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On the derivations of generalized Witt algebras over a field of characteristic zero

Dedicated to the memory of Professor Shigeaki Tôgô

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1. Introduction

In this paper we consider the derivations of a generalized Witt algebra $W(G, I)$ over a field $\mathfrak{k}$ of characteristic zero, where I is a non-empty index set, G is an additive submonoid of $\prod_{i \in I} \mathfrak{k}_i^+$, and $\mathfrak{k}_i^+$ $(i \in I)$ are copies of the additive group $\mathfrak{k}^+$. $W(G, I)$ is a Lie algebra which has a basis $\{w(a, i) | a \in G, i \in I\}$ and the multiplication

$$[w(a, i), w(b, j)] = a_j w(a + b, i) - b_i w(a + b, j),$$

where $i, j \in I$ and $a = (a_i)_{i \in I}$, $b = (b_i)_{i \in I} \in G$.

Generalized Witt algebras have been considered by many authors over fields of positive characteristic (e.g., [4], [6], [8]) and over fields of characteristic zero (e.g., [1], [5]). We shall show that any derivation of $W(G, I)$ is a sum of a locally inner derivation and a derivation of degree zero (Theorem 1). In the case of $G = \bigoplus_{i \in I} \boldsymbol{Z}_i$ the Lie algebra $W(G, I)$ has only locally inner derivations, in particular if $|I| < \infty$ then the derivations of $W(G, I)$ are inner (Theorem 2). Concerning the above results it is known that if G is a group and L is a finitely generated G-graded Lie algebra which admits a weight space decomposition $\bigoplus_{a \in G} L_a$ with finite dimensional L_a, then a derivation of L is a sum of inner derivation and a derivation of degree zero [2, p. 36].

For every $a \in G$ let W_a be the subspace of W spanned by $\{w(a, i) | i \in I\}$. We say that a derivation δ of $W(G, I)$ has degree b if $W_a \delta \subset W_{a+b}$ for any $a \in G$, and hence every W_a is invariant under a derivation of degree zero. Let L be a Lie algebra over $\mathfrak{k}$. A derivation δ of L is a locally inner derivation if for any finite subset F of L there exist a finite-dimensional subspace V of L containing F and $x \in W$ such that $\delta|_V = \mathrm{ad}\, x|_V$ [3]. We denote by $\mathrm{Der}(L)$, $\mathrm{Inn}(L)$, $\mathrm{Lin}(L)$ and $\mathrm{Der}(L)_0$ respectively the derivations of L, the inner derivations of L, the locally inner derivations of L and the derivations of L of degree zero.

2. The derivations of $W(G, I)$

Let δ be a derivation of $W(G, I)$, and suppose that

$$w(a, i)\delta = \sum_{s \in G, h \in I} c(a, i; s, h) w(s, h) \qquad (a \in G, i \in I), \tag{2.1}$$

where $c(a, i; s, h) \in \mathfrak{k}$ and is equal to 0 except for a finite number of s and h. Since

$$[w(0, i), w(0, j)] = 0 \qquad (i, j \in I), \tag{2.2}$$

we have

$$\begin{aligned} 0 &= [w(0, i)\delta, w(0, j)] + [w(0, i), w(0, j)\delta] \\ &= \sum_{s \in G, h \in I} c(0, i; s, h)[w(s, h), w(0, j)] - \sum_{s \in G, h \in I} c(0, j; s, h)[w(s, h), w(0, i)] \\ &= \sum_{s \in G, h \in I} (s_j c(0, i; s, h) - s_i c(0, j; s, h)) w(s, h) . \end{aligned} \tag{2.3}$$

Hence if $s_i \neq 0$ and $s_j \neq 0$, then

$$\frac{c(0, i; s, h)}{s_i} = \frac{c(0, j; s, h)}{s_j} . \tag{2.4}$$

If $s \neq 0$ then $s_i \neq 0$ for some i, and we can put

$$\alpha(s, h) = -\frac{c(0, i; s, h)}{s_i} \qquad (h \in I), \tag{2.5}$$

which is well defined by (2.4). For each $s \neq 0$ we have

$$\alpha(s, h) = 0 \tag{2.6}$$

except for a finite number of h. Thus we can define an element

$$x_s = \sum_{h \in I} \alpha(s, h) w(s, h) \tag{2.7}$$

of $W(G, I)$ for $s \in G \backslash \{0\}$.

