an isometry. Then $f : \mathbf{D} \to (D, \lambda_D)$ is η-quasisymmetric if and only if $\mathrm{id} : \mathbf{D} \to (\mathbf{D}, d_{\varrho_f})$ is η-quasisymmetric. Thus by Proposition 2.11 in [3] $\varrho_f(z) = |f'(z)|$ is doubling if and only if $f : \mathbf{D} \to (D, \lambda_D)$ is quasisymmetric. By Section 7 in [17] (see also Väisälä [21]) this happens if and only if D is a John disk. □

Proof of Theorem 1. For the necessity, let D be a c-John disk and let $f : \mathbf{D} \to D$ be a conformal map. By Lemma 4 $\varrho_f(z) = |f'(z)|$ is a doubling conformal density in $\mathbf{D}$. By Lemma 3 there are constants A and K, depending only on ϱ_f (hence only on c), and a K-quasiconformal map $g : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$A^{-1}\varrho_f(z) \leq J_g(z)^{1/2} \leq A\varrho_f(z) \text{ a.e.}$$

Let $\Omega = g(\mathbf{D})$. Then by definition Ω is a K-quasidisk. Because f is conformal, $\varrho_f(z) = |f'(z)| = J_f^{1/2}$ and we have that

$$A^{-2}J_f(z) \leq J_g(z) \leq A^2 J_f(z) \text{ a.e.}$$

By Lemma 1 the map $F = f \circ g^{-1}$ is locally L-bilipschitz for some $L = L(c)$. Since F is a homeomorphism, and thus in particular injective, we have that

$$L^{-1}\lambda_\Omega(z_1, z_2) \leq \lambda_D(F(z_1), F(z_2)) \leq L\lambda_\Omega(z_1, z_2).$$

The sufficiency is not hard to prove from first principles: it is easy to see that a local bilipschitz image of a John domain is another John domain. One can also appeal to the fact that quasidisks are John disks, and to Theorem 3.6 in [17], which says that the John condition is preserved under maps that are quasisymmetric with respect to internal distances. □

Remark 1. In general we cannot use the unit disk as the quasidisk Ω in Theorem 1 because then D must be a so-called *internal chord-arc domain*, and thus in particular must have rectifiable boundary. See Väisälä [20, 21]. Also note that, in general, the bilipschitz constant cannot be too close to 1, since then D would have to be a quasidisk [8].

It is well known that in a John disk the *internal diameter distance* δ_D, defined by using diameters instead of lengths in (1), is equivalent to the λ_D metric [14]. We will show next that any locally Euclidean metric that is quasisymmetrically equivalent to λ_D is in fact equivalent to it.

By a *locally Euclidean metric* in a domain $D \subset \mathbf{R}^n$ we will mean a metric d on D such that for every z in D there is a ball $B \subset D$ in which we have $d(z,w) = |z-w|$. For all locally Euclidean metrics d we clearly have that $d \le \lambda_D$. Both λ_D and δ_D are locally Euclidean. Two metrics d_1 and d_2 on D are *quasisymmetrically equivalent* if the map $\mathrm{id}: (D,d_1) \to (D,d_2)$ is quasisymmetric. The metrics d_1 and d_2 are *equivalent* if $\mathrm{id}: (D,d_1) \to (D,d_2)$ is bilipschitz.

Theorem 2. *Let $D \subset \mathbf{R}^2$ be a c-John disk. Let d be a metric in D such that d is locally Euclidean and $d(z,w) \ge |z-w|$. If d is quasisymmetrically equivalent to λ_D, then d is equivalent to λ_D.*

Proof. Recall that in a John disk one may always use hyperbolic segments γ as the curves in the definition [12], [17]. Remember also the Gehring-Hayman theorem that says that if $\gamma[z,w]$ is the hyperbolic geodesic joining z and w in D, then

$$\ell(\gamma[z,w]) \le K\lambda_D(z,w) \tag{3}$$

for a universal constant K, the Gehring-Hayman constant. See e.g. Pommerenke [18], Theorem 4.20 or [4], Theorem 3.1.

Take z and w in D. Since D is a John disk we have that

$$\ell(\gamma[z,z_0]) = \ell(\gamma[w,z_0]) \le c\,\mathrm{dist}(z_0,\partial D), \tag{4}$$

where z_0 is the mid-point—with respect to Euclidean arc length—of the hyperbolic geodesic joining z and w in D. By the quasisymmetric equivalence of d and λ_D we have that

$$\frac{d(z,w)}{d(z,z_0)} \le \eta\left(\frac{\lambda_D(z,w)}{\lambda_D(z,z_0)}\right) \le \eta\left(K\frac{\ell(\gamma[z,w])}{\ell(\gamma[z,z_0])}\right) = \eta(2K) = a$$

where K is the Gehring-Hayman constant. The same argument with $\frac{d(z,z_0)}{d(z,w)}$ gives

$$b^{-1} \le \frac{d(z,w)}{d(z,z_0)} \le a, \tag{5}$$

where $b = \eta(K/2)$.

