Function of Generalized Scalar Operators

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**Introduction.** In the author's previous work [4], generalized scalar and spectral operators were defined and studied on a separated locally convex space E for which L(E) is quasi-complete. The present paper studies some sufficient conditions for a function, especially a polynomial, of two commuting generalized scalar or spectral operators to be again of the same type. In this respect, this paper is a kind of supplement to [4] and we shall use the definitions and results given in [4] without their detailed descriptions.

We shall be especially interested in the case where basic algebras are \( C^\omega \) or \( C^\omega \) (the case considered by Foias [1] on a Banach space) and the results for this case will be stated after corresponding theorems. As a special case, we shall see that sum and product of \( C^\omega \)-scalar operators are again \( C^\omega \)-scalar; sum and product of \( C^\omega \)-spectral operators with compact spectrum are \( C^\omega \)-spectral, under certain assumptions on commutativity.

Finally, one should remark that the theory can be easily extended to a function or a polynomial of a finite number of commuting generalized scalar or spectral operators.

§1. \( \Phi \)-proper functions.

Let \( X \) be a set and \( \Psi \) be an algebra of functions on \( X \) containing constants and having a locally convex topology. Given a basic algebra ([4], Def. 1.1) \( \Phi \), we consider the following notion:

**Definition 1.1.** A function \( f \) on \( X \) will be called \( \Phi \)-proper with respect to \( \Psi \) if it satisfies the following three conditions:

i) \( \phi \circ f \in \Psi \) for all \( \phi \in \Phi \),

ii) \( \phi \rightarrow \phi \circ f \) is continuous from \( \Phi \) into \( \Psi \), and

iii) \( 1 \in \{ \phi \circ f ; \ \phi \in \Phi \} \).

We remark that if \( \Psi \) is \( \Phi \)-admissible ([4], Def. 1.4), then any bounded function \( f \in \Psi \) is \( \Phi \)-proper; if, in addition, \( \Phi \) contains constants, then any function \( f \in \Psi \) is \( \Phi \)-proper.

**Proposition 1.1.** Let \( X \) be a locally compact space (resp. a \( C^\omega \)-manifold), let \( \Psi = C^\omega(X) \) or \( C^\omega(X) \oplus C \) (resp. \( C^\omega(X) \) or \( C^\omega(X) \oplus C \)) and let \( \Phi = C^\omega \) or \( C^\omega \) (resp. \( C^\omega \) or \( C^\omega \)). Then, \( f \) is \( \Phi \)-proper w.r.t. \( \Psi \) if and only if \( f \in \Psi \).

**Proof:** “If” part is obvious from the above remark and [4], Example 1.1–1.4. We omit the proof of “only if” part here, since it is not essential in
Theorem 1.1 and Proposition 2.6 of [4] can be modified by our notion of \(\Phi\)-proper function as follows:

**Proposition 1.2.** Suppose there is a continuous homomorphism \(V\) of \(\Psi\) into \(L(E)\) such that \(V(I)=I\). If \(f\) is \(\Phi\)-proper w.r.t. \(\Psi\), then \(U_f\) defined by \(U_f(\phi) = V(\phi f)\) is a \(\Phi\)-spectral representation on \(E\). ([4], Def. 1.3.) If, in addition, \(f\) is \(\Phi\)-proper w.r.t. \(\Psi\), then \(V(f)\) is \(\Phi\)-scalar.

§2. Tensor product of two commuting representations.

