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Finding disjoint incompressible spanning surfaces for a link

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Introduction

In this paper we shall consider the problem of finding disjoint non-equivalent incompressible spanning surfaces for a link. It is known that there are many links in the 3-sphere which have plural non-equivalent incompressible spanning surfaces ([1], [10], [3], [8] etc.). We shall associate to each link L a certain simplicial complex $IS(L)$ whose vertex set is the set $\mathscr{I}\mathscr{S}(L)$ of the equivalence classes of incompressible spanning surfaces for L. We also introduce a 'distance' on $\mathscr{I}\mathscr{S}(L)$. Using this distance, we prove that the complex $IS(L)$ is connected. As an application of this result, the complexes $IS(L)$ for composite knots are determined under some additional conditions.

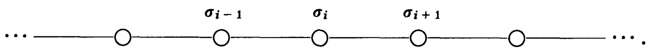
Let L be an oriented link in the 3-sphere S^3, and let $E(L) = S^3 - \operatorname{Int} N(L)$ be its exterior where $N(L)$ is a fixed tubular neighborhood of L. We shall use the term "spanning surface" for L to denote a surface $S = \Sigma \cap E(L)$ where Σ is an oriented surface in S^3 such that $\partial\Sigma = L$, Σ has no closed component and is possibly disconnected and that $\Sigma \cap N(L)$ is a collar of $\partial\Sigma$ in Σ. Two spanning surfaces for L are said to be *equivalent* if they are ambient isotopic in $E(L)$ to each other. A spanning surface S is *incompressible* (resp. *of minimal genus*) if each component of S is incompressible in $E(L)$ (resp. the Euler number $\chi(S)$ is maximum among all spanning surfaces for L). Let $\mathscr{S}(L)$ denote the set of equivalence classes of spanning surfaces for L, and $\mathscr{I}\mathscr{S}(L)$ and $\mathscr{M}\mathscr{S}(L)$ the subsets of $\mathscr{S}(L)$ consisting of those classes of incompressible and of minimal genus ones respectively.

Now we associate to each non-split oriented link L a simplicial complex $IS(L)$ as follows: The vertex set of $IS(L)$ is $\mathscr{I}\mathscr{S}(L)$, and vertices $\sigma_0, \sigma_1, \dots, \sigma_k \in \mathscr{I}\mathscr{S}(L)$ span a k-simplex if there are representatives $S_i \in \sigma_i$, $0 \leq i \leq k$, so that $S_i \cap S_j = \emptyset$ for all $i < j$. Replacing $\mathscr{I}\mathscr{S}(L)$ with $\mathscr{M}\mathscr{S}(L)$, we obtain another simplicial complex $MS(L)$, and $MS(L)$ becomes a full subcomplex of $IS(L)$. In §1 we define a 'distance' on $\mathscr{S}(L)$, and in §2 we prove the main theorem (Theorem 2.1) which is formulated in terms of the distance. The main theorem implies the following

THEOREM A. *Let L be a non-split oriented link. Then both $IS(L)$ and $MS(L)$ are connected.*

Scharlemann and Thompson [12, Prop. 5] proved the connectedness of $MS(L)$ in the case when L is a knot. We have a feeling that Theorem A is useful for the classification of the incompressible spanning surfaces for a given link. For example, Eisner [3] proved that a composite knot of two non-fibred knots has infinitely many non-equivalent minimal genus spanning surfaces. In §3 we prove the following theorem by using Theorem A.

THEOREM B. *Let K be a composite knot of two knots K_1 and K_2. Suppose that, for each $i = 1$ and 2, K_i is not fibred and the incompressible spanning surfaces for K_i are unique. Then $IS(K) = MS(K)$ and this complex is in the form of*



In Theorem B the vertices σ_i $(i \in \boldsymbol{Z})$ are represented by the surfaces constructed by Eisner [3]: See §3.

Recently we have gotten the classification of the incompressible spanning surfaces for each prime knot of ≤ 10 crossings [9]; Theorem A is extensively used in its proof.

1. Distance on $\mathscr{S}(L)$

Let $L \subset S^3$ be an oriented link, $E = E(L)$ its exterior and $\mathscr{S}(L)$ the set of equivalence classes of spanning surfaces for L. In this section, we will define a distance on $\mathscr{S}(L)$.

Consider the infinite cyclic covering $p: (\tilde{E}, a_0) \to (E, a)$ such that $p_*\pi_1(\tilde{E}, a_0)$ is the augmentation subgroup of $\pi_1(E, a)$ where $a \in E$ is a base point (cf. [2]), and let τ denote a generator of the covering transformation group. Let $S \subset E$ be a spanning surface for L, and let E_0 denote the closure of a lift of $E - S$ to $\tilde{E}$ (note that $E - S$ is connected since S has no closed component). Put $E_j = \tau^j(E_0)$ and $S_j = E_{j-1} \cap E_j$ $(j \in \boldsymbol{Z})$. Then we see that

(1.1) $\tilde{E} = \bigcup_{j \in \boldsymbol{Z}} E_j$, $p^{-1}(S) = \bigcup_{j \in \boldsymbol{Z}} S_j$ and $p|S_j: S_j \longrightarrow S$ is a homeomorphism.