We observe that coefficients $c(a, i; s, h)$ satisfy several relations. Applying δ to

$$[w(a, i), w(0, j)] = a_j w(a, i) \qquad (a \in G, i, j \in I), \tag{2.8}$$

we have

$$\begin{aligned} &\sum_{s \in G, h \in I} a_j c(a, i; s, h) w(s, h) \\ &= [w(a, i)\delta, w(0, j)] + [w(a, i), w(0, j)\delta] \end{aligned} \tag{2.9}$$

$$= \sum_{s\in G,h\in I} c(a, i; s, h)[w(s, h), w(0, j)]$$

$$- \sum_{s\in G,h\in I} c(0, j; s, h)[w(s, h), w(a, i)]$$

$$= \sum_{s\in G,h\in I} s_j c(a, i; s, h)w(s, h)$$

$$- \sum_{s\in G,h\in I} c(0, j; s, h)(s_i w(a + s, h) - a_h w(a + s, i))$$

$$= \sum_{s\notin a+G,h\in I} s_j c(a, i; s, h)w(s, h) + \sum_{s\in G,h\in I} (a + s)_j c(a, i; a + s, h)w(a + s, h)$$

$$- \sum_{s\in G,h\in I} s_i c(0, j; s, h)w(a + s, h) + \sum_{s\in G,h\in I} a_h c(0, j; s, h)w(a + s, i)$$

$$= \sum_{s\notin a+G,h\in I} s_j c(a, i; s, h)w(s, h)$$

$$+ \sum_{s\in G,h\in I} ((a + s)_j c(a, i; a + s, h) - s_i c(0, j; s, h))w(a + s, h)$$

$$+ \sum_{s\in G,h\in I} a_h c(0, j; s, h)w(a + s, i).$$

It follows that

$$(2.10) \qquad a_j c(a, i; s, h) = s_j c(a, i; s, h) \qquad (s \notin a + G),$$

$$(2.11) \quad a_j c(a, i; a + s, h) = (a + s)_j c(a, i; a + s, h) - s_i c(0, j; s, h) \qquad (h \neq i),$$

$$(2.12) \quad a_j c(a, i; a + s, i) = (a + s)_j c(a, i; a + s, i) - s_i c(0, j; s, i) + \sum_{h\in I} a_h c(0, j; s, h).$$

If $s \notin a + G$ then $a_j \neq s_j$ for some j. Hence by (2.10)

$$(2.13) \qquad c(a, i; s, h) = 0 \qquad (s \notin a + G),$$

and from (2.11) and (2.12)

$$(2.14) \qquad s_j c(a, i; a + s, h) = s_i c(0, j; s, h) \qquad (i, j, h \in I, h \neq i),$$

$$(2.15) \qquad s_j c(a, i; a + s, i) = s_i c(0, j; s, i) - \sum_{h\in I} a_h c(0, j; s, h) \qquad (i, j \in I).$$

We note here that (2.1) can be written by (2.13) as follows:

$$(2.16) \qquad w(a, i)\delta = \sum_{s\in G,h\in I} c(a, i; a + s, h)w(a + s, h) \qquad (a \in G, i \in I).$$

If $h \neq i$ and $s_i \neq 0$, then by (2.14) we have

$$(2.17) \qquad c(a, i; a + s, h) = c(0, i; s, h).$$

If $h \neq i$, $s \neq 0$ and $s_i = 0$, then $s_j \neq 0$ for some j, and from (2.14) we have

(2.18) $$c(a, i; a + s, h) = 0\,.$$

Hence by (2.17), (2.18) and (2.5) we obtain

(2.19) $$c(a, i; a + s, h) = -s_i\alpha(s, h) \qquad (s \neq 0, h \neq i)\,.$$

If $h = i$ and $s_i \neq 0$, then by (2.15) we have

(2.20) $$c(a, i; a + s, i) = c(0, i; s, i) - \sum_{h\in I} \frac{a_h}{s_i} c(0, i; s, h)\,.$$

If $h = i$, $s \neq 0$ and $s_i = 0$, then $s_j \neq 0$ for some j, and from (2.15) we have

(2.21) $$c(a, i; a + s, i) = -\sum_{h\in I} \frac{a_h}{s_j} c(0, j; s, h)\,.$$

Hence by (2.20), (2.21) and (2.5) we obtain

(2.22) $$c(a, i; a + s, i) = -s_i\alpha(s, i) + \sum_{h\in I} a_h\alpha(s, h) \qquad (s \neq 0)\,.$$