Let \(\Phi_1\) and \(\Phi_2\) be two basic algebras contained in \(B(C)\). The complete inductive tensor product \(\Phi_1 \otimes \Phi_2\) (This notation is due to L. Schwartz [5]. Grothendieck [2] denoted it \(\Phi_1 \otimes \Phi_2\)). of these two algebras can be regarded as a subalgebra of \(B(C^2)\), the space of all locally bounded complex valued functions (Borel measurable) on \(C^2=R^4\), provided that the topologies of \(\Phi_1\) and \(\Phi_2\) are stronger than the induced topologies from \(B(C)\). Let \(\mathcal{I}(\Phi_1, \Phi_2)\) be the subalgebra of \(B(C^2)\) generated by \(\Phi_1 \otimes \Phi_2\) and \(C\) (the constant functions). Then, it is easy to see that

\[
\begin{align*}
(\text{i}) & \quad \mathcal{I}(\Phi_1, \Phi_2) = \Phi_1 \otimes \Phi_2 \quad \text{if} \quad 1 \in \Phi_1 \otimes \Phi_2, \\
(\text{ii}) & \quad \mathcal{I}(\Phi_1, \Phi_2) = (\Phi_1 \otimes \Phi_2) \oplus C \quad \text{if} \quad 1 \in \Phi_1 \otimes \Phi_2.
\end{align*}
\]

In the latter case, we introduce the topology of direct sum in \(\mathcal{I}(\Phi_1, \Phi_2)\).

We say that a \(\Phi_1\)-spectral representation \(U_1\) and a \(\Phi_2\)-spectral representation \(U_2\) are commuting if

\[
U_1(\phi_1)U_2(\phi_2) = U_2(\phi_2)U_1(\phi_1) \quad \text{for all} \quad \phi_1 \in \Phi_1 \quad \text{and} \quad \phi_2 \in \Phi_2.
\]

**Proposition 2.1.** If \(U_1\) and \(U_2\) are commuting \(\Phi_1\)- and \(\Phi_2\)-spectral representations respectively, then there is a continuous homomorphism \(V\) of \(\mathcal{I}(\Phi_1, \Phi_2)\) into \(L(E)\) such that

\[
\begin{align*}
1) & \quad V(\phi_1 \otimes \phi_2) = U_1(\phi_1)U_2(\phi_2) \quad \text{for} \quad \phi_1 \in \Phi_1, \phi_2 \in \Phi_2, \\
2) & \quad V(I) = I.
\end{align*}
\]

**Proof:** \(V = U_1 \otimes U_2\) on \(\Phi_1 \otimes \Phi_2\) is defined by the equation 1). It is a homomorphism on \(\Phi_1 \otimes \Phi_2\). Since the mapping \(\phi_1 \otimes \phi_2 \mapsto V(\phi_1 \otimes \phi_2)\) is separately continuous from \(\Phi_1 \times \Phi_2\) into \(L(E)\), the mapping \(\phi_1 \otimes \phi_2 \mapsto V(\phi_1 \otimes \phi_2)\) is continuous with respect to the inductive tensor product topology on \(\Phi_1 \otimes \Phi_2\). (See [2] or [5].) Hence, \(V\) can be extended continuously over \(\Phi_1 \otimes \Phi_2\). To prove 2), we consider the two cases:

(\text{i}) The case \(\mathcal{I}(\Phi_1, \Phi_2) = \Phi_1 \otimes \Phi_2\).

Choose \(\{\phi_a\} \subseteq \Phi_1\) and \(\{\psi_\beta\} \subseteq \Phi_2\) such that \(U_1(\phi_a) \to I\) and \(U_2(\psi_\beta) \to I\). For any \(x \in E\),

\[
V(1)x = V(1) \lim_\alpha U_1(\phi_a)x = V(1) \lim_\alpha U_1(\phi_a) \lim_\beta U_2(\psi_\beta)x
\]
\[= \lim_{\alpha} \lim_{\beta} V(1) U_1(\varphi_{\alpha}) U_2(\psi_{\beta}) x\]
\[= \lim_{\alpha} \lim_{\beta} V(1) V(\varphi_{\alpha} \otimes \psi_{\beta}) x\]
\[= \lim_{\alpha} \lim_{\beta} V(\varphi_{\alpha} \otimes \psi_{\beta}) x\]
\[= \lim_{\alpha} \lim_{\beta} U_1(\varphi_{\alpha}) U_2(\psi_{\beta}) x = x.\]

Hence, \(V(1) = I\).

(ii) The case \(\mathcal{V}(\Phi_1, \Phi_2) = (\Phi_1 \otimes \Phi_2) \oplus \mathbb{C}\).