Let $S' \subset E$ be another spanning surface for L. Then we have a similar description of $\tilde{E}$:

(1.2) $$\tilde{E} = \bigcup_{k \in \mathbf{Z}} E'_k,\ E'_{k-1} \cap E'_k = S'_k,\ p^{-1}(S') = \bigcup_{k \in \mathbf{Z}} S'_k \quad \text{and} \quad E'_k = \tau^k(E'_0).$$

We set

$$m = \min\{k \in \mathbf{Z} | E_0 \cap E'_k \neq \emptyset\},\ r = \max\{k \in \mathbf{Z} | E_0 \cap E'_k \neq \emptyset\} \text{ and}$$
$$d(S, S') = r - m.$$

It is easy to see that

(1.3) (a) $d(S, S') \geq 1$,

(b) $d(S, S') = 1$ if and only if $S \cap S' = \emptyset$,

(c) $E_j \cap E'_k \neq \emptyset$ if and only if $m \leq k - j \leq r$, and

(d) $E_0 \subset \bigcup_{m \leq k \leq r} E'_k,\ S_1 \subset \bigcup_{m+1 \leq k \leq r} E'_k$.

Now, for σ, $\sigma' \in \mathscr{S}(L)$, we define $d(\sigma, \sigma') \in \mathbf{Z}_+$ (the set of non-negative integers) by

$$d(\sigma, \sigma') = \begin{cases} 0 & \text{if } \sigma = \sigma', \\ \min_{S \in \sigma, S' \in \sigma'} d(S, S') & \text{if } \sigma \neq \sigma'. \end{cases}$$

PROPOSITION 1.4. *The function $d\colon \mathscr{S}(L) \times \mathscr{S}(L) \to \mathbf{Z}_+$ satisfies the axioms of distance, i.e. for every $\sigma, \sigma', \sigma'' \in \mathscr{S}(L)$,*

(i) *$d(\sigma, \sigma') = 0$ if and only if $\sigma = \sigma'$,*
(ii) *$d(\sigma, \sigma') = d(\sigma', \sigma)$ and*
(iii) *$d(\sigma, \sigma'') \leq d(\sigma, \sigma') + d(\sigma', \sigma'')$.*

PROOF. (i) follows from (1, 3) (a).

(ii) Suppose that $\sigma \neq \sigma'$ and $d(\sigma, \sigma') = d(S, S')$ for some $S \in \sigma$, $S' \in \sigma'$. By (1.3) (c), $E'_0 \cap E_j \neq \emptyset$ if and only if $-r \leq j \leq -m$. Hence $d(\sigma', \sigma) \leq d(S', S) \leq (-m) - (-r) = d(\sigma, \sigma')$. Similarly we have $d(\sigma', \sigma) \geq d(\sigma, \sigma')$, and hence $d(\sigma, \sigma') = d(\sigma', \sigma)$.

(iii) It suffices to verify the inequality in the case that $\sigma \neq \sigma'$ and $\sigma' \neq \sigma''$. Suppose that $d(\sigma, \sigma') = d(S, S')$ for $S \in \sigma$, and $S' \in \sigma'$. Then we can take $S'' \in \sigma''$ so that $d(\sigma', \sigma'') = d(S', S'')$, and $\tilde{E}$ has the following description associated with S'':

$$\tilde{E} = \bigcup_{i \in \mathbf{Z}} E''_i,\ E''_{i-1} \cap E''_i = S''_i,\ p^{-1}(S'') = \bigcup_{i \in \mathbf{Z}} S''_i \quad \text{and} \quad E''_i = \tau^i(E''_0).$$

Now suppose that $E_j \cap E'_k \neq \emptyset$ if and only if $m \leq k - j \leq r$, and that $E'_k \cap E''_i \neq \emptyset$ if and only if $m' \leq i - k \leq r$. This implies that $d(\sigma, \sigma') = r - m$ and

$d(\sigma', \sigma'') = r' - m'$. If $E_0 \cap E_i'' \neq \emptyset$, by (1.3) (c) there is $k_0 (m \leq k_0 \leq r)$ so that $E'_{k_0} \cap E_i'' \neq \emptyset$. Since $m' \leq i - k_0 \leq r'$, and $m + m' \leq i \leq r + r'$. This implies that $d(\sigma, \sigma') \leq d(S, S'') \leq (r + r') - (m + m') = d(\sigma, \sigma') + d(\sigma', \sigma'')$. □

2. Main theorem

The following Theorem 2.1 is the main theorem in this paper, from which Theorem A follows directlt. For a spanning surface S, its equivalence class will be denoted by $[S] \in \mathscr{S}(L)$.

THEOREM 2.1. *Let $L \subset S^3$ be a non-split link and $S, S' \subset E(L)$ two incompressible* (*resp. minimal genus*) *spanning surfaces for L. Suppose that $n = d([S], [S']) \geq 1$. Then there is a sequence of incompressible* (*resp. minimal genus*) *spanning surfaces $S = F_0, F_1, \ldots, F_n$ such that*

(1) $[F_n] = [S']$,
(2) $F_{i-1} \cap F_i = \emptyset$ *for each* $1 \leq i \leq n$, *and*
(3) $d([S], [F_i]) = i$ *for each* $0 \leq i \leq n$.