Now we consider a locally inner derivation of $W(G, I)$. For any fixed $a \in G$ and $i \in I$ we put

(2.23) $$S_{a,i} = \{s \in G\backslash\{0\} \mid c(a, i; a + s, h) \neq 0 \text{ for some } h \in I\}\,.$$

Clearly $S_{a,i}$ is a finite subset of G, and we can define a linear map $\hat{\delta}: W(G, I) \to W(G, I)$ as follows:

(2.24) $$w(a, i)\hat{\delta} = w(a, i)\,\mathrm{ad}\left(\sum_{s\in S} x_s\right),$$

where $S = S_{a,i}$ and $x_s = \sum_{h\in I}\alpha(s, h)w(s, h)$ as in (2.7). Let T be a finite subset of $G\backslash\{0\}$ which contains $S = S_{a,i}$. Then we have

(2.25) $$\begin{aligned} & w(a, i)\,\mathrm{ad}\left(\sum_{s\in T} x_s\right) \\ &= \sum_{s\in T, h\in I} \alpha(s, h)[w(a, i), w(s, h)] \\ &= \sum_{s\in T, h\in I} \alpha(s, h)(a_h w(a + s, i) - s_i w(a + s, h)) \\ &= \sum_{s\in T}\left(\sum_{h\neq i}(-s_i)\alpha(s, h)w(a + s, h) + (-s_i\alpha(s, i) + \sum_{h} a_h\alpha(s, h))w(a + s, i)\right) \\ &= \sum_{s\in T, h\in I} c(a, i; a + s, h)w(a + s, h) \qquad \text{(by (2.19) and (2.22))} \\ &= \sum_{s\in T, h\in I} c(a, i; a + s, h)w(a + s, h)\,. \end{aligned}$$

For any finite subset F of $W(G, I)$ we can take a finite number of $w(a, i)$ which span a subspace of $W(G, I)$ containing F. Let T be a finite subset of $G\backslash\{0\}$ containing the corresponding $S_{a,i}$'s. Then by (2.24) and (2.25) we have

$$y\hat{\delta} = y \operatorname{ad}\left(\sum_{s \in T} x_s\right) \qquad (y \in F), \tag{2.26}$$

and $\hat{\delta}$ is a locally inner derivation of $W(G, I)$.

We conclude by (2.16) and (2.26) that for any $a \in G$ and $i \in I$

$$\begin{aligned} w(a, i)\delta &= \sum_{s \in G, h \in I} c(a, i; a + s, h)w(a + s, h) \\ &= \sum_{h \in I} c(a, i; a, h)w(a, h) + \sum_{s \in S, h \in I} c(a, i; a + s, h)w(a + s, h) \\ &= \sum_{h \in I} c(a, i; a, h)w(a, h) + w(a, i)\hat{\delta}, \end{aligned} \tag{2.27}$$

and that $\delta - \hat{\delta}$ is a derivation of degree 0. If $|I|$ is finite, then

$$x = \sum_{s \neq 0, h \in I} \alpha(s, h)w(s, h) \tag{2.28}$$

is an element of $W(G, I)$, since for each $h \in I$ the coefficients $\alpha(s, h)$ are 0 except for a finite number of s. It is easy to see that $\delta - \operatorname{ad} x$ is a derivation of degree 0 in a similar way to (2.25) and (2.27). Thus we have the following

THEOREM 1. *Let G be an additive submonoid of $\prod_{i \in I} \mathfrak{k}_i^+$, and let $W = W(G, I)$. Then*

$$\operatorname{Der}(W) = \operatorname{Lin}(W) + \operatorname{Der}(W)_0 .$$

Furthermore if $|I|$ is finite, then

$$\operatorname{Der}(W) = \operatorname{Inn}(W) + \operatorname{Der}(W)_0 .$$

3. The case of $G = \bigoplus \boldsymbol{Z}_i$

In this section we consider a degree zero derivation δ of $W(G, I)$. Throughout this section we assume that $G = \bigoplus_{i \in I} \boldsymbol{Z}_i$ is a direct sum of $\boldsymbol{Z}_i$, where $\boldsymbol{Z}_i$ is a copy of $\boldsymbol{Z}$ and I is not necessarily a finite set. Suppose that

$$w(a, i)\delta = \sum_{h \in I} c(a, i, h)w(a, h) \qquad (a \in G, i \in I), \tag{3.1}$$

where $c(a, i, h) \in \mathfrak{k}$ and is equal to 0 except for a finite number of h.