We extend \(V\) over \(\mathcal{V}(\Phi_1, \Phi_2)\) by
\[V(\varphi + c) = V(\varphi) + cI \quad \text{for} \quad \varphi \in \Phi_1 \otimes \Phi_2.\]

Then, it is easy to see that \(V\) is a continuous homomorphism on \(\mathcal{V}(\Phi_1, \Phi_2)\) and \(V(1) = I\).

\textbf{Q. E. D.}

\textbf{THEOREM I.} Let \(U_1\) and \(U_2\) be commuting \(\Phi_1\)- and \(\Phi_2\)-spectral representations respectively. If \(f\) is \(\Phi\)-proper w.r.t. \(\mathcal{V}(\Phi_1, \Phi_2)\), then
\[W_f : W_f(\varphi) = V(\varphi \circ f)\]
is a \(\Phi\)-spectral representation, where \(V\) is the homomorphism defined in the previous proposition. If, in addition, \(f \in \mathcal{V}(\Phi_1, \Phi_2)\), then \(V(f)\) is \(\Phi\)-scalar.

\textbf{PROOF:} This is an immediate consequence of Proposition 1.2 and the previous proposition.

\textbf{Corollary.} If \(U_1\) and \(U_2\) are commuting \(C^*\)-spectral (resp. \(C^\infty_z\)-spectral) representations, then \(W_f\) defined above is a \(C^*\)-spectral representation for any \(f \in C^*(R^4)\) (resp. \(f \in C^\infty_z(R^4) \oplus \mathbb{C}\)) and \(V(f)\) is \(C^*\)-scalar for such a function \(f\).

\textbf{PROOF:} Grothendieck [2] (II, p. 84) and L. Schwartz [5] (I, p. 94, II, p. 17) showed that \(C^\infty_z \otimes C^* = C^*(R^4)\) and \(C^\infty_z \otimes C^\infty_z = C^\infty_z(R^4)\). Hence \(\mathcal{V}(C^\infty_z, C^*) = C^*(R^4)\) and \(\mathcal{V}(C^\infty_z, C^\infty_z) = C^\infty_z(R^4) \oplus \mathbb{C}\). We know by proposition 1.1. that any function \(f \in \mathcal{V}\) is \(C^*\)-proper in these cases. Therefore, the corollary follows from the theorem.

\textbf{Remark:} The above corollary does not hold for \(C^0\)-spectral or \(C^0_z\)-spectral representations. The example by Kakutani [3] gives an indication of this fact. The difficulty appears in the fact that the topology of \(C^0 \otimes C^0\) is strictly stronger than the topology of \(C^0(R^4)\).

\section*{§3 Polynomials of two commuting scalar operators.}

Let \(S_i, i = 1, 2\), be \(\Phi_i\)-scalar operators on \(E\) with commuting \(\Phi_i\)-spectral representations \(U_i\). Let \(P(z_1, z_2)\) be a polynomial in two variables. Then \(P(S_1, S_2)\) is formally given as an element of \(L(E)\). Is it scalar again? The answer is partially given by the following proposition.
PROPOSITION 3.1.

(i) If \( sp(S_i) \) are compact \((i = 1, 2)\), then \( P(S_1, S_2) \) is \( \Phi \)-scalar whenever \( \Psi(\Phi_1, \Phi_2) \) is \( \Phi \)-admissible.

(ii) Suppose both \( \Phi_1 \) and \( \Phi_2 \) contain polynomials, so that \( S_i = U_i(z) \) \((i = 1, 2)\), and suppose \( \Psi(\Phi_1, \Phi_2) \) is \( \Phi \)-admissible and \( \Phi \) contains constants. Then, \( P(S_1, S_2) \) is \( \Phi \)-scalar.

\[ \text{PROOF:} \]

(i) We can choose \( \{\Phi_i\} \) \((i = 1, 2)\) such that \( \Phi_i = 1 \) on a neighborhood of \( sp(S_i) \). Let \( f(z_1, z_2) = P(z_1 \phi_1(z_1), z_2 \phi_2(z_2)) \). Then \( f \) is bounded and \( f \in \Phi_1 \otimes \Phi_2 \), so that \( f \) is \( \Phi \)-proper w.r.t. \( \Psi(\Phi_1, \Phi_2) \). Hence,
\[
V(f) = P(U_1(z_1 \phi_1(z_1)), U_2(z_2 \phi_2(z_2))) = P(S_1, S_2)
\]
is \( \Phi \)-scalar by Theorem I.

