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Recursive Games with Infinitely Many Strategies

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In our previous paper [1] in which certain stochastic games were shown to be strictly determined, we have first defined a dummy game which may be considered a linearlization of stochastic games, and then by using the principle of contraction we have shown the existence of the principal value vector thereof which turned out to be the value vector of the stochastic games. This paper is a continuation of the one [1] just cited, and by the same method as above we shall show that some recursive games (a recursive game is originated by *H.* Everett [2]) are strictly determined.

Throughout this paper we use the notations and some of the results which were obtained in [1].

First we begin with the definition of recursive games. Suppose we are given N positions 1, 2, $\cdots$, N. To each position k we consider a game

$$\Gamma_k = (A_k, B_k, g_k, \mathfrak{M}_k, \mathfrak{N}_k)$$

called a component game of the recursive game, which will be defined below. Let Players 1 and 2 choose a pair $(a, b) \in A_k \times B_k$. Then the transition probabilities $p_{kl}(a, b)$ and the stop probabilities $p_{k0}(a, b)$ are given. Here $p_{kl}(a, b)$ denotes the probability with which the game Γ_k moves to the next one Γ_l and $p_{k0}(a, b)$ denotes the probability with which the game stops at this position k. Let $\mathfrak{A}_k$ (resp. $\mathfrak{B}_k$) be a σ-algebra of subsets of A_k (resp. B_k) such that $\mathfrak{A}_k$ (resp. $\mathfrak{B}_k$) contains one point set (a) for any $a \in A$ (resp. (b) for any $b \in B$), and $\mathfrak{C}_k$ be the smallest σ-algebra of subsets of $A_k \times B_k$ which contains the Cartesian product $\mathfrak{A}_k \times \mathfrak{B}_k$. We assume that $p_{kl}(a, b)$, $p_{k0}(a, b)$ are bounded and $\mathfrak{C}_k$-measurable. The recursive game $\boldsymbol{\Gamma}$ is defined as the collection of all Γ_k, p_{kl}, p_{k0}, k, $l=1, 2, \cdots, N$, where we assume

(i) the pay-offs $g_k(a, b)$ are bounded and $\mathfrak{C}_k$-measurable over $A_k \times B_k$ for every k,

(ii) payments can take place when and only when the game stops,

(iii) strategy spaces (A_k, δ^{g_k}), $(A_k, \delta^{p_{kl}})$ are precompact for every $k, l=1, 2, \cdots, N$,

and

(iv) the transition probabilities $p_{kl}(a, b)$ are $\mathfrak{C}_k$-measurable over $A_k \times B_k$ for every k, l, and the stop probabilities $p_{k0}(a, b)$ are non-negative for every k.

Now we shall define dummy games associated with $\boldsymbol{\Gamma}$. Suppose Player 1 (resp. Player 2) select mixed strategies μ_j (resp. ν_j) in each component game Γ_j of the recursive game. We consider the infinite game $\vec{\Gamma}_k$ beginning with Γ_k which has the expected value $G_k(\vec{\mu}, \vec{\nu})$ of the gains of Player 1:

$$G_k(\vec{\mu}, \vec{\nu}) = g_k\, p_{k0}(\mu_k, \nu_k) + \sum_{l_1=1}^{N} p_{kl_1}(\mu_k, \nu_k)\left(g_{l_1} p_{l_1 0}(\mu_{l_1}, \nu_{l_1})\right) + \sum_{l_1=1}^{N}\sum_{l_2=1}^{N} p_{kl_1}(\mu_k, \nu_k) p_{l_1 l_2}(\mu_{l_1}, \nu_{l_1})\left(g_{l_2} p_{l_2 0}(\mu_{l_2}, \nu_{l_2})\right) + \cdots, \tag{1}$$

where

$$g_k p_{k0}(\mu_k, \nu_k) = \int_{B_k}\int_{A_k} g_k(a, b) p_{k0}(a, b)\, d\mu_k(a)\, d\nu_k(b).$$

The right hand series of (1) is absolutely convergent. For, let

$$|g_k(a, b)| \leqq M,$$

then we have

$$|G_k(\vec{\mu}, \vec{\nu})| \leqq M\{p_{k0}(\mu_k, \nu_k) + \sum_{l_1=1}^{N} p_{kl_1}(\mu_k, \nu_k) p_{l_1 0}(\mu_{l_1}, \nu_{l_1}) + \sum_{l_1=1}^{N}\sum_{l_2=1}^{N} p_{kl_1}(\mu_k, \nu_k) p_{l_1 l_2}(\mu_{l_1}, \nu_{l_1}) p_{l_2 0}(\mu_{l_2}, \nu_{l_2}) + \cdots\}.$$

Now using the relations

$$\sum_{l=1}^{N} p_{kl} + p_{k0} = 1,$$

we have

$$M\{p_{k0} + \sum_{l_1} p_{kl_1} p_{l_1 0} + \cdots + \sum_{l_1, l_2, \cdots, l_{n+1}} p_{kl_1} p_{l_1 l_2} \cdots p_{l_n l_{n+1}} p_{l_{n+1} 0}\}$$
$$\leqq M\{p_{k0} + \sum_{l_1} p_{kl_1} p_{l_1 0} + \cdots + \sum_{l_1, l_2, \cdots, l_n} p_{kl_1} p_{l_1 l_2} \cdots p_{l_{n-1} l_n}\} = M.$$