PROOF. We prove the theorem by induction on $n = d([S], [S'])$. In the case of $n = 1$, S' is equivalent to F with $S \cap F = \emptyset$ by (1.3) (b), and the conclusion is clear. Thus we assume that the theorem holds for $n \leq q - 1$ $(q \geq 2)$ and then will prove it for $n = q$. Moving S' by an ambient isotopy of $E = E(L)$, we may assume that

(2.2) $\quad d(S, S') = q$, $\partial S \cap \partial S' = \phi$ and S intersects S' transversely.

Note that E is irreducible since L is non-splittable. From this together with the incompressibility of S and S' we can further assume that

(2.3) $\quad$ each circle of $S \cap S'$ is essential on S and S'.

We will find an incompressible (resp. minimal genus) spanning surface $S'' \subset E$ which satisfies the condition

(2.4) $\quad S'' \cap S' = \emptyset$ and $d([S], [S'']) = q - 1$.

We use the same notation $\tilde{E}$, (1.1), (1.2), etc. for E, S, S' as in the beginning of §1. Consider E_r' where $r = \max\{k \in \boldsymbol{Z} \mid E_0 \cap E_k' \neq \emptyset\}$. We note that $E_0 \cap S'_{r+1} = \emptyset$ and $E_q \cap S_r' = \emptyset$ by (1.3). By (2.2) and (2.3), S_j intersects S_k' transversely and each circle of $S_j \cap S_k'$ is essential on S_j and S_k'. Hence

(2.5) $\quad$ each component of $S_1 \cap E_r'$ and $S_q \cap E_r'$ is incompressible in E_r'.

Let X be a regular neighborhood of $S_r' \cup (E_0 \cap E_r')$ in E_r' with $X \cap E_q = \emptyset$. Let Y be the closure of the component of $E_r' - X$ containing S'_{r+1}, and put

$R = X \cap Y$. Then R is a surface in E'_r which is disjoint from E_0, E_q, S'_r and S'_{r+1}. R inherits the orientation from S_1 and S'_r, and $p(R) \subset E$ is a spanning surface for L with $p(R) \cap S' = \emptyset$. Now we consider the two cases that both S and S' are of minimal genus and that both S and S' are incompressible separately.

Case 1: Both S and S' are of minimal genus. We see that $p(R)$ is also of minimal genus as follows. Put $Z = (E_0 \cup E_1) \cap (\bigcup_{k \leq r-1} E'_k)$. Let V be a regular neighborhood of $(E_1 \cup S'_r) \cap Z$ in Z, and W the closure of the component of $Z - V$ containing S_0 (note that $S_0 \subset Z$). Put $Q = V \cap W$. Then Q inherits the orientation from S_1 and S'_r. $p: Q \to E$ is an embedding since $Q \subset E_0 - (S_0 \cup S_1)$, and hence $p(Q)$ is a spanning surface for L. By the constructions of Q and R together with (2.3), we see that $\chi(Q) + \chi(R) \geq \chi(S_1) + \chi(S'_r) = \chi(S) + \chi(S') = 2\chi(S)$. This implies that $\chi(Q) = \chi(R) = \chi(S)$ and $p(R)$ is of minimal genus since so is S. We put $S'' = p(R)$.

Case 2: Both S and S' are incompressible. In this case R is not necessarily incompressible in E'_r. We will modify R to be incompressible.

Put $X' = \mathrm{Cl}\,(E'_r - Y)$. By applying a finite number of *simple moves* due to McMillan [11] to X' in E'_r, we obtain a 3-submanifold X'' so that each component of $\mathrm{Cl}\,(\partial X'' \cap \mathrm{Int}\, E'_r)$ is incompressible in E'_r. This means that there is a finite sequence of 3-submanifolds of E'_r, $X' = X_0, X_1, \ldots, X_k = X''$ such that, for each $1 \leq i \leq k$, one of the following conditions (i)–(iv) holds:

(i) X_i is obtained from X_{i-1} by adding a 2-handle whose core is a 2-disk $D \subset \mathrm{Int}\, E'_r$ such that $D \cap X_{i-1} = \partial D \subset \mathrm{Cl}\,(\partial X_{i-1} \cap \mathrm{Int}\, E'_r)$ and ∂D is essential in $\mathrm{Cl}\,(\partial X_{i-1} \cap \mathrm{Int}\, E'_r)$.

(ii) There is a 3-ball $C \subset \mathrm{Int}\, E'_r$ such that $X_i = X_{i-1} \cup C$ and $X_{i-1} \cap C = \partial C \subset \mathrm{Cl}\,(\partial X_{i-1} \cap \mathrm{Int}\, E'_r)$.