(ii) Under our assumptions, \( P \in \Phi_1 \otimes \Phi_2 \). Since \( \Phi \) contains constants, \( P \) is \( \Phi \)-proper w.r.t. \( \Psi(\Phi_1, \Phi_2) \) (see the remark after Def. 1.1). Hence, again by Theorem I, \( V(P) = P(U_1(z_1), U_2(z_2)) = P(S_1, S_2) \) is \( \Phi \)-scalar.

COROLLARY. (i) Let \( S_1 \) and \( S_2 \) be \( C^\ast \)-scalar operators with commuting \( C^\ast \)-spectral representations. Then, \( P(S_1, S_2) \) is \( C^\ast \)-scalar for any polynomial \( P \).

(ii) Let \( S_1 \) and \( S_2 \) be \( C^\ast \)-scalar operators with commuting \( C^\ast \)-spectral representations. If \( sp(S_i) \) are compact, then \( P(S_1, S_2) \) is \( C^\ast \)-scalar for any polynomial \( P \). (cf. Foias [1], Theorem 4)

REMARK. In the case \( sp(S_i) \) \((i = 1, 2)\) are compact, we can define (uniquely) \( f(S_1, S_2) \) for any function \( f(z_1, z_2) \) in two variables, holomorphic in a neighborhood of \( sp(S_1) \times sp(S_2) \). (Waelbroeck [6]) Here, we may assume that \( f \in \Psi(C^\ast, C^\ast) \), so that \( f(S_1, S_2) \) is \( C^\ast \)-scalar.

\[ \text{§4 Polynomials of two commuting spectral operators.} \]

For generalized spectral operators, the following theorem is an easy consequence of the previous section.

\[ \text{THEOREM II.} \]

Let \( T_i \) be \( \Phi_i \)-spectral operators with \( \Phi_i \)-spectral representations \( U_i \) \((i = 1, 2)\). Suppose that \( T_1, T_2, U_1(\phi_1) \) and \( U_2(\phi_2) \) belong to a same commutative subalgebra of \( L(E) \) and suppose \( \Psi(\Phi_1, \Phi_2) \) is \( \Phi \)-admissible.

If \( sp(T_i), i = 1, 2, \) are compact, then \( P(T_1, T_2) \) is \( \Phi \)-spectral for any polynomial \( P \).

\[ \text{PROOF:} \]

Let \( i = 1 \) or \( 2 \). If \( sp(T_i) \) is compact, then \( T_i = S_i + Q_i \), where \( S_i = U(z \phi_i) \) and \( Q_i \) is quasi-nilpotent on \( E \). Then, \( T_i, S_i, Q_i \) \((i = 1, 2)\) commute each other, so that
\[
P(T_1, T_2) = P(S_1, S_2) + R_1(S_1, S_2, Q_1, Q_2)Q_1 + R_2(S_1, S_2, Q_1, Q_2)Q_2,
\]
where \( R_1 \) and \( R_2 \) are polynomials.

By Proposition 3.1, \( P(S_1, S_2) \) is \( \Phi \)-scalar and its spectrum is compact. Since the quasi-nilpotent operators form an ideal in \( L_r(E) \) (the algebra of all ele-
ments of \( L(E) \) with compact spectrum), \( R_1Q_1 + R_2Q_2 \) is again quasi-nilpotent. Hence, by Th. 4.2 of [4], \( P(T_1, T_2) \) is a \( \Phi \)-spectral operator.