We shall show that $w(a, i)\delta = c(a, i, i)w(a, i)$. We assume that $|I| \geq 2$ since the assertion is obvious for $|I| = 1$. Since

$$[w(a, i), w(b, i)] = (a_i - b_i)w(a + b, i) \qquad (a, b \in G, i \in I), \tag{3.2}$$

we have

$$(3.3)\qquad (a_i - b_i)w(a+b, i)\delta = [w(a, i)\delta, w(b, i)] + [w(a, i), w(b, i)\delta]\,.$$

Hence by (3.1)

$$(3.4)\qquad \sum_{h\in I} (a_i - b_i)c(a+b, i, h)w(a+b, h)$$
$$= \sum_{h\in I} c(a, i, h)[w(a, h), w(b, i)] - \sum_{h\in I} c(b, i, h)[w(b, h), w(a, i)]$$
$$= \sum_{h\in I} c(a, i, h)(a_i w(a+b, h) - b_h w(a+b, i))$$
$$- \sum_{h\in I} c(b, i, h)(b_i w(a+b, h) - a_h w(a+b, i))$$
$$= \sum_{h\in I} (a_i c(a, i, h) - b_i c(b, i, h))w(a+b, h)$$
$$+ \sum_{h\in I} (a_h c(b, i, h) - b_h c(a, i, h))w(a+b, i)\,.$$

It follows that

$$(3.5)\quad (a_i - b_i)c(a+b, i, h) = a_i c(a, i, h) - b_i c(b, i, h) \qquad (h \neq i)\,,$$

$$(3.6)\quad (a_i - b_i)c(a+b, i, i) = a_i c(a, i, i) - b_i c(b, i, i) + \sum_{h\in I} (a_h c(b, i, h) - b_h c(a, i, h))$$
$$= (a_i - b_i)(c(a, i, i) + c(b, i, i))$$
$$+ \sum_{h\neq i} (a_h c(b, i, h) - b_h c(a, i, h))\,.$$

Suppose that $a_i \neq 0$. If $a_i = b_i$ then from (3.5)

$$(3.7)\qquad c(a, i, h) = c(b, i, h) \qquad (h \neq i)\,.$$

Let $h \neq i$ and choose an element $b \in G$ such that $b_h \neq a_h$ and $b_l = a_l$ for $l \neq h$. Then by (3.6) and (3.7)

$$(3.8)\qquad a_h c(b, i, h) - b_h c(a, i, h) = (a_h - b_h)c(a, i, h) = 0\,,$$

whence

$$(3.9)\qquad c(a, i, h) = 0 \qquad (a_i \neq 0, h \neq i)\,.$$

Suppose that $a_i = 0$. Let e_i be an element of G with the i-th component is 1 and the other components are 0. Then

$$(3.10)\qquad [w(a+e_i, i), w(-e_i, i)] = 2w(a, i)\,.$$

Applying δ to (3.10) we have by (3.9)

$$(3.11)\qquad 2\sum_{h\in I} c(a, i, h)w(a, h) = c(a + e_i, i, i)[w(a + e_i, i), w(-e_i, i)] + c(-e_i, i, i)[w(a + e_i, i), w(-e_i, i)] = 2(c(a + e_i, i, i) - c(e_i, i, i))w(a, i)\,,$$

whence

$$(3.12)\qquad c(a, i, h) = 0 \qquad (a_i = 0, h \neq i)\,.$$

Thus by (3.9) and (3.12)

$$(3.13)\qquad w(a, i)\delta = c(a, i, i)w(a, i)$$

and it follows from (3.6) that

$$(3.14)\qquad c(a + b, i, i) = c(a, i, i) + c(b, i, i) \qquad (a_i \neq b_i)\,.$$

Now we may assume from (3.13) and (3.14) that

$$(3.15)\qquad w(a, i)\delta = c(a, i)w(a, i)\,,$$

where $c(a, i) \in \mathfrak{k}$, and that

$$(3.16)\qquad c(a + b, i) = c(a, i) + c(b, i) \qquad (a_i \neq b_i)\,.$$

We show that (3.16) holds even for $a_i = b_i$. Let $a_i = b_i$ and choose $d \in G$ such that $d_i \neq 0, a_i, 2a_i$. Then by (3.16) we have

$$(3.17)\qquad c(a + b, i) + c(d, i) = c(a + b + d, i) = c(a, i) + c(b + d, i) = c(a, i) + c(b, i) + c(d, i)\,.$$

Therefore

$$(3.18)\qquad c(a + b, i) = c(a, i) + c(b, i) \qquad (a, b \in G, i \in I)\,,$$

and $c(\cdot, i): G \to \mathfrak{k}^+$ is a homomorphism.