(iii) X_i is obtained from X_{i-1} by splitting at a 2-disk $D \subset X_{i-1}$ such that $\partial D = D \cap \mathrm{Cl}\,(\partial X_{i-1} \cap \mathrm{Int}\, E'_r)$ and ∂D is essential in $\mathrm{Cl}\,(\partial X_{i-1} \cap \mathrm{Int}\, E'_r)$.

(iv) There is a component C of X_{i-1} such that C is a 3-ball and $X_i = X_{i-1} - C$.

Claim 2.6. We can take X'' so that $X'' \cap E_q = \emptyset$ and $E_0 \cap E'_r \subset X''$.

Consider the above sequence $X' = X_0, X_1, \ldots, X_k = X''$. We will show that each X_i can be taken so that $X_i \cap E_q = \emptyset$ and $E_0 \cap E'_r \subset X_i$ by induction on i. By the definition of X', X_0 satisfies the condition. We suppose that X_{i-1} satisfies the desired condition, and consider X_i. If X_i is obtained by a simple move of type (ii), the added 3-ball C is disjoint from E_q since $C \subset \mathrm{Int}\, E'_r$ and since there is no component of $E_q \cap E'_r$ which is contained in $\mathrm{Int}\, E'_r$. Hence

X_i satisfies the desired condition. Similarly, if X_i is obtained by a simple move of type (iv), then the removed 3-ball is disjoint from E_0, and X_i satisfies the condition. In the case that X_i is obtained by a simple move of type (i), we can modify the 2-disk D, a core of the added 2-handle, so that $D \cap E_q = \emptyset$. In fact since each component of $S_q \cap E'_r$ is incompressible in E'_r by (2.5), this modification can be done by using the standard cut and paste argument. Hence we can take X_i to be satisfy the desired condition. Similarly, in the case that X_i is obtained by a simple move of type (iii), we can take the splitting 2-disk D to be disjoint from E_0 by (2.5). Hence we can take X_i to be satisfy the desired condition. Thus Claim 2.6 follows.

Let Z be the union of the components of X'' containing some components of S'_r and put $F = \mathrm{Cl}(\partial Z \cap \mathrm{Int}\, E'_r)$. Clearly $Z \cap E_q = \emptyset$ by Claim 2.6. Claim 2.6 further implies that $E_0 \cap E'_r \subset Z$ since there is no component of $E_0 \cap E'_r$ which is disjoint from S'_r. Moreover F is incompressible in E'_r and $p(F)$ becomes an incompressible spanning surface for L which is disjoint from S'. In this case we put $S'' = p(F)$.

Now we consider the two cases together, and show the following assertion

$$d([S], [S'']) = q - 1. \tag{2.7}$$

We have $d([S'], [S'']) \leq 1$ by $S' \cap S'' = \emptyset$. From this and by the assumption that $d([S], [S']) = q$ together with Proposition 1.4 (iii), we have $d([S], [S'']) \geq d([S], [S']) - d([S'], [S'']) \geq q - 1$. On the other hand, we consider the description of $\tilde{E}$ associated with S'' as (1.1) in §1:

$$\tilde{E} = \bigcup_{i \in \mathbf{Z}} E''_i, \ E''_{i-1} \cap E'_i = S''_i \quad \text{and} \quad p^{-1}(S'') = \bigcup_{i \in \mathbf{Z}} S''_i.$$

By the construction of S'', we may assume that $S''_r = F$ in Case 2 (resp. $S''_r = R$ in Case 1). Then we see that $E_0 \subset \bigcup_{r-q \leq i \leq r-1} E''_i$. Hence $d([S], [S'']) \leq d(S, S'') \leq q - 1$, and (2, 7) follows. Thus $S'' \subset E$ is an incompressible (resp. minimal genus) spanning surface for L satisfying the condition (2.4).

Now we will define the desired sequence of incompressible (resp. minimal genus) spanning surfaces $S = F_0, F_1, \ldots, F_q$. Since S'' satisfies (2.4), by the inductive assumption, there is a sequence of incompressible (resp. minimal genus) spanning surfaces $S = F_0, F_1, \ldots, F_{q-1}$ such that

(1′) $[F_{q-1}] = [S'']$,
(2′) $F_{i-1} \cap F_i = \emptyset$ for each $1 \leq i \leq q - 1$, and
(3′) $d([S], [F_i]) = i$ for each $0 \leq i \leq q - 1$.

Let $\{h_t\}$ be an isotopy of E such that $h_0 = \mathrm{id}$ and $h_1(S'') = F_{q-1}$. Put

$F_q = h_1(S')$. Then $[F_q] = [S']$, $F_{q-1} \cap F_q = \emptyset$ since $S'' \cap S' = \emptyset$, and $d([S], [F_q]) = d([S], [S']) = q$ by the assumption. Thus the theorem holds for $n = q$.