**Remark.** This proof can not be applied to the case where the \( sp(T_i) \) are not compact, due to the following fact: “Let \( Q \) be a quasi-nilpotent operator. If \( S \in L(E) \) has non-compact spectrum, then \( SQ \) is not necessarily quasi-nilpotent even if \( S \) and \( Q \) commute.” (cf. Appendix).

If, however, \( Q \) is nilpotent, then \( SQ \) is again nipotent whenever \( S \) and \( Q \) commute. Therefore, the following proposition is an immediate consequence of Proposition 3.1, (ii):

**Proposition 4.1.** Let \( T_i \) be as in the previous theorem except that \( sp(T_i) \) may not be compact. Suppose \( \Phi_i \) contains polynomials, \( \Phi \) contains constants and \( T_i = U_i(z) + Q_i \) with nilpotent operators \( Q_i (i = 1, 2) \), then \( P(T_1, T_2) \) is \( \Phi \)-spectral.

**Corollary to Theorem I.** Let \( T_i (i = 1, 2) \) be regular \( C^\infty \)-spectral operators with \( C^\infty \)-spectral representations \( U_i \) such that \( T_1, T_2, U_1(q_1), U_2(q_2); q_1, q_2 \in C^\infty \) belong to a same commutative subalgebra of \( L(E) \). Then \( P(T_1, T_2) \) is \( C^\infty \)-spectral for any polynomial \( P \).

**Remark:** The corresponding statement in \( C^\infty \) to Proposition 4.1 is a triviality, since, in this case, \( T_i \) are \( C^\infty \)-scalar. (See [1].)

**Appendix.** An example of a quasi-nilpotent operator \( Q \) and a non-regular operator \( S \) which are commutative but \( SQ \) is not quasi-nilpotent.

Let us consider the space
\[
E = \{ f(x, y) \in C^\infty([0,1] \times R); (\partial^k f/\partial x^k)(0, y) = 0, k = 0, 1, \ldots, f(., y) \in S_y(R) \}.
\]
Here, \( S_y(R) \) is the space of rapidly decreasing functions in \( y \). The space \( E \) is Fréchet with a countable number of norms \( p_{k,m,q}: (k, m, q = 0, 1, \ldots) \)
\[
 p_{k,m,q}(f) = \sup_{x \in [0,1], y \in R} |y^k(\partial^q f/\partial x^m \partial y^q)(x, y)|.
\]
Let
\[
Sf(x, y) = yf(x, y), \quad Qf(x, y) = \int_0^x f(t, y) dt.
\]
It is easy to see that \( S, Q \in L(E) \), \( Q \) is quasi-nilpotent and \( SQ = QS \). Now,
\[
(SQ)^n f(x, y) = y^n \int_0^x (x - t)^n f(t, y) dt.
\]
Taking the function \( f(x, y) = \exp \left( -\frac{\sqrt{1 + y^2}}{x} \right) \in E \), let us compute \( a_n = \left[p_{0,0,0} (SQ)^n f(x, y)]^{1/n} \). If \( SQ \) were quasi-nilpotent, then \( a_n \to 0 \) \( (n \to \infty) \). We shall show this is not the case.
\[ \alpha_n = \sup_y |y| \frac{1}{(n!)^{1/n}} \left( \int_0^1 (1 - t)^n f(t, y) \, dt \right)^{1/n} \]
\[ \geq \alpha_n |\gamma| \frac{1}{(n!)^{1/n}} \left( \int_1^\infty \frac{(1 - t)^n}{t} \exp \left( - \frac{\sqrt{1 + y^2}}{t} \right) \, dt \right)^{1/n} \]
\[ \geq K \sup_y |y| \frac{1}{(n!)^{1/n}} \exp \left( - 3 |y| \sqrt{n} \right) \]
\[ \geq K \frac{n}{(n!)^{1/n}} \quad \text{(taking } y = n) \]
\[ \rightarrow eK_1 \quad (n \to \infty). \]

Hence, \( SQ \) cannot be quasi-nilpotent.

**References**