If $|z - z_0| < \mathrm{dist}(z_0, \partial D)$, we have $d(z, z_0) \geq |z - z_0| = \lambda_D(z, z_0)$. Then using (5) and (3) it follows that

$$d(z,w) \geq \frac{d(z,z_0)}{b} \geq \frac{\lambda_D(z,z_0)}{b} \geq \frac{\ell(\gamma[z,z_0])}{Kb} = \frac{\ell(\gamma[z,w])}{2Kb} \geq \frac{\lambda_D(z,w)}{2Kb}.$$

If $|z - z_0| \geq \mathrm{dist}(z_0, \partial D)$ we have by (5) and (4)

$$d(z,w) \geq \frac{d(z,z_0)}{b} \geq \frac{|z-z_0|}{b} \geq \frac{\mathrm{dist}(z_0,\partial D)}{b} \geq \frac{\ell(\gamma[z,z_0])}{bc} \geq \frac{\lambda_D(z,w)}{2bc}. \qquad \square$$

Remark 2. The assumption that $d(z, w) \geq |z - w|$ in the formulation of Theorem 2 is non-trivial in the sense that a locally Euclidean metric does not have to be greater than Euclidean distance globally. Let ε be any positive number with $\varepsilon < \mathrm{diam}\, D$. Then $d(z, w) = \min\{|z - w|, \varepsilon\}$ does not satisfy the assumption in Theorem 2, and $d(z, w) \geq |z - w|$ does not necessarily hold even when D is convex. With the assumptions in Theorem 2, however, we will always have $d(z, w) = |z - w|$ in any convex subdomain of D.

Next we give some examples to illustrate the use of Theorem 1 when proving necessary conditions for John disks.

For the first result, let $\mathscr{S}$ denote Rohde's class of *snowflake-like* curves, as constructed in Section 3 of [19]. The classic von Koch snowflake is a prototype of such a curve. In [19], Theorem 1.1, Rohde proved that every quasicircle is a Euclidean (global) bilipschitz image of some curve S in $\mathscr{S}$. Combining Rohde's result with Theorem 1, we immediately obtain the following.

Corollary 1. *A bounded, simply connected domain $D \subset \mathbf{R}^2$ is a John disk if and only if there are $S \in \mathscr{S}$ and a bilipschitz map $F : (\Omega, \lambda_\Omega) \to (D, \lambda_D)$, where Ω is the bounded component of $\mathbf{R}^2 \backslash S$.*

Thus the collection of domains bounded by some $S \in \mathscr{S}$ contains *all* bounded John disks up to application of a locally bilipschitz map.

Corollary 1 immediately gives information about the Assouad- and Hausdorff dimensions of the metric space $(\partial D, \lambda_D)$ for a Jordan John domain D. In this case λ_D defines a natural metric at the boundary ∂D because all boundary points of a John domain are rectifiably accessible. See [17], Remark 6.6 and [7]. The Assouad- and Hausdorff dimensions are invariant under bilipschitz maps, so that $(\partial D, \lambda_D)$ has the same dimension as the boundary of the corresponding curve S in Corollary 1. See [4], Section 7 for more general results.

The next result illustrates well how easy it is to obtain necessary conditions for John disks using Theorem 1. It is a one sided version of a result due to Gehring for quasidisks [9], Theorem 2.11.

Corollary 2 ([7], Theorem 4.7). *Let $D \subset \mathbf{R}^2$ be a bounded Jordan c-John domain. Suppose that $f : (\partial D, \lambda_D) \to (\partial D', \lambda_{D'})$ is an L-bilipschitz map, where D' is a bounded Jordan domain. Then f has an extension $\tilde{f} : (\overline{D}, \lambda_D) \to (\overline{D'}, \lambda_{D'})$ which is also bilipschitz.*

Gehring's result says that a Euclidean bilipschitz map of a quasicircle extends as a bilipschitz map of the circle union its interior domain. Thus Corollary 2 readily follows by combining this result with Theorem 1. In [7] there is a version also for unbounded John disks. It would still be interesting to know whether the converse to Corollary 2 holds or not. (It does in the bounded quasicircle case [9].)

Finally, we present a new characterization of John disks in terms of a bound for hyperbolic distance [6]. There has been some interest in this lately [22]. Denote hyperbolic distance in a simply connected domain D by $h_D(z_1, z_2)$, and define a metric j'_D by

$$j'_D(z_1, z_2) = \log\left(1 + \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_1, \partial D)}\right)\left(1 + \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_2, \partial D)}\right).$$

This is an internal distance version of the classic j_D metric [13]. In [16], Theorem 4.1, Kim and Langmeyer gave the following characterization of John disks.

Lemma 5 ([16], Theorem 4.1). *Let D be a bounded simply connected domain. Then D is a c-John disk if and only if*

$$h_D(z_1, z_2) \le b j'_D(z_1, z_2) \qquad \text{for all } z_1, z_2 \in D.$$

The constants b and c depend only on each other.