The proof of Theorem 2.1 is now completed. □

3. Simplicial complexes $IS(L)$ and $MS(L)$

In this section we first note some properties of the complexes $IS(L)$ and $MS(L)$, and then prove Theorem B. Let L be a non-split oriented link. Then the dimension of $IS(L)$ is finite by Haken's finiteness theorem [5, p. 48]. However the example described in [8] shows that $IS(L)$ is not necessarily locally finite in general. By Theorem A we can define $\ell_I(\sigma, \sigma')$ (resp. $\ell_M(\sigma, \sigma')$) for $\sigma, \sigma' \in \mathcal{IS}(L)$ (resp. $\mathcal{MS}(L)$) by the minimum length of edge paths in $\mathcal{IS}(L)$ (resp. $MS(L)$) connecting σ to σ'. Then we have

PROPOSITION 3.1. (1) *$\ell_I(\sigma, \sigma') = d(\sigma, \sigma')$ for $\sigma, \sigma' \in \mathcal{IS}(L)$.*
(2) *$\ell_M(\sigma, \sigma') = d(\sigma, \sigma')$ for $\sigma, \sigma' \in \mathcal{MS}(L)$.*

PROOF. We give the proof of (1) only because the proof of (2) is similar. First note that $\ell_I(\sigma, \sigma') = 1$ is equivalent to $d(\sigma, \sigma') = 1$. Also Theorem 2.1 shows that $\ell_I(\sigma, \sigma') \leq d(\sigma, \sigma')$. Conversely, if $\ell_I(\sigma, \sigma') = n$, then by the definition there is a finite sequence $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_n = \sigma'$ in $\mathcal{IS}(L)$ so that $\ell_I(\sigma_{i-1}, \sigma_i) = 1$ for all $1 \leq i \leq n$. Hence

$$\begin{aligned} \ell_I(\sigma, \sigma') &= \ell_I(\sigma_0, \sigma_1) + \cdots + \ell_I(\sigma_{n-1}, \sigma_n) \\ &= d(\sigma_0, \sigma_1) + \cdots + d(\sigma_{n-1}, \sigma_n) \\ &\geq d(\sigma_0, \sigma_n) = d(\sigma, \sigma'). \end{aligned}$$

Thus we get $\ell_I(\sigma, \sigma') = d(\sigma, \sigma')$. □

Now let K be a composite knot of two non-fibred knots K_1 and K_2. We will determine the simplicial complexes $IS(K)$ and $MS(K)$ under the assumption that the incompressible spanning surfaces for K_i are unique for $i = 1$ and 2. We note that there are many non-fibred 2-bridge knots whose incompressible spanning surfaces are unique (cf. [6]). Also there are many non-fibred and non-2-bridge prime knots of ≤ 10 crossings whose incompressible spanning surfaces are unique ([9]).

In [3] and [4] Eisner constructed infinitely many non-equivalent minimal genus spanning surfaces for K. We review the construction. We may assume that $E(K) = E(K_1) \cup E(K_2)$ and the intersection $A = E(K_1) \cap E(K_2) = \partial E(K_1) \cap \partial E(K_2)$ is an annulus. Let $S \subset E(K)$ be a minimal genus spanning surface for K such that so is $R_i = S \cap E(K_i)$ for K_i $(i = 1, 2)$. Note that $S = R_1 \cup R_2$ and the intersection $I = R_1 \cap R_2 = S \cap A$ is an arc. We fix an identification

$$A = \{(e^{2\pi i\theta}, s) | 0 \leq \theta \leq 1,\ 0 \leq s \leq 1\}$$

so that $I = \{(1, s) | 0 \leq s \leq 1\}$ and the loop $m: [0, 1] \to E(K)$, $\theta \mapsto (e^{2\pi i\theta}, 1)$ represents a meridian element $\mu \in \pi_1(E(K), a)$ where $a = (1, 1) \in \partial I \subset E(K)$. Let $A \times [0, 1] \subset E(K_1)$ be an embedding such that $A = A \times \{1\}$ and $(A \times [0, 1]) \cap \partial E(K) = \partial A \times [0, 1]$. We define a homeomorphism $f: E(K) \to E(K)$ by

$$\text{(3.2)} \qquad f | E(K_2) = \mathrm{id},\ f | (E(K_1) - (A \times [0, 1])) = \mathrm{id} \text{ and}$$
$$f(e^{2\pi i\theta}, s, t) = (e^{2\pi i(\theta + t)}, s, t) \text{ on } A \times [0, 1].$$

Now we put $S^{(n)} = f^n(S)$ for each $n \in \boldsymbol{Z}$. Then we see that each $S^{(n)}$ is a minimal genus spanning surface for K which satisfies the following properties:

(3.3)
(a) $S^{(n)} \cap A = I$.
(b) $S^{(n)} \cap E(K_2) = R_2$.
(c) $S^{(n)} \cap E(K_1)$ is a minimal genus spanning surface for K_1 and equivalent to R_1.
(d) $S^{(k)} = f^{k-n}(S^{(n)})$ for each $k \in \boldsymbol{Z}$.

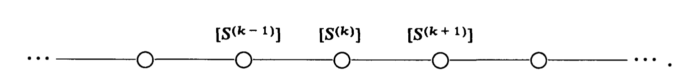
PROPOSITION 3.4 ([3], [4]). *$S^{(k)}$ is not equivalent to $S^{(n)}$ for all $k \neq n$.*

Moreover we show the following proposition; Theorem B in the introduction follows from this together with Proposition 3.1.