Namely the right hand series of (1) is absolutely convergent, and therefore convergent. Now it is clear that $G_k(\vec{\mu}, \vec{\nu})$ is a solution of the N equations in N unknowns $v_1, v_2, \cdots, v_N$:

$$v_k = g_k p_{k0}(\mu_k, \nu_k) + \sum_{l=1}^{N} p_{kl}(\mu_k, \nu_k) v_l \quad \text{for} \quad k = 1, 2, \cdots, N.$$

Put $\quad h_k(a, b, \vec{v}) = g_k(a, b) p_{k0}(a, b) + \sum_{l=1}^{N} p_{kl}(a, b) v_l,$

where

$$\vec{v} = (v_1, v_2, \cdots, v_N).$$

The game

$$(A_k,\ B_k,\ h_k(a,\ b,\ \vec{v}),\ \mathfrak{M}_k,\ \mathfrak{N}_k),\quad k=1,\ 2,\ \cdots,\ N$$

is called a dummy game, and denoted by $\Gamma_k^d(\vec{v})$.

The following lemmas shall be needed for later purpose.

LEMMA 1. Let $\Gamma_1=(A, B, K, \mathfrak{M}, \mathfrak{N})$ and $\Gamma_2=(A, B, H, \mathfrak{M}, \mathfrak{N})$ be two games. If the strategy spaces $(A,\ \delta^K)$ and $(A,\ \delta^H)$ are precompact, then $(A,\ \delta^{K\cdot H})$ is also precompact.

PROOF. A space is precompact if and only if any sequence of A contains a Cauchy subsequence. Then the statement of our lemma is obvious from the inequality:

$$\delta^{K\cdot H}(a,\ a')\leqq c_1\delta^K(a,\ a')+c_2\delta^H(a,\ a')\quad \text{for any}\quad a,\ a'\in A,$$

where

$$c_1\geqq|H(a,\ b)|,\quad c_2\geqq|K(a,\ b)|.$$

As an immediate consequence of Lemma 1 we have

LEMMA 2. Let the strategy spaces $(A,\ \delta^K)$ and $(A,\ \delta^H)$ be precompact. Then the game $(A, B, K\cdot H, \mathfrak{M}, \mathfrak{N})$ is strictly determined.

It is to be noticed that each dummy game $\Gamma_k^d(\vec{v})$ is strictly determined by Lemma 2, and Lemma 2 in [1].

We shall consider the cases where there exists a positive number c with $p_{k0}\geqq c$, $k=1, 2, \cdots, N$ (Case 1) or some $p_{k0}=0$ (Case 2). In Case 2 we shall impose further conditions which will be described below. We call Case 1 (resp. Case 2) normal (resp. semi-normal). If all p_{k0} vanish identically, then the recursive game is practically non-terminating, which amounts to meaningless.

It is our main purpose to prove that each $\vec{\Gamma}$ is strictly determined, i.e.

$$\sup_{\mu}\inf_{\nu} G_k(\vec{\mu},\ \vec{\nu})=\inf_{\nu}\sup_{\mu} G_k(\vec{\mu},\ \vec{\nu})$$

in both cases 1 and 2.

CASE 1. First we consider a principal value vector $\vec{v}^*$ of dummy games $\Gamma_k^d(\vec{v}^*)$ $(k=1, 2, \cdots, \mathrm{N})$. It is defined as follows:

$$v_k^*=\sup_{\mu_k}\inf_{\nu_k}\int_{B_k}\int_{A_k} h_k(a,\ b,\ \vec{v}^*)d\mu_k(a)d\nu_k(b)=\inf_{\nu_k}\sup_{\mu_k}\int_{B_k}\int_{A_k}\cdots,$$

where

$$h_k(a,\ b,\ \vec{v}^*)=g_k(a,\ b)p_{k0}(a,\ b)+\sum_{l_1=1}^{N}p_{kl_1}(a,\ b)v_{l_1}^*\quad \text{for every } k.$$

The existence of $\vec{v}^*$ can be shown as follows:

Consider the value transformation T: $\vec{v}^0 \to \vec{v}^1$ defined by

$$v_k^1 = \sup_{\mu_k} \inf_{\nu_k} \int_{B_k} \int_{A_k} h_k(a, b, \vec{v}^0) d\mu_k(a) d\nu_k(b) \quad \text{for} \quad k = 1, 2, \cdots, N.$$

Define the norm of $\vec{v}$ by $\|\vec{v}\| = \max_k |v_k|$. Then we have

$$\begin{aligned} (2) \qquad \|T\vec{w} - T\vec{v}\| &= \max_k |\text{value of } \Gamma_k^d(\vec{w}) - \text{value of } \Gamma_k^d(\vec{v})| \\ &\leqq \max_k [\sup_{a,b} |h_k(a, b, \vec{w}) - h_k(a, b, \vec{v})|] \\ &= \sup_{k,a,b} |\sum_{l=1}^N p_{kl}(a, b)(w_l - v_l)| \\ &\leqq \sup_{k,a,b} |\sum_{l=1}^N p_{kl}(a, b)| \max_l |w_l - v_l| \\ &\leqq (1-c)\|\vec{w} - \vec{v}\|. \end{aligned}$$

Then by the principle of contraction, there exists a unique $\vec{v}^*$ which satisfies $T\vec{v}^* = \vec{v}^*$, the principal value vector $\vec{v}^*$.

Next we introduce the notion of ϵ-optimal strategies of the dummy games, ϵ being any non-negative number. Let $\vec{v}^*$ denote the principal value vector as before. For each k, any pair $(\mu_k^\epsilon, \nu_k^\epsilon) \in \mathfrak{M}_k \times \mathfrak{