We claim that

$$(3.19)\qquad c(a, i) = c(a, j) \qquad (a \in G, i, j \in I)\,.$$

Since

$$(3.20)\qquad [w(e_h, i), w(e_h, h)] = w(2e_h, i) \qquad (h \neq i)\,,$$

$$(3.21)\qquad w(2e_h, i)\delta = c(2e_h, i)w(2e_h, i) = 2c(e_h, i)w(2e_h, i)$$

and

$$(3.22)\qquad [w(e_h, i), w(e_h, h)]\delta = (c(e_h, i) + c(e_h, h))[w(e_h, i), w(e_h, h)] = (c(e_h, i) + c(e_h, h))w(2e_h, i)\,,$$

we have $c(e_h, h) = c(e_h, i)$ for any $h \neq i$, which holds clearly for $h = i$. Thus

$$c(e_h, i) = c(e_h, h) \qquad (i, h \in I)\,. \tag{3.23}$$

Since G is generated by $\{e_i \mid i \in I\}$ and $c(\cdot, i)$ is a homomorphism, we have (3.19) from (3.23).

From (3.15) and (3.19) we can put

$$w(a, i)\delta = c(a)w(a, i) \qquad (a \in G, i \in I)\,, \tag{3.24}$$

where $c: G \to \mathfrak{k}^+$ is a homomorphism. For any finite subset F of $W(G, I)$ there exists a finite subset J of I satisfying

$$F \subseteq \textstyle\bigoplus_{a \in S} W_a\,, \tag{3.25}$$

where

$$S = \{a = (a_h)_{h \in I} \in G \mid a_h = 0 \text{ for any } h \in I \backslash J\}\,. \tag{3.26}$$

Put

$$x = \sum_{j \in J} c(e_j)w(0, j)\,, \tag{3.27}$$

and let $y = \sum_{a \in G} y_a$ be any element of F, where $y_a \in W_a$. Then by (3.24)

$$y\delta = \sum_{a \in G} y_a\delta = \sum_{a \in G} c(a)y_a\,, \tag{3.28}$$

and on the other hand

$$y \operatorname{ad} x = \sum_{a \in G, j \in J} c(e_j)[y_a, w(0, j)] = \sum_{a \in G, j \in J} a_j c(e_j) y_a = \sum_{a \in G} c(a)y_a \tag{3.29}$$

since $c: G \to \mathfrak{k}^+$ is a homomorphism. Therefore

$$y\delta = y \operatorname{ad} x \qquad (y \in F)\,, \tag{3.30}$$

and δ is a locally inner derivation of $W(G, I)$.

In the case of $G = \boldsymbol{Z}^n$ we put

$$x = \sum_{i=1}^{n} c(e_i)w(0, i)\,. \tag{3.31}$$

In a similar way to the above we have $\delta = \operatorname{ad} x$, and δ is an inner derivation of $W(G, I)$.

Thus by using Theorem 1 and [5, Corollary 3.3] we have the following

THEOREM 2. *Let* $G = \bigoplus_{i \in I} \boldsymbol{Z}_i$, *and let* $W = W(G, I)$. *Then* W *is simple and*

$$\operatorname{Der}(W) = \operatorname{Lin}(W)\,.$$

In particular if $|I|$ is finite and $G = \boldsymbol{Z}^n$, then

$$\operatorname{Der}(W) = \operatorname{Inn}(W).$$

REMARK. In Theorem 2 if $|I| = \infty$, then $\operatorname{Der}(W) \neq \operatorname{Inn}(W)$ in general. For example, a derivation δ can be defined by

$$w(a, i)\delta = \left(\sum_{h \in I} a_h\right) w(a, i),$$

since $a \in \bigoplus_{i \in I} \boldsymbol{Z}_i$. But δ is not an inner derivation. A Lie algebra $\mathfrak{sl}(\infty, \mathfrak{k}) = \bigcup_n \mathfrak{sl}(n, \mathfrak{k})$ is another example of a locally finite simple Lie algebra [7], and it is not hard to see that $\mathfrak{sl}(\infty, \mathfrak{k})$ has an outer derivation.

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