PROPOSITION 3.5. *Let K be a composite knot of two non-fibred knots K_1 and K_2, and let $\{S^{(n)}\}_{n \in \boldsymbol{Z}}$ be the spanning surfaces for K constructed above. Suppose in addition that, for $i = 1, 2$, the incompressible spanning surfaces for K_i are unique. Then*

(i) *any incompressible spanning surface for K is equivalent to some $S^{(n)}$, and*
(ii) *$d([S^{(n)}], [S^{(k)}]) = n - k$ for all $n \geq k$.*

PROOF. By the construction of $\{S^{(k)}\}$, we can move $S^{(k+1)}$ by a tiny isotopy of $E(K)$ so that $S^{(k+1)}$ is disjoint from $S^{(k)}$. Hence $d([S^{(k)}], [S^{(k+1)}]) = 1$. It follows from this together with Proposition 3.4 that $IS(K)$ contains the following complex as a subcomplex:



If there is an incompressible spanning surface for K which is not equivalent to any $S^{(k)}$, then by Theorem A, there is an incompressible spanning surface which is not equivalent to any $S^{(k)}$ and disjoint from some $S^{(n)}$. Thus we prove (i) by showing the following assertion for each $n \in \boldsymbol{Z}$.

(3.6) Let F be an incompressible spanning surface for K which is disjoint from

$S^{(n)}$. Then F is equivalent to $S^{(n-1)}$, $S^{(n)}$ or $S^{(n+1)}$.

Moreover it suffices to show (3.6) for $n=0$ by (3.3).

Let F be an incompressible spanning surface for K which is disjoint from $S^{(0)}$. We can move F by an isotopy of $E(K)$ so that F intersects A transeversely in an arc J since F is incompressible. Note that J is properly embedded in A and parallel to I in A. Hence $F_i = F \cap E(K_i)$ becomes an incompressible spanning surface for K_i $(i=1, 2)$. We may assume that $J = \{(-1, s) | 0 \leq s \leq 1\}$ $(\subset A)$. By the uniqueness of the incompressible spanning surfaces for K_i, F_i is parallel to R_i in $E(K_i)$ $(i=1, 2)$. Let $e^{(i)}: F_i \times [0, 1] \to E(K_i)$ be an embedding such that $e^{(i)}|F_i \times \{0\} = \mathrm{id}$ and $e^{(i)}|F_i \times \{1\}$ is a homeomorphism $F_i \to R_i$ $(i=1, 2)$. We can take $e^{(i)}$ so that $e^{(i)}(J \times [0, 1]) = A \cap e^{(i)}(F_i \times [0, 1])$ $(i=1, 2)$ in addition. Hence $e^{(i)}(J \times [0, 1]) = A_+$ or $=A_-$ where $A_+ = \{(e^{2\pi i\theta}, s) | 0 \leq \theta \leq 1/2, 0 \leq s \leq 1\}$ and $A_- = \{(e^{2\pi i\theta}, s) | 1/2 \leq \theta \leq 1, 0 \leq s \leq 1\}$. Thus there are four cases (1)–(4):

(1) $e^{(1)}(J \times [0, 1]) = e^{(2)}(J \times [0, 1]) = A_+$. In this case $F = F_1 \cup F_2$ is parallel to $S = R_1 \cup R_2$.

(2) $e^{(1)}(J \times [0, 1]) = e^{(2)}(J \times [0, 1]) = A_-$. In this case F is also parallel to S.

(3) $e^{(1)}(J \times [0, 1]) = A_+$ and $e^{(2)}(J \times [0, 1]) = A_-$. In this case we see that F is equivalent to $S^{(1)} = f(S)$.

(4) $e^{(1)}(J \times [0, 1]) = A_-$ and $e^{(2)}(J \times [0, 1]) = A_+$. In this case F is equivalent to $S^{(-1)} = f^{-1}(S)$.

Thus (3.6) and hence (i) are proved.

Next we prove (ii). It follows from (i) that if $d([S^{(k)}], [S^{(n)}]) < n-k$ for some $k<n$, then $d([S^{(i)}], [S^{(j)}]) = 1$ for some i, j with $j-i \geq 2$. Thus, to prove (ii) it suffices to show the following assertion

(3.7) $$d([S^{(k)}], [S^{(n)}]) \geq 2 \qquad \text{for all } k, n \text{ with } n-k \geq 2.$$

Moreover it suffices to show (3.7) for $k=0$ by (3.3).