Compare Gehring-Osgood [13], Bonk, Heinonen and Koskela [2] and Kim [15]. The same bound, with j_D instead of j'_D, characterizes quasidisks [10]. Gehring and Hag [11] have also characterized quasidisks in terms of the bound:

$$h_D(z_1, z_2) \le k a_D(z_1, z_2) \qquad \text{for all } z_1, z_2 \in D, \tag{6}$$

for hyperbolic distance. Here a_D is the Apollonian metric [1], defined by

$$a_D(z_1, z_2) = \sup_{w_1, w_2 \in \partial D} \log\left(\frac{|z_1 - w_1|}{|z_2 - w_1|}\frac{|z_2 - w_2|}{|z_1 - w_2|}\right). \tag{7}$$

(Generally speaking a_D is only a pseudo metric if ∂D is a proper subset of some circle.) Let us introduce

$$a'_D(z_1, z_2) = \sup_{w_1, w_2 \in \partial_r D} \log\left(\frac{\lambda_D(z_1, w_1)}{\lambda_D(z_2, w_1)}\frac{\lambda_D(z_2, w_2)}{\lambda_D(z_1, w_2)}\right).$$

The supremum here is taken over the rectifiably accessible boundary points $\partial_r D$. For a John domain $\partial_r D = \partial D$. The following inequalities always hold:

$$a_D \le j_D, \qquad a'_D \le j'_D. \tag{8}$$

To prove e.g. the right inequality we note that for any two $w_1, w_2 \in \partial D$ we have that

$$\frac{\lambda_D(z_1, w_j)}{\lambda_D(z_2, w_j)} \le \frac{\lambda_D(z_1, z_2) + \lambda_D(z_2, w_j)}{\lambda_D(z_2, w_j)} = \frac{\lambda_D(z_1, z_2)}{\lambda_D(z_2, w_j)} + 1 \le \frac{\lambda_D(z_1, z_2)}{\operatorname{dist}(z_2, \partial D)} + 1.$$

Because w_1, w_2 were chosen arbitrarily (8) follows by symmetry. Inequality (8) shows that a'_D is always a pseudo metric, even if the supremum defining it a priori could be infinite. We now have the following, which was proved in the author's thesis [6], but with a more tedious proof.

COROLLARY 3. *A bounded, simply connected Jordan domain D in $\mathbf{R}^2$ is a c-John disk if and only if there are constants b and d such that*

$$h_D(z_1, z_2) \le b a'_D(z_1, z_2) + d \qquad \textit{for all } z_1, z_2 \in D. \tag{9}$$

PROOF. If (9) holds, then it follows from (8) that $h_D(z_1, z_2) \le b j'_D(z_1, z_2) + d$. By reasoning as in [10] we may remove the additive constant (this is written out in detail in [6], Section 6.2), and hence D is a John disk by Lemma 5.

Conversely, let D be a c-John disk. Let Ω be the K-quasidisk guaranteed by Theorem 1 and denote by F the corresponding map. Note that F can be taken to be L-bilipschitz with respect to both internal and hyperbolic distances; it is not difficult to see that F is bilipschitz with respect to the quasihyperbolic metric, and the quasihyperbolic metric is equivalent to the hyperbolic metric. Next let $z_1, z_2 \in D$ and take $\zeta_j \in \Omega$ such that $F(\zeta_j) = z_j$. Take ω_1 and ω_2 in $\partial\Omega$ such that the supremum (7) for $a_\Omega(\zeta_1, \zeta_2)$ is achieved. Let $w_j = F(\omega_j)$. Since Ω is a quasidisk, we know from (6) that there is a constant k such that $h_\Omega(\zeta_1, \zeta_2) \le k a_\Omega(\zeta_1, \zeta_2)$. Using the above facts, we have that

$$\begin{aligned} h_D(z_1, z_2) &\le L h_\Omega(\zeta_1, \zeta_2) \\ &\le k L a_\Omega(\zeta_1, \zeta_2) \\ &= kL \log\left(\frac{|\zeta_1 - \omega_1|}{|\zeta_2 - \omega_1|}\,\frac{|\zeta_2 - \omega_2|}{|\zeta_1 - \omega_2|}\right) \\ &\le kL \log\left(L^4 \frac{\lambda_D(z_1, w_1)}{\lambda_D(z_2, w_1)}\,\frac{\lambda_D(z_2, w_2)}{\lambda_D(z_1, w_2)}\right) \end{aligned}$$

$$= kL \log\left(\frac{\lambda_D(z_1, w_1)}{\lambda_D(z_2, w_1)} \frac{\lambda_D(z_2, w_2)}{\lambda_D(z_1, w_2)}\right) + 4kL \log L$$

$$\leq kLa'_D(z_1, z_2) + 4kL \log L.$$

Thus we can let $b = kL$ and $d = 4kL \log L$. □

The above proof also shows that whenever D is a bounded John disk, then a'_D is in fact a metric, and not just a pseudo metric. Besides, a similar argument can be used to prove the necessity in Lemma 5.

Remark 3. Unfortunately it is not possible to remove the additive constant in the above corollary. See the examples in [22].

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