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On the oscillation of solutions of second order quasilinear ordinary differential equations

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0. Introduction

This paper is concerned with the oscillatory (and nonoscillatory) behavior of solutions of second order quasilinear ordinary differential equations of the type

$$\text{(A)} \qquad (r(t)\psi(y'))' + f(t, y) = 0\,, \qquad t \geq t_0\,,$$

subject to the hypotheses:

$$\text{(B)} \quad \begin{cases} \text{(a)} & r : [t_0, \infty) \to (0, \infty) \text{ is continuous, and } \lim_{t\to\infty} r(t) = \infty\,; \\ \text{(b)} & \psi : \boldsymbol{R} \to \boldsymbol{R} \text{ is continuously differentiable, } \psi(-s) = -\psi(s) \text{ and} \\ & \psi'(s) > 0 \text{ for } s \in \boldsymbol{R}\,; \\ \text{(c)} & f : [t_0, \infty) \times \boldsymbol{R} \to \boldsymbol{R} \text{ is continuous, } \operatorname{sgn} f(t, y) = \operatorname{sgn} y, \text{ and} \\ & f(t, y) \text{ is nondecreasing in } y \text{ for each fixed } t \geq t_0\,. \end{cases}$$

By a solution of (A) is meant a function $y : [T_y, \infty) \to \boldsymbol{R}$, $T_y \geq t_0$, such that y and $r(t)\psi(y')$ are continuously differentiable and satisfy the equation (A) for $t \geq T_y$. Those solutions of (A) which vanish in a neighborhood of infinity will be precluded from our consideration. A solution of (A) is said to be oscillatory if it has infinitely many zeros tending to infinity, and nonoscillatory otherwise.

Our objective is to establish criteria for (A) to have various types of nonoscillatory solutions, as well as criteria for all solutions of (A) to be oscillatory. The desired criteria will be developed in Sections 1 and 2 dealing respectively with the cases

$$\int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty \qquad \text{and} \qquad \int_{t_0}^{\infty} \frac{dt}{r(t)} < \infty\,.$$

In Section 3, results for (A) are shown to be applicable to derive information about the oscillatory properties of partial differential equations of generalized mean curvature type of the form

$$\operatorname{div}\left[\frac{Du}{(1 + |Du|^2)^m}\right] + g(x, u) = 0\,, \qquad 0 < m \leq 1/2\,,$$

in exterior domains in $\boldsymbol{R}^N$, where $x = (x_1, \ldots, x_N) \in \boldsymbol{R}^N$, $Du = (D_1 u, \ldots, D_N u)$, $D_i = \partial/\partial x_i$, $i = 1, \ldots, N$, and $|Du| = (\sum_{i=1}^N |D_i u|^2)^{1/2}$.

For closely related results the reader is referred to the papers [4, 6, 7, 8] in which equations of the form (A) with different nonlinear functions ψ are considered and oscillation theory for such equations is so designed as to apply to the partial differential equation

$$\operatorname{div}(|Du|^{p-2}Du) + g(x, u) = 0\,, \qquad p > 1\,,$$

in exterior domains in $\boldsymbol{R}^N$.

1. The case $\int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty$

1.1. Nonoscillation theorems

We begin with a simple observation regarding the function ψ in (A). By the hypothesis (B)–(b), ψ has the inverse function defined on $\psi(\boldsymbol{R})$; $\psi(\boldsymbol{R})$ may or may not be identical with $\boldsymbol{R}$. Throughout the paper the inverse function of ψ is denoted by ϕ. It is clear that $\phi(-s) = -\phi(s)$ and $\phi'(s) > 0$ for all $s \in \operatorname{dom} \phi = \psi(\boldsymbol{R})$, and that for any positive $\alpha \in \operatorname{dom} \phi$ there exist positive constants $a(\alpha)$ and $b(\alpha)$ such that

$$a(\alpha)s \le \phi(s) \le b(\alpha)s \qquad \text{for } 0 \le s \le \alpha\,. \tag{1.1}$$

First we consider the equation (A) in which $r(t)$ satisfies

$$\int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty\,. \tag{1.2}$$

We make use of the function

$$R(t, \tau) = \int_{\tau}^{t} \frac{ds}{r(s)}\,, \qquad t \ge \tau \ge t_0\,; \qquad R(t) \equiv R(t, t_0)\,. \tag{1.3}$$

Clearly, $R(t, \tau) \to \infty$ as $t \to \infty$ for any fixed $\tau \ge t_0$.

Three types of asymptotic behavior at infinity are possible for nonoscillatory solutions of (A) as the following lemma shows.

LEMMA 1.1. *One and only one of the following cases occurs for each nonoscillatory solution $y(t)$ of* (A):

(I) $\lim_{t\to\infty} y(t)/R(t) = \text{const} \neq 0$;
(II) $\lim_{t\to\infty} y(t)/R(t) = 0$, $\lim_{t\to\infty} |y(t)| = \infty$;
(III) $\lim_{t\to\infty} y(t) = \text{const} \neq 0$.

PROOF. Let $y(t)$ be a nonoscillatory solution of (A). With no loss of generality we may assume that $y(t)$ is positive for $t \geq t_1$ $(\geq t_0)$. The function $r(t)\psi(y'(t))$ is decreasing for $t \geq t_1$, since $[r(t)\psi(y'(t))]' = -f(t, y(t)) < 0$ for $t \geq t_1$. We claim that $r(t)\psi(y'(t)) > 0$, $t \geq t_1$, so that the finite limit $\lim_{t\to\infty} r(t)\psi(y'(t)) \geq 0$ exists. In fact, if $r(t_2)\psi(y'(t_2)) = -k < 0$ for some $t_2 \geq t_1$ and $k > 0$, then $r(t)\psi(y'(t)) \leq -k$ for $t \geq t_2$, or

$$\psi(y'(t)) \leq -\frac{k}{r(t)} \qquad \text{for } t \geq t_2. \tag{1.4}$$

Since $k/r(t) \to 0$ as $t \to \infty$ by (B) − (a), for any fixed positive $\alpha \in \operatorname{dom} \phi$ there is $t_3 \geq t_2$ such that $k/r(t) \leq \alpha$ for $t \geq t_3$. From (1.4) and (1.1) we then have

$$y'(t) \leq -\phi\left(\frac{k}{r(t)}\right) \leq -\frac{a(\alpha)k}{r(t)}, \qquad t \geq t_3 .$$

Integrating the above inequality from t_3 to t and letting $t \to \infty$, we conclude in view of (1.2) that $y(t) \to -\infty$ as $t \to \infty$, which contradicts the assumed positivity of $y(t)$. Therefore $r(t)\psi(y'(t)) > 0$ for $t \geq t_1$, and $\lim_{t\to\infty} r(t)\psi(y'(t)) =$ const ≥ 0. This implies that $\psi(y'(t)) \to 0$ and $y'(t) \to 0$ as $t \to \infty$ because $r(t) \to \infty$ as $t \to \infty$.