We now assume that, for some $n \geq 2$, there is an isotopy $h: E(K) \times [0, 1] \to E(K)$ so that $h_0 = \mathrm{id}$ and $h_1(S^{(n)}) \cap S = \emptyset$, and then we will show that this implies a contradiction. Let $p: (\tilde{E}, a_0) \to (E(K), a)$ be the infinite cyclic covering. Putting $\tilde{E}(K_i) = p^{-1}(E(K_i))$, we see that the restriction $p: \tilde{E}(K_i) \to E(K_i)$ is the infinite cyclic covering for K_i, $\tilde{E} = \tilde{E}(K_1) \cup \tilde{E}(K_2)$ and $\tilde{A} = \tilde{E}(K_1) \cap \tilde{E}(K_2) = p^{-1}(A)$ is homeomorphic to $I \times (-\infty, \infty)$. Also $\tilde{E}$ has the following description (see §1):

(3.8) $$\tilde{E} = \bigcup_{k \in \boldsymbol{Z}} E_k,\ E_{k-1} \cap E_k = S_k,\ p^{-1}(S) = \bigcup_{k \in \boldsymbol{Z}} S_k,$$

$$a_0 \in S_0 \text{ and } (E_k, S_k, a_k) = \tau^k(E_0, S_0, a_0)$$

where τ is the covering transformation corresponding to the meridian element $\mu \in \pi_1(E(K), a)$. Putting $(E_k)_i = E_k \cap \tilde{E}(K_i)$ and $(S_k)_i = S_k \cap \tilde{E}(K_i)$, we have a description of $\tilde{E}(K_i)$ $(i = 1, 2)$:

(3.9) $$\tilde{E}(K_i) = \bigcup_{k \in \boldsymbol{Z}} (E_k)_i, \ (E_{k-1})_i \cap (E_k)_i = (S_k)_i \text{ and } p^{-1}(R_i) = \bigcup_{k \in \boldsymbol{Z}} (S_k)_i.$$

Now consider the lift $(S_0^{(n)}, a_0)$ of $(S^{(n)}, a)$. We can identify $S_0^{(n)}$ with the surface obtained as follows: Set $H = (\bigcup_{0 \le k \le n-1} (E_k)_1) \cap \partial\tilde{E}(K_1)$ and $R = H \cup (S_n)_1$. We push R into $\bigcup_{0 \le k \le n-1} (E_k)_1$ by a tiny isotopy keeping $\partial R = \partial(S_0)_1$ fixed so that the resulting surface R' satisfies the condition $R' \cap \partial E(K_1) = \partial R' = \partial(S_0)_1$. Then by the definition of $S_0^{(n)}$ we can identify $S_0^{(n)}$ with $R' \cup (S_0)_2$ (see Figure 1).

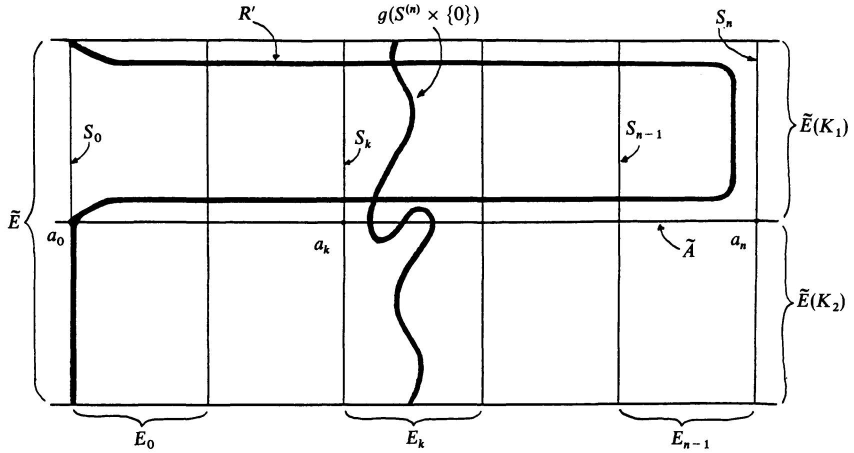


Figure 1

We next consider the lift $g: (S^{(n)} \times [0, 1], a_0 \times \{0\}) \to (\tilde{E}, a_0)$ of the restriction $h: (S^{(n)} \times [0, 1], a_0 \times \{0\}) \to (E(K), a_0)$. Note that $g(S^{(n)} \times \{0\}) = S_0^{(n)}$ and that $g(S^{(n)} \times \{1\})$ is contained in E_k for some $k \in \boldsymbol{Z}$ since $h(S^{(n)}) \cap S = \emptyset$. We move g if necessary so that g is transverse relative to $\tilde{A}$. Thus $A' = g^{-1}(\tilde{A})$ is a properly embedded surface in $S^{(n)} \times [0, 1]$ which satisfies the following

(3.10) There is a unique pair of component A_0' of A' and component C of $\partial A_0'$ so that $A' \cap (S^{(n)} \times \{0\}) = A_0' \cap (S^{(n)} \times \{0\}) = I \subset C$ and $\partial A' - C \subset S^{(n)} \times \{1\}$ (cf. (3.3)).

Since $\tilde{E}(K_i)$ $(i = 1, 2)$ are aspherical and since $S^{(n)} \times [0, 1]$ is irreducible, by the standard technique (cf. [7, Lemma 6.5]), we can modify g into a homotopy $g'\colon S^{(n)} \times [0, 1] \to \tilde{E}$ such that $g'|S^{(n)} \times \{0\} = g|S^{(n)} \times \{0\}$, $g'(S^{(n)} \times \{1\}) \subset E_k$, and that (3, 10) remains valid for $A' = g'^{-1}(\tilde{A})$ and each component of A' is incompressible in $S^{(n)} \times [0, 1]$ in addition. Hence, by Haken [5, Lemma in §8], A'_0 must be a disk, A' has no closed component and each component of $A' - A'_0$ is parallel to a surface in $S^{(n)} \times \{1\}$. It follows from this that we can further eliminate all components of $A' - A'_0$ from $g'^{-1}(\tilde{A})$ by moving g'. Thus the resulting g' satisfies the condition that $g'^{-1}(\tilde{A})$ is a disk which is isotopic to $I \times [0, 1]$ in $S^{(n)} \times [0, 1]$. Now we have two cases. Note that either $n - k \geq 2$ or $k \geq 1$ since $n \geq 2$.

CASE 1: $n - k \geq 2$. In this case we will show that $((E_{n-1})_1, (S_{n-1})_1, (S_n)_1)$ is homeomorphic to $(S_n)_1 \times ([0, 1], 0, 1)$: This contradicts the assumption that K_1 is not fibred. Firstly, using the above homotopy g', we get a homotopy $\tilde{g}\colon R' \times [0, 1] \to \tilde{E}(K_1)$ such that

(3.11) $\tilde{g}|R' \times \{0\} = \mathrm{id}$, $\tilde{g}(\partial R' \times [0, 1]) \subset \partial\tilde{E}(K_1)$, $T = \tilde{g}(R' \times \{1\})$ is a properly embedded surface in $\tilde{E}(K_1)$ and $T \subset (\tilde{E}_k)_1 - ((S_k)_1 \cup (S_{k+1})_1)$ (see Figure 2).

We also note that

(3.12) the surface $R'' = R' \cap (E_{n-1})_1$ is parallel to $\mathrm{Cl}\,(\partial(E_{n-1})_1 - (S_{n-1})_1)$ in $(E_{n-1})_1$, and in particular $\partial R''$ is parallel to $\partial(S_{n-1})_1$ in $(S_{n-1})_1$.

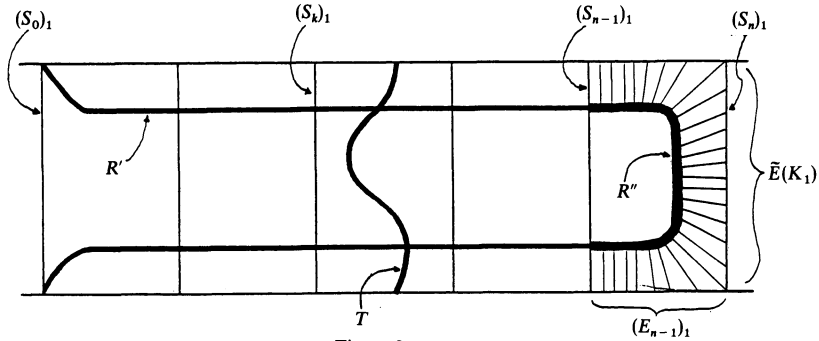


Figure 2

We now move $\tilde{g}$ to be transverse relative to $(S_{n-1})_1$. Then $X = \tilde{g}^{-1}((S_{n-1})_1)$ is a surface in $R' \times [0, 1]$, and there is only one component X_0 of X so that $X \cap \partial(R' \times [0, 1]) = X_0 \cap \partial(R' \times [0, 1]) \subset R' \times \{0\}$. Moreover $X_0 \cap \partial(R' \times [0, 1])$ is the circle $\partial R'' \times \{0\}$. We can further modify $\tilde{g}$ so that each component of $X = \tilde{g}^{-1}((S_{n-1})_1)$ is incompressible in $R' \times [0, 1]$ by [7, Lemma 6.5]. Hence, by Haken [5, Lemma in §8], $X = X_0$ and X_0 is parallel to $R'' \times \{0\}$ in $R' \times [0, 1]$. Thus the region Z bounded by $(R'' \times \{0\}) \cup X_0$ is homeomorphic to $R'' \times [0, 1]$. By using the restriction $\tilde{g}|Z$, we get a homotopy $\alpha\colon R'' \times [0, 1] \to \bigcup_{k \geq n-1} (E_k)_1$ so that $\alpha_0 = \mathrm{id}$ and $\alpha(\partial R''$

$\times [0, 1] \cup R'' \times \{1\}) \subset (S_{n-1})_1$. Thus by Waldhausen [13, Lemma 5.3], R'' is parallel to the surface in $(S_{n-1})_1$ bounded by $\partial R''$. From this together with (3.12) we see that $((E_{n-1})_1, (S_{n-1})_1, (S_n)_1)$ is homeomorphic to $(S_n)_1 \times ([0, 1], 0, 1)$; this contradicts the assumption that K_1 is not fibred.

CASE 2: $k \geq 1$. In this case, by using similar argument as in the case 1, we can show that $((E_0)_2, (S_0)_2, (S_1)_2)$ is homeomorphic to $(S_0)_2 \times ([0, 1], 0, 1)$. This contradicts the assumption that K_2 is not fibred.

Thus (3.7) and hence (ii) are proved. The proof of Proposition 3.5 is now completed. □

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