Suppose that $\lim_{t\to\infty} r(t)\psi(y'(t)) = c > 0$, then we have by L'Hospital's rule

$$\lim_{t\to\infty} \frac{y(t)}{R(t)} = \lim_{t\to\infty} r(t)y'(t) = \lim_{t\to\infty} r(t)\psi(y'(t))\frac{y'(t)}{\psi(y'(t))} = c\phi'(0) > 0 ,$$

which implies that $y(t)$ is of the type (I).

Suppose next that $\lim_{t\to\infty} r(t)\psi(y'(t)) = 0$. Then, we have $\lim_{t\to\infty} y(t)/R(t) = 0$ as above. Since $y'(t) > 0$ for $t \geq t_1$, $y(t)$ is an increasing function which either grows to infinity or tends to a finite positive limit as $t \to \infty$. In the former case $y(t)$ is of the type (II), and in the latter $y(t)$ is of the type (III). This completes the proof.

We want to obtain criteria for the existence of nonoscillatory solutions of (A) of the types (I), (II) and (III).

THEOREM 1.2. *The equation* (A) *has a nonoscillatory solution of the type* (I) *if and only if*

$$\int_{t_0}^{\infty} |f(t, cR(t))|\, dt < \infty \tag{1.5}$$

for some nonzero constant c.

THEOREM 1.3. *The equation* (A) *has a nonoscillatory solution of the type* (III) *if and only if*

$$\int_{t_0}^{\infty} R(t)|f(t, c)|\,dt < \infty \tag{1.6}$$

for some nonzero constant c.

PROOF OF THEOREM 1.2. (The "only if" part) Suppose that (A) has a nonoscillatory solution $y(t)$ of the type (I). We may suppose that $y(t) > 0$ for $t \geq t_1$, since a parallel argument holds if $y(t)$ is supposed to be negative. An integration of (A) shows that

$$\int_{t_1}^{\infty} f(s, y(s))ds < r(t_1)\psi(y'(t_1)),$$

which, combined with the inequality $y(t) \geq c_1 R(t)$, $t \geq t_1$, $c_1 > 0$ being a constant, yields $\int_{t_1}^{\infty} f(s, c_1 R(s))ds < \infty$.

(The "if" part) We may suppose that the constant c in (1.5) is positive. Let α be a fixed positive constant in $\operatorname{dom}\phi$. Let $a(\alpha)$ and $b(\alpha)$ be the constants appearing in (1.1). Take constants $k \geq 1$ and $l > 0$ such that

$$b(\alpha) \leq ka(\alpha) \qquad \text{and} \qquad (k+1)l \leq c,$$

and choose $T > t_0$ so large that

$$\frac{1}{r(t)}\left(\frac{l}{a(\alpha)} + \int_t^{\infty} f(s, (k+1)lR(s, T))ds\right) \leq \alpha, \qquad t \geq T,$$

and

$$b(\alpha)\left(\frac{l}{a(\alpha)} + \int_T^{\infty} f(s, (k+1)lR(s, T))ds\right) \leq (k+1)l.$$

We now define the set $Y \subset C[T, \infty)$ and the mapping $\mathscr{F}: Y \to C[T, \infty)$ by

$$Y = \{y \in C[T, \infty): lR(t, T) \leq y(t) \leq (k+1)lR(t, T), t \geq T\}$$

and

$$\mathscr{F}y(t) = \int_T^t \phi\left(\frac{1}{r(s)}\left[\frac{l}{a(\alpha)} + \int_s^{\infty} f(\sigma, y(\sigma))d\sigma\right]\right)ds, \qquad t \geq T.$$

Clearly, Y is a closed convex subset of the Fréchet space $C[T, \infty)$ with the topology of uniform convergence on compact subintervals of $[T, \infty)$. It is a matter of routine computation to show that $\mathscr{F}$ is a continuous mapping which sends Y into a compact subset of Y. Therefore, the Schauder-Tychonoff fixed point theorem (see e.g. [3, 5]) is applicable, and $\mathscr{F}$ has a fixed point y in Y. This fixed point satisfies the integral equation

$$y(t) = \int_T^t \phi\left(\frac{1}{r(s)}\left[\frac{l}{a(\alpha)} + \int_s^\infty f(\sigma, y(\sigma))d\sigma\right]\right)ds, \qquad t \geq T,$$

from which it follows that $y(t)$ is a solution of the equation (A) on $[T, \infty)$ and satisfies $\lim_{t\to\infty} y(t)/R(t) = \phi'(0)l/a(\alpha) > 0$. This completes the proof.

PROOF OF THEOREM 1.3. (The "only if" part) Suppose that (A) has a positive solution $y(t)$ of the type (III) on $[t_1, \infty)$. Then $y'(t) > 0$ for $t \geq t_1$ and $\lim_{t\to\infty} y'(t) = 0$, and so there exist positive constants c_1, c_2 and c_3 such that

$$\text{(1.7)} \qquad c_1 \leq y(t) \leq c_2 \quad \text{and} \quad \psi(y'(t)) \leq c_3 y'(t) \quad \text{for } t \geq t_1 .$$

We now integrate (A) over $[t, \infty)$, $t \geq t_1$. Using (1.7) and the fact that $r(t)\psi(y'(t)) \to 0$ as $t \to \infty$, we see that

$$\frac{1}{r(t)}\int_t^\infty f(s, c_1)ds \leq \frac{1}{r(t)}\int_t^\infty f(s, y(s))ds = \psi(y'(t)) \leq c_3 y'(t)$$

for $t \geq t_1$. Integrating the above over $[t_1, \infty)$, we obtain

$$\int_{t_1}^\infty \frac{1}{r(t)}\int_t^\infty f(s, c_1)dsdt \leq c_2 c_3 ,$$

which clearly implies (1.6). Likewise, the existence of an eventually negative solution of the type (III) leads to the inequality (1.6) with a suitable negative constant c.

(The "if" part) Suppose that $c > 0$ in (1.6). Let a positive $\alpha \in \operatorname{dom} \phi$ be fixed. Choose $T > t_0$ so large that

$$\frac{1}{r(t)}\int_t^\infty f(s, c)ds \leq \alpha , \qquad t \geq T,$$

and

$$b(\alpha)\int_T^\infty \frac{1}{r(s)}\int_s^\infty f(\sigma, c)d\sigma ds \leq \frac{c}{2}.$$

If we define

$$Z = \{z \in C[T, \infty) : c/2 \leq z(t) \leq c, t \geq T\}$$

and

$$\mathcal{G}z(t) = c - \int_t^\infty \phi\left(\frac{1}{r(s)}\int_s^\infty f(\sigma, z(\sigma))d\sigma\right)ds, \qquad t \geq T,$$

then it can be shown via the Schauder-Tychonoff fixed point theorem that $\mathcal{G}$ has a fixed point z in Z, i.e.,

$$z(t) = c - \int_t^\infty \phi\left(\frac{1}{r(s)} \int_s^\infty f(\sigma, z(\sigma))d\sigma\right)ds\,, \qquad t \geq T.$$

It is easy to see that $z(t)$ is a solution of (A) on $[T, \infty)$ satisfying $\lim_{t\to\infty} z(t) = c > 0$.

If $c < 0$ in (1.6), the same procedure as above establishes the existence of a negative solution of the type (III) of (A). The proof is complete.

Unlike the solutions of the types (I) and (III) it is not easy to characterize the type (II) solutions of (A). Only sufficient conditions for the existence of such solutions will be given below.

THEOREM 1.4. *The equation* (A) *has a nonoscillatory solution of type* (II) *if*

$$\int_{t_0}^\infty |f(t, cR(t))|dt < \infty \tag{1.8}$$

for some nonzero constant c, and

$$\int_{t_0}^\infty R(t)|f(t, d)|dt = \infty \tag{1.9}$$

for all nonzero constants d with cd > 0.

PROOF. We may assume that $c > 0$ in (1.8). Let $\alpha \in \operatorname{dom} \phi$ and $l \in (0, c)$ be fixed. Let $T > t_0$ be such that

$$\frac{1}{r(t)} \int_t^\infty f(s, l + lR(s, T))ds \leq \alpha\,, \qquad t \geq T,$$

and

$$b(\alpha) \int_T^\infty f(s, l + lR(s, T))ds \leq l\,.$$

Then, applying the Schauder-Tychonoff fixed point theorem, we can show that the mapping $\mathscr{H}$ defined by

$$\mathscr{H}w(t) = l + \int_T^t \phi\left(\frac{1}{r(s)} \int_s^\infty f(\sigma, w(\sigma))d\sigma\right)ds\,, \qquad t \geq T,$$

possesses a fixed element w in the set W given by

$$W = \{w \in C[T, \infty) : l \leq w(t) \leq l + lR(t, T), t \geq T\}\,.$$

From the integral equation for w

$$w(t) = l + \int_T^t \phi\left(\frac{1}{r(s)} \int_s^\infty f(\sigma, w(\sigma))d\sigma\right)ds\,, \qquad t \geq T,$$

it follows that

$$(1.10)\quad \psi(w'(t)) = \frac{1}{r(t)} \int_t^\infty f(s, w(s))ds \le \frac{1}{r(t)} \int_t^\infty f(s, l + lR(s))ds, \qquad t \ge T,$$

and

$$(1.11)\qquad w(t) \ge l + a(\alpha) \int_T^t \frac{1}{r(s)} \int_s^\infty f(\sigma, l)d\sigma ds, \qquad t \ge T.$$

From (1.10) we see that $\lim_{t\to\infty} w(t)/R(t) = 0$, and from (1.11) we conclude that $\lim_{t\to\infty} w(t) = \infty$. Thus $w(t)$ is a positive type (II) solution of (A). This completes the proof.

1.2. Oscillation theorems

We are now interested in the situation in which all solutions of the equation (A) are oscillatory. In view of the Atkinson-Belohorec oscillation theory [1, 2] one may expect that a characterization for such a situation for (A) can be obtained under additional conditions on the nonlinear function $f(t, y)$.

DEFINITION 1.5. (i) $f(t, y)$ is said to be *strongly superlinear* if there is a constant $\gamma > 1$ such that $|y|^{-\gamma}|f(t, y)|$ is nondecreasing in $|y|$ for each fixed $t \ge t_0$.

(ii) $f(t, y)$ is said to be *strongly sublinear* if there is a constant δ, $0 < \delta < 1$, such that $|y|^{-\delta}|f(t, y)|$ is nonincreasing in $|y|$ for each fixed $t \ge t_0$.

THEOREM 1.6. *Let $f(t, y)$ be strongly superlinear. All solutions of* (A) *are oscillatory if and only if*

$$(1.12)\qquad \int_{t_0}^\infty R(t)|f(t, c)|dt = \infty$$

for every nonzero constant c.

THEOREM 1.7. *Let $f(t, y)$ be strongly sublinear. All solutions of* (A) *are oscillatory if and only if*

$$(1.13)\qquad \int_{t_0}^\infty |f(t, cR(t))|dt = \infty$$

for every nonzero constant c.

PROOF OF THEOREM 1.6. The "only if part" of Theorem 1.6 readily follows from Theorem 1.3.

To prove the "if" part, suppose that (A) has a nonoscillatory solution $y(t)$ on $[t_1, \infty)$, $t_1 \geq t_0$. We may assume that $y(t) > 0$ for $t \geq t_1$. As was seen in the proof of Lemma 1.1, $y'(t) > 0$ for $t \geq t_1$ and $y'(t) \to 0$ as $t \to \infty$, and so there are two positive constants c_1, c_2 such that

$$y(t) \geq c_1 \quad \text{and} \quad \psi(y'(t)) \leq c_2 y'(t) \quad \text{for } t \geq t_1 . \tag{1.14}$$

An integration of (A) over $[t, \infty)$, $t \geq t_1$, gives

$$r(t)\psi(y'(t)) \geq \int_t^\infty f(s, y(s))ds , \qquad t \geq t_1 ,$$

which combined with the second inequality in (1.14) implies

$$c_2 y'(t) \geq \frac{1}{r(t)} \int_t^\infty f(s, y(s))ds , \qquad t \geq t_1 . \tag{1.15}$$

On the other hand, from the strong superlinearity of $f(t, y)$ and the first inequality in (1.14) it follows that

$$\begin{aligned} f(t, y(t)) &= [y(t)]^{-\gamma} f(t, y(t))[y(t)]^\gamma \\ &\geq c_1^{-\gamma} f(t, c_1)[y(t)]^\gamma , \qquad t \geq t_1 . \end{aligned} \tag{1.16}$$

Using (1.16) in (1.15) and noting that $y(t)$ is increasing, we have

$$\begin{aligned} c_2 y'(t) &\geq \frac{c_1^{-\gamma}}{r(t)} \int_t^\infty f(s, c_1)[y(s)]^\gamma ds \\ &\geq \frac{c_1^{-\gamma}}{r(t)} \int_t^\infty f(s, c_1)ds \cdot [y(t)]^\gamma , \qquad t \geq t_1 . \end{aligned}$$

Dividing the above inequality by $[y(t)]^\gamma$ and integrating from t_1 to ∞, we obtain

$$\begin{aligned} c_1^{-\gamma} \int_{t_1}^\infty \frac{1}{r(t)} \int_t^\infty f(s, c_1)dsdt &= c_1^{-\gamma} \int_{t_1}^\infty R(s, t_1) f(s, c_1)ds \\ &\leq \frac{c_2[y(t_1)]^{1-\gamma}}{\gamma - 1} < \infty , \end{aligned}$$

which contradicts (1.12). This completes the proof of the "if" part of Theorem 1.6.

PROOF OF THEOREM 1.7. We need only to prove the "if" part, since the "only if" part follows from Theorem 1.2. Let $y(t)$ be a nonoscillatory solution of (A) which may be assumed to be positive on $[t_1, \infty)$ without loss of generality. As in the proof of Theorem 1.6 the inequality (1.15) holds for some constant $c_2 > 0$. Let $t_2 > t_1$ be fixed and integrate (1.15) over $[t_1, t]$, $t \geq t_2$. We then see that

$$c_2^* y(t) \geq R(t) \int_t^\infty f(s, y(s))ds, \qquad t \geq t_2 \tag{1.17}$$

for some constant $c_2^* > 0$. Since there is a constant $c_1 > 0$ such that $y(t) \leq c_1 R(t)$ for $t \geq t_2$, the strong sublinearity of $f(t, y)$ implies that

$$\begin{aligned} f(t, y(t)) &= [y(t)]^{-\delta} f(t, y(t))[y(t)]^\delta \\ &\geq c_1^{-\delta} f(t, c_1 R(t)) \left(\frac{y(t)}{R(t)} \right)^\delta, \qquad t \geq t_2 . \end{aligned} \tag{1.18}$$

Combining (1.17) with (1.18), we have

$$\frac{c_2^* y(t)}{R(t)} \geq c_1^{-\delta} \int_t^\infty f(s, c_1 R(s)) \left(\frac{y(s)}{R(s)} \right)^\delta ds, \qquad t \geq t_2,$$

from which, denoting the right-hand side by $z(t)$, we find

$$(-[z(t)]^{1-\delta})' \geq (1 - \delta)(c_1 c_2^*)^{-\delta} f(t, c_1 R(t)), \qquad t \geq t_2 .$$

An integration of this inequality shows that

$$\int_{t_2}^\infty f(t, c_1 R(t))dt < \infty,$$

which contradicts (1.13). This completes the proof.

EXAMPLE 1.8. Consider the equation

$$(t \sinh y')' + q(t)|y|^\lambda \operatorname{sgn} y = 0, \qquad t \geq 1, \tag{1.19}$$

where $\lambda > 0$ and $q : [1, \infty) \to \boldsymbol{R}_+$ is continuous. The function $R(t)$ defined by (1.3) is taken to be $R(t) = \log t$. Applying the above theorems to (1.19) we have the following statements.

(i) There exists a nonoscillatory solution $y(t)$ of (1.19) such that $\lim_{t\to\infty} y(t)/\log t = \text{const} \neq 0$ if and only if

$$\int_1^\infty q(t)(\log t)^\lambda dt < \infty . \tag{1.20}$$

(ii) There exists a nonoscillatory solution $y(t)$ of (1.19) such that $\lim_{t\to\infty} y(t) = \text{const} \neq 0$ if and only if

$$\int_1^\infty q(t) \log t\, dt < \infty . \tag{1.21}$$

(iii) There exists a nonoscillatory solution $y(t)$ of (1.19) such that $\lim_{t\to\infty} y(t)/\log t = 0$ and $\lim_{t\to\infty} y(t) = \infty$ (or $-\infty$) if (1.20) holds and

$$\int_1^\infty q(t)\log t dt = \infty . \tag{1.22}$$

Note that (1.20) and (1.22) are consistent only if $0 < \lambda < 1$.

(iv) All solutions of (1.19) with $\lambda > 1$ are oscillatory if and only if (1.22) holds.

(v) All solutions of (1.19) with $0 < \lambda < 1$ are oscillatory if and only if

$$\int_1^\infty q(t)(\log t)^\lambda dt = \infty. \tag{1.23}$$

2. The case $\int_{t_0}^\infty \frac{dt}{r(t)} < \infty$

2.1. Nonoscillation theorems

We now turn to the case where the function $r(t)$ in (A) satisfies

$$\int_{t_0}^\infty \frac{dt}{r(t)} < \infty . \tag{2.1}$$

We define the function $\rho(t)$ by

$$\rho(t) = \int_t^\infty \frac{ds}{r(s)}, \qquad t \ge t_0 . \tag{2.2}$$

Throughout this section we make the additional assumption that the inverse function ϕ of ψ in (A) satisfies $\inf\{\phi(s)/s : s \in \operatorname{dom}\phi,\ s > 0\} > 0$, that is, there is a constant $k > 0$ such that

$$\phi(s) \ge ks \qquad \text{for } s \in \operatorname{dom}\phi,\ s > 0 . \tag{2.3}$$

LEMMA 2.1. *If $y(t)$ is a positive solution of* (A) *such that $y'(t)$ is eventually negative, then*

$$y(t) \ge -kr(t)\psi(y'(t))\rho(t) \tag{2.4}$$

for all sufficiently large t, where k is a constant in (2.2).

PROOF. Suppose that $y(t) > 0$ and $y'(t) < 0$ for $t \ge t_1$. Since $r(t)\psi(y'(t))$ is decreasing, $r(t)\psi(y'(t)) \ge r(s)\psi(y'(s))$ for $s \ge t$, i.e.,

$$\psi(-y'(s)) \ge \frac{r(t)\psi(-y'(t))}{r(s)}, \qquad s \ge t \ge t_1 .$$

We operate the function ϕ on both sides of the above inequality, obtaining in view of (2.3)

$$-y'(s) \geq \frac{kr(t)\psi(-y'(t))}{r(s)}, \qquad s \geq t \geq t_1 .$$

An integration of this inequality yields

$$y(t) \geq kr(t)\psi(-y'(t))\rho(t), \qquad t \geq t_1 .$$

The asymptotic behavior of possible nonoscillatory solutions of (A) is described in the following lemma.

LEMMA 2.2. *If $y(t)$ is a nonoscillatory solution of* (A), *then there exist positive constants c_1, c_2 and $t_1 \geq t_0$ such that*

$$c_1 \rho(t) \leq |y(t)| \leq c_2 \qquad \textit{for } t \geq t_1 . \tag{2.5}$$

PROOF. Let $y(t)$ be a nonoscillatory solution of (A). Assume that $y(t)$ is eventually positive. Then (A) implies that $r(t)\psi(y'(t))$ is eventually decreasing, so that $y'(t)$ is eventually of constant sign. If $y'(t) > 0$ for $t \geq t_1$, then from the inequality $r(t)\psi(y'(t)) \leq r(t_1)\psi(y'(t_1)) \equiv k_1$, $t \geq t_1$, we have $\psi(y'(t)) \leq k_1/r(t)$, $t \geq t_1$. Noting that $k_1/r(t) \to 0$ as $t \to \infty$ and using (1.1), we see that there are constants $k_2 > 0$ and $t_2 \geq t_1$ such that $y'(t) \leq k_2/r(t)$ for $t \geq t_2$, which implies $y(t) \leq k_2\rho(t_2) + y(t_2)$ for $t \geq t_2$. This proves the second inequality in (2.5). If $y'(t) < 0$ for $t \geq t_1$, then noting that (2.4) holds for $t \geq t_2$, $t_2 > t_1$ being sufficiently large, and $r(t)\psi(y'(t)) \leq r(t_2)\psi(y'(t_2))$, $t \geq t_2$, we see that $y(t) \geq -kr(t_2)\psi(y'(t_2))\rho(t)$, $t \geq t_2$, proving the first inequality in (2.5). A parallel argument holds if $y(t)$ is an eventually negative solution of (A).

Lemma 2.2 shows that the following three types of asymptotic behavior at infinity are possible for nonoscillatory solutions $y(t)$ of (A) subject to (2.1) and (2.3):

(I) $\lim_{t\to\infty} y(t) = \text{const} \neq 0$;

(II) $\lim_{t\to\infty} y(t) = 0$, $\limsup_{t\to\infty} |y(t)|/\rho(t) = \infty$;

(III) $0 < \liminf_{t\to\infty} |y(t)|/\rho(t)$, $\limsup_{t\to\infty} |y(t)|/\rho(t) < \infty$.

We begin by characterizing the solutions of the type (III) of (A).

THEOREM 2.3. *The equation* (A) *has a nonoscillatory solution of type* (III) *if and only if*

$$\int_{t_0}^{\infty} |f(t, c\rho(t))|\, dt < \infty \tag{2.6}$$

for some nonzero constant c.

PROOF. (The "only if" part) Let $y(t)$ be a nonoscillatory solution of type (III) of (A). We may assume that $y(t)$ is eventually positive, in which case $y'(t)$ is eventually negative. So, there is $t_1 \geq t_0$ such that $y(t) > 0$ and $y'(t) < 0$ for $t \geq t_1$. Integrating (A) from t_1 to t and using (2.4), we have

$$\frac{y(t)}{\rho(t)} \geq k \int_{t_1}^{t} f(s, y(s))ds\,, \qquad t \geq t_1\,. \tag{2.7}$$

Since $c_1\rho(t) \leq y(t) \leq c_2\rho(t)$, $t \geq t_1$, for some positive constants c_1 and c_2, it follows from (2.7) that

$$k \int_{t_1}^{\infty} f(s, c_1\rho(s))ds \leq c_2\,.$$

(The "if" part) We may suppose that the constant c in (2.6) is positive. Let $\alpha \in \operatorname{dom}\phi$, $\alpha > 0$, be fixed and let $k \geq 1$ be such that $b(\alpha) \leq ka(\alpha)$, where $a(\alpha)$ and $b(\alpha)$ are given by (1.1). Choose $l > 0$ and $T > t_0$ so that $(k+1)l \leq c$,

$$\frac{1}{r(t)}\left(\frac{l}{a(\alpha)} + \int_T^t f(s, (k+1)l\rho(s))ds\right) \leq \alpha\,, \qquad t \geq T,$$

$$b(\alpha)\left(\frac{l}{a(\alpha)} + \int_T^{\infty} f(s, (k+1)l\rho(s))ds\right) \leq (k+1)l\,,$$

and consider the set Y and the mapping $\mathscr{F}$ defined by

$$Y = \{y \in C[T, \infty) : l\rho(t) \leq y(t) \leq (k+1)l\rho(t),\ t \geq T\}$$

and

$$\mathscr{F}y(t) = \int_t^{\infty} \phi\left(\frac{1}{r(s)}\left[\frac{l}{a(\alpha)} + \int_T^s f(\sigma, y(\sigma))d\sigma\right]\right)ds\,, \qquad t \geq T.$$

It is not difficult to verify that $\mathscr{F}$ is a continuous mapping which sends Y into a compact subset of Y. Therefore, the Schauder-Tychonoff fixed point theorem ensures the existence of an element $y \in Y$ such that $y = \mathscr{F}y$, which gives rise to a positive solution of type (III) of (A). This completes the proof.

The next theorem concerns the solutions of the type (I) of (A).

THEOREM 2.4. *The equation* (A) *has a nonoscillatory solution of the type* (I) *if and only if there exist constants* $c \neq 0$ *and* $t_1 \geq t_0$ *such that*

$$\frac{1}{r(t)} \int_{t_1}^{t} |f(s, c)|\,ds \in \operatorname{dom}\phi\,, \qquad t \geq t_1\,, \tag{2.8}$$

and

$$\int_{t_1}^{\infty} \phi\left(\frac{1}{r(t)} \int_{t_1}^{t} |f(s, c)| ds\right) dt < \infty . \tag{2.9}$$

PROOF. (The "only if" part) Let $y(t)$ be a type (I) solution of (A) which is positive on $[t_1, \infty)$. There is a constant $c > 0$ such that $y(t) \geq c$ for $t \geq t_1$.

Suppose first that $y'(t) > 0$ for $t \geq t_1$. An integration of (A) then shows that

$$\int_{t_1}^{\infty} f(s, c) ds \leq \int_{t_1}^{\infty} f(s, y(s)) ds \leq r(t_1) \psi(y'(t_1)) < \infty ,$$

from which both (2.8) and (2.9) follow immediately.

Next suppose that $y'(t) < 0$ for $t \geq t_1$. We then have

$$\psi(-y'(t)) \geq \frac{1}{r(t)} \int_{t_1}^{t} f(s, y(s)) ds \geq \frac{1}{r(t)} \int_{t_1}^{t} f(s, c) ds , \qquad t \geq t_1 , \tag{2.10}$$

which clearly implies $[r(t)]^{-1} \int_{t_1}^{t} f(s, c) ds \in \psi(\boldsymbol{R}_+) \subset \operatorname{dom} \phi$ for $t \geq t_1$. Integrating the inequality

$$-y'(t) \geq \phi\left(\frac{1}{r(t)} \int_{t_1}^{t} f(s, c) ds\right), \qquad t \geq t_1 ,$$

which follows from (2.10), we conclude that

$$\int_{t_1}^{\infty} \phi\left(\frac{1}{r(t)} \int_{t_1}^{t} f(s, c) ds\right) dt \leq y(t_1) < \infty .$$

(The "if" part) We suppose that $c > 0$ in (2.8) and (2.9). Let $T \geq t_1$ be large enough so that

$$\frac{1}{r(t)} \int_{T}^{t} f(s, c) ds \in \operatorname{dom} \phi , \qquad t \geq T,$$

and

$$\int_{T}^{\infty} \phi\left(\frac{1}{r(s)} \int_{T}^{s} f(\sigma, c) d\sigma\right) dt \leq \frac{c}{2} .$$

The desired positive type (I) solution of (A) will be obtained as a fixed point of the mapping

$$\mathscr{G} z(t) = \frac{c}{2} + \int_{t}^{\infty} \phi\left(\frac{1}{r(s)} \int_{T}^{s} f(\sigma, z(\sigma)) d\sigma\right) ds , \qquad t \geq T,$$

in the set $Z = \{z \in C[T, \infty) : c/2 \leq z(t) \leq c, t \geq T\}$. The verification is left to the reader. This finishes the proof.

Sufficient conditions for the existence of a type (II) solution are given below.

THEOREM 2.5. *Let* $L^* = \sup(\mathrm{dom}\,\phi)$. *Suppose that there are constants* $L_0 \in (0, L^*)$, $c \neq 0$ *and* $t_1 \geq t_0$ *such that*

$$\frac{1}{r(t)}\int_{t_1}^{t} |f(s, c)|\,ds \leq L_0\,, \qquad t \geq t_1\,, \tag{2.11}$$

and

$$\int_{t_1}^{\infty} \frac{1}{r(t)}\int_{t_1}^{t} |f(s, c)|\,ds dt < \infty\,. \tag{2.12}$$

If furthermore

$$\int_{t_1}^{\infty} |f(t, d\rho(t))|\,dt = \infty \tag{2.13}$$

for every $d \neq 0$ *with* $cd > 0$, *then the equation* (A) *has a nonoscillatory solution of the type* (II).

PROOF. Suppose that $c > 0$ in (2.11) and (2.12). Take an $L \in (L_0, L^*)$ and let $l > 0$ be fixed. Choosing $T \geq t_1$ so that

$$l\rho(T) \leq c\,,$$

$$\frac{1}{r(t)}\left(\frac{l}{a(L)} + \int_{T}^{t} f(s, c)ds\right) \leq L\,, \qquad t \geq T,$$

$$b(L)\left(\frac{l}{a(L)}\rho(T) + \int_{T}^{\infty} \frac{1}{r(s)}\int_{T}^{s} f(\sigma, c)d\sigma ds\right) \leq c\,,$$

where $a(L)$ and $b(L)$ are constants appearing in (1.1) with $\alpha = L$, we define

$$W = \{w \in C[T, \infty) : l\rho(t) \leq y(t) \leq c,\, t \geq T\}$$

and

$$\mathscr{H}w(t) = \int_{t}^{\infty} \phi\left(\frac{1}{r(s)}\left[\frac{l}{a(L)} + \int_{T}^{s} f(\sigma, w(\sigma))d\sigma\right]\right)ds\,, \qquad t \geq T.$$

Let $w \in W$ be a fixed point of $\mathscr{H}$ guaranteed by the Schauder-Tychonoff theorem. Then, $w = w(t)$ is a positive solution of (A) on $[T, \infty)$. It is clear that $w(t) \to 0$ as $t \to \infty$. That $w(t)/\rho(t) \to 0$ as $t \to \infty$ follows from L'Hospital's rule and (2.13) as follows:

$$\lim_{t\to\infty}\frac{w(t)}{\rho(t)}=\lim_{t\to\infty}\frac{w'(t)}{\rho'(t)}=\lim_{t\to\infty}r(t)\phi\left(\frac{1}{r(t)}\left[\frac{l}{a(L)}+\int_T^t f(\sigma,w(\sigma))d\sigma\right]\right)$$
$$\geq\lim_{t\to\infty}\left(l+a(L)\int_T^t f(\sigma,l\rho(\sigma))d\sigma\right)=\infty\,.$$

This completes the proof.

2.2. Oscillation theorems

The purpose of this subsection is to present criteria for the oscillation of all solutions of the equation (A) which is either strongly superlinear or strongly sublinear in the sense of Definition 1.5. The superlinear case is easy to investigate.

THEOREM 2.6. *Let $f(t, y)$ be strongly superlinear. All solutions of* (A) *are oscillatory if and only if*

$$\int_{t_0}^{\infty}|f(t,c\rho(t))|\,dt=\infty \tag{2.14}$$

for every nonzero constant c.

PROOF. The "only if" part follows from Theorem 2.3. Suppose that (2.14) holds and that (A) has a nonoscillatory solution $y(t)$. It suffices to consider the cases: $\{y(t)>0,\ y'(t)>0,\ t\geq t_1\}$ and $\{y(t)>0,\ y'(t)<0,\ t\geq t_1\}$. In the first case, it can be shown that $\int_{t_1}^{\infty}f(t,c_1)dt<\infty$ for some constant $c_1>0$, which implies $\int_{t_1}^{\infty}f(t,c_1\rho(t))dt<\infty$, a contradiction to (2.14). In the second case, we note that (2.7) holds:

$$\frac{y(t)}{\rho(t)}\geq k\int_{t_1}^{t}f(s,y(s))ds\,,\qquad t\geq t_1\,. \tag{2.7}$$

By Lemma 2.2 there is a constant $c_2>0$ such that $y(t)/\rho(t)\geq c_2$ for $t\geq t_1$, and so

$$f(t,y(t))\geq c_2^{-\gamma}f(t,c_2\rho(t))\left(\frac{y(t)}{\rho(t)}\right)^{\gamma},\qquad t\geq t_1\,,$$

by the strong superlinearity of $f(t, y)$. Combining this with (2.7), we have

$$\frac{y(t)}{\rho(t)}\geq kc_2^{-\gamma}\int_T^t f(s,c_2\rho(s))\left(\frac{y(s)}{\rho(s)}\right)^{\gamma}ds\,,\qquad t\geq t_1\,.$$

If we denote by $z(t)$ the right-hand side of the above inequality, we obtain

$$z'(t)\geq kc_2^{-\gamma}f(t,c_2\rho(t))[z(t)]^{\gamma}\,,\qquad t\geq t_1\,,$$

an integration of which yields for any $t_2 > t_1$

$$kc_2^{-\gamma}\int_{t_2}^{\infty} f(s, c_2\rho(s))ds \leq \frac{[z(t_2)]^{1-\gamma}}{\gamma-1} < \infty .$$

This again contradicts (2.14), and the proof of the "if" part is complete.

For the sublinear case of (A) only a sufficient condition for the oscillation of all solutions will be presented.

THEOREM 2.7. *Let $f(t, y)$ be strongly sublinear. All solutions of* (A) *are oscillatory if*

$$\int_{t_0}^{\infty} \frac{1}{r(t)}\int_{t_0}^{t} |f(s, c)| dsdt = \infty \tag{2.15}$$

for every nonzero constant c.

REMARK 2.8. The condition (2.15) is equivalent to

$$\int_{t_0}^{\infty} \rho(t)|f(t, c)| dt = \infty . \tag{2.16}$$

PROOF OF THEOREM 2.7. Let $y(t)$ be a positive solution of (A) on $[t_1, \infty)$. If $y'(t) > 0$ for $t \geq t_1$, then there is a constant $c_1 > 0$ such that $\int_{t_1}^{\infty} f(s, c_1)ds < \infty$, which contradicts (2.15). If $y'(t) < 0$ for $t \geq t_1$, then we have (2.10), from which after operating ϕ, we see that

$$-y'(t) \geq \frac{k}{r(t)}\int_{T}^{t} f(s, y(s))ds , \qquad t \geq t_1 . \tag{2.17}$$

Noting that $y(t) \leq c_2$, $t \geq t_1$, for some constant $c_2 > 0$, and using the strong sublinearity of $f(t, y)$, we obtain from (2.17)

$$\begin{aligned} -y'(t) &\geq \frac{k}{r(t)}\int_{t_1}^{t} [y(s)]^{-\delta} f(s, y(s))[y(s)]^{\delta} ds \\ &\geq \frac{kc_2^{-\delta}}{r(t)}[y(t)]^{\delta}\int_{t_1}^{t} f(s, c_2)ds , \qquad t \geq t_1 . \end{aligned}$$

Divide the above by $[y(t)]^{\delta}$ and integrate from t_1 to ∞. We then have

$$kc_2^{-\delta}\int_{t_1}^{\infty}\frac{1}{r(t)}\int_{t_1}^{t} f(s, c_2)dsdt \leq \frac{[y(t_1)]^{1-\delta}}{1-\delta} < \infty ,$$

which contradicts (2.15). This completes the proof.

EXAMPLE 2.9. Consider the equation

(2.18) $$[e^t \log(y' + \sqrt{1 + (y')^2})]' + q(t)|y|^\lambda \operatorname{sgn} y = 0, \qquad t \geq 0,$$

where $\lambda > 0$ is a constant and $q(t)$ is a positive continuous function on $\boldsymbol{R}_+$. Since the inverse function of $\psi(s) = \log(s + \sqrt{1 + s^2})$ is $\phi(s) = \sinh s$, which clearly satisfies (2.3), all the theorems of this section can be applied to this equation. Noting that the function $\rho(t)$ defined by (2.2) is $\rho(t) = e^{-t}$, we have the following statements.

(i) The equation (2.18) possesses a nonoscillatory solution $y(t)$ satisfying

$$0 < \liminf_{t \to \infty} e^t |y(t)|, \qquad \limsup_{t \to \infty} e^t |y(t)| < \infty$$

if and only if

(2.19) $$\int_0^\infty e^{-\lambda t} q(t) dt < \infty.$$

(ii) Suppose that

(2.20) $$\int_0^t q(s) ds = O(e^t) \qquad \text{as } t \to \infty$$

and

(2.21) $$\int_0^\infty e^{-t} \int_0^t q(s) ds dt < \infty.$$

Then the equation (2.18) possesses a nonoscillatory solution $y(t)$ such that $\lim_{t \to \infty} y(t) = \text{const} \neq 0$. Note that under (2.20) the condition (2.21) is equivalent to

(2.22) $$\int_0^\infty e^{-t} q(t) dt < \infty.$$

(iii) Suppose that (2.20) and (2.22) hold. If in addition

(2.23) $$\int_0^\infty e^{-\lambda t} q(t) dt = \infty,$$

then the equation (2.18) possesses a nonoscillatory solution $y(t)$ such that $\lim_{t \to \infty} y(t) = 0$ and $\limsup_{t \to \infty} e^t |y(t)| = \infty$. Note that (2.22) and (2.23) are consistent only if $0 < \lambda < 1$.

(iv) All solutions of (2.18) with $\lambda > 1$ are oscillatory if and only if (2.23) holds.

(v) All solutions of (2.18) with $0 < \lambda < 1$ are oscillatory if

$$\int_0^\infty e^{-t} q(t) dt = \infty.$$

3. Application to partial differential equations

3.1. The oscillation theory for (A) developed in the preceding sections will now be applied to derive information about the oscillatory behavior of solutions of quasilinear elliptic equations of the type

$$\operatorname{div}\left[\frac{Du}{(1+|Du|^2)^m}\right] + F(|x|, u) = 0 \tag{3.1}$$

subject to the hypotheses:

(3.2) (a) m is a constant with $0 < m \leq 1/2$;
(b) $F(t, u)$ is continuous on $[a, \infty) \times \boldsymbol{R}$, $\operatorname{sgn} F(t, u) = \operatorname{sgn} u$ and $F(t, u)$ is nondecreasing in u for each fixed $t \geq a$.

The importance of such equations has been widely recognized in connection with the study of capillarity and surfaces with prescribed mean curvature; see e.g., [6, 7, 8].

Our consideration will be restricted to radial solutions of (3.1) defined in exterior domains of the form $\Omega_R = \{x \in \boldsymbol{R}^N : |x| \geq R\}$, $R \geq a$. Basic to the subsequent discussion is the observation that the equation (3.1) for radial functions $u = y(|x|)$ in Ω_R reduces to the ordinary differential equation

$$\left[\frac{t^{N-1}y'}{(1+(y')^2)^m}\right]' + t^{N-1}F(t, y) = 0 \tag{3.3}$$

for $t \geq R$, where a prime denotes differentiation with respect to t. This equation is a special case of (A) with

$$r(t) = t^{N-1}, \qquad \psi(s) = \frac{s}{(1+s^2)^m}, \qquad f(t, y) = t^{N-1}F(t, y), \tag{3.4}$$

for which the conditions in (B) are clearly satisfied. Let ϕ denote the inverse function of ψ appearing in (3.4). Then $\operatorname{dom}\phi = \boldsymbol{R}$ if $0 < m < 1/2$, $\operatorname{dom}\phi = (-1, 1)$ if $m = 1/2$, and we see that $\inf\{\phi(s)/s : s \in \operatorname{dom}\phi,\ s > 0\} = 1$, implying that the condition (2.3) is satisfied for (3.3). We also see that the function $r(t)$ in (3.4) satisfies (1.2) or (2.1) according as $N = 2$ or $N \geq 3$, and that the functions $R(t)$ and $\rho(t)$ defined by (1.3) and (2.2) can be taken to be

$$R(t) = \log t \quad (N = 2) \qquad \text{and} \qquad \rho(t) = t^{2-N} \quad (N \geq 3).$$

The above observation enables us to apply the theorems of Sections 1 and 2 to the two-dimensional and higher-dimensional cases of (3.3), respectively.

3.2. The two-dimensional case: $N = 2$. In this subsection we give a list of results for the two-dimensional equation (3.1) which follows from Theorems 1.2–1.4, 1.6 and 1.7 applied to (3.3) with $N = 2$.

(i) The equation (3.1) (with $N = 2$) has a nonoscillatory radial solution $u(x)$ which is defined in an exterior domain Ω_R, $R \geq a$, and satisfies

$$\lim_{|x|\to\infty} \frac{u(x)}{\log|x|} = \text{const} \neq 0 \tag{3.5}$$

if and only if there is a nonzero constant c such that

$$\int_1^\infty t|F(t, c\log t)|\,dt < \infty\,. \tag{3.6}$$

(ii) The equation (3.1) (with $N = 2$) has a nonoscillatory radial solution $u(x)$ which is defined in Ω_R, $R \geq a$, and satisfies

$$\lim_{|x|\to\infty} u(x) = \text{const} \neq 0 \tag{3.7}$$

if and only if there is a nonzero constant c such that

$$\int_1^\infty t\log t \cdot |F(t, c)|\,dt < \infty\,. \tag{3.8}$$

(iii) The equation (3.1) (with $N = 2$) has a nonoscillatory radial solution $u(x)$ which is defined in Ω_R, $R \geq a$, and satisfies

$$\lim_{|x|\to\infty} \frac{u(x)}{\log|x|} = 0\,, \qquad \lim_{|x|\to\infty} |u(x)| = \infty \tag{3.9}$$

if (3.6) holds for some nonzero constant c and

$$\int_1^\infty t\log t \cdot |F(t, d)|\,dt = \infty \tag{3.10}$$

for every nonzero constant d with $cd > 0$.

(iv) Let the function $F(t, y)$ be strongly superlinear. All radial solutions of (3.1) ($N = 2$) are oscillatory if and only if (3.10) is satisfied for every nonzero constant d.

(v) Let the function $F(t, y)$ be strongly sublinear. All radial solutions of (3.1) ($N = 2$) are oscillatory if and only if

$$\int_1^\infty t|F(t, d\log t)|\,dt = \infty \tag{3.11}$$

for every nonzero constant d.

3.3. The higher-dimensional case: $N \geq 3$. This subsection is concerned with radial solutions of the higher dimensional equation (3.1). The results following from Theorems 2.3–2.7 are listed below.

(i) The equation (3.1) ($N \geq 3$) has a nonoscillatory radial solution $u(x)$ defined in Ω_R satisfying

$$k_1 |x|^{2-N} \leq |u(x)| \leq k_2 |x|^{2-N}, \qquad |x| \geq R \tag{3.12}$$

for some positive constants k_1, k_2 and R if and only if

$$\int_1^\infty t^{N-1} |F(t, ct^{2-N})| \, dt < \infty \tag{3.13}$$

for some nonzero constant c.

(ii) The equation (3.1) ($N \geq 3$) has a nonoscillatory radial solution $u(x)$ which is defined in Ω_R and satisfies

$$\lim_{|x| \to \infty} u(x) = \text{const} \neq 0 \tag{3.14}$$

if and only if there exist constants $c \neq 0$ and $t_1 \geq a$ such that

$$t^{1-N} \int_{t_1}^t s^{N-1} |F(s, c)| \, ds \in \operatorname{dom} \phi, \qquad t \geq t_1 \tag{3.15}$$

and

$$\int_{t_1}^\infty \phi\left(t^{1-N} \int_{t_1}^t s^{N-1} |F(s, c)| \, ds \right) dt < \infty. \tag{3.16}$$

(iii) Let $L^* = \sup(\operatorname{dom} \phi)$. Suppose that there are constants $L_0 \in (0, L^*)$, $c \neq 0$ and $t_1 \geq a$ such that

$$t^{1-N} \int_{t_1}^t s^{N-1} |F(s, c)| \, ds \leq L_0, \qquad t \geq t_1 \tag{3.17}$$

and

$$\int_{t_1}^\infty t^{1-N} \int_{t_1}^t s^{N-1} |F(s, c)| \, ds dt < \infty. \tag{3.18}$$

If in addition

$$\int_{t_1}^\infty t^{N-1} |F(s, dt^{2-N})| \, dt = \infty \tag{3.19}$$

for every nonzero constant d with $cd > 0$, then (3.1) ($N \geq 3$) has a nonoscillatory radial solution $u(x)$ such that

$$\lim_{|x| \to \infty} u(x) = 0, \qquad \lim_{|x| \to \infty} |x|^{N-2} |u(x)| = \infty. \tag{3.20}$$

(iv) Let $F(t, u)$ be strongly superlinear. All radial solutions of (3.1) ($N \geq 3$) are oscillatory if and only if (3.19) holds for every nonzero constant d.

(v) Let $F(t, u)$ be strongly sublinear. All radial solutions of (3.1) ($N \geq 3$) are oscillatory if

$$\int_a^\infty t^{1-N} \int_a^t s^{N-1} |F(s, d)| \, ds dt = \infty \tag{3.21}$$

for every nonzero constant d.

REMARK 3.1. Assume that there is a constant $c \neq 0$ such that

$$\int_a^\infty t |F(t, c)| \, dt < \infty . \tag{3.22}$$

Then, the conditions (3.15) and (3.17) are satisfied, since for any $t_1 \geq a$

$$t^{1-N} \int_{t_1}^t s^{N-1} |F(s, c)| \, ds \leq \int_{t_1}^t |F(s, c)| \, ds \leq \int_{t_1}^\infty |F(s, c)| \, ds , \qquad t \geq t_1 ,$$

and the last integral tends to zero as $t_1 \to \infty$ because of (3.22). Note that both (3.16) and (3.18) are equivalent to (3.22), and that (3.21) is equivalent to

$$\int_a^\infty t |F(t, d)| \, dt = \infty . \tag{3.23}$$

EXAMPLE 3.2. Oscillation and nonoscillation criteria will be given for the particular equation of mean curvature type

$$\operatorname{div}\left[\frac{Du}{(1 + |Du|^2)^{1/2}}\right] + q(|x|)|u|^\lambda \operatorname{sgn} u = 0 , \tag{3.24}$$

where λ is a positive constant and $q(t)$ is a positive continuous function on $[a, \infty)$.

The desired criteria are selected from the following list:

$$\int_1^\infty t(\log t)^\lambda q(t) \, dt < \infty , \qquad N = 2 ; \tag{3.25}$$

$$\int_1^\infty t(\log t)^\lambda q(t) \, dt = \infty , \qquad N = 2 ; \tag{3.26}$$

$$\int_1^\infty t \log t \cdot q(t) \, dt < \infty , \qquad N = 2 ; \tag{3.27}$$

$$\int_1^\infty t \log t \cdot q(t) \, dt = \infty , \qquad N = 2 ; \tag{3.28}$$

$$\int_1^\infty t q(t) \, dt < \infty , \qquad N \geq 3 ; \tag{3.29}$$

(3.30) $$\int_1^\infty tq(t)dt = \infty\,, \qquad N \geq 3\,;$$

(3.31) $$\int_1^\infty t^{N-1-\lambda(N-2)}q(t)dt < \infty\,, \qquad N \geq 3\,;$$

(3.32) $$\int_1^\infty t^{N-1-\lambda(N-2)}q(t)dt = \infty\,, \qquad N \geq 3\,.$$

From the results of the above subsections we have the following statements regarding radial solutions of (3.24) in exterior domains.

(i) (3.25) is a necessary and sufficient condition for (3.24) to have a nonoscillatory radial solution $u(x)$ satisfying (3.5).

(ii) (3.27) is a necessary and sufficient condition for (3.24) to have a nonoscillatory radial solution $u(x)$ satisfying (3.7).

(iii) (3.25) and (3.28) are sufficient conditions for the sublinear equation (3.24) ($0 < \lambda < 1$) to have a nonoscillatory radial solution $u(x)$ satisfying (3.9).

(iv) (3.28) is a necessary and sufficient condition for all radial solutions of the superlinear equation (3.24) ($\lambda > 1$) to be oscillatory.

(v) (3.26) is a necessary and sufficient condition for all radial solutions of the sublinear equation (3.24) ($0 < \lambda < 1$) to be oscillatory.

(vi) (3.31) is a necessary and sufficient condition for (3.24) to have a nonoscillatory radial solution $u(x)$ satisfying (3.12) for some positive constants k_1, k_2 and R.

(vii) (3.29) is a sufficient condition for (3.24) to have a nonoscillatory radial solution $u(x)$ satisfying (3.14).

(viii) (3.29) and (3.32) are sufficient conditions for the sublinear equation (3.24) ($0 < \lambda < 1$) to have a nonoscillatory radial solution $u(x)$ satisfying (3.21).

(ix) (3.32) is a necessary and sufficient condition for all radial solutions of the superlinear equation (3.24) ($\lambda > 1$) to be oscillatory.

(x) (3.30) is a necessary and sufficient condition for all radial solutions of the sublinear equation (3.24) ($0 < \lambda < 1$) to be oscillatory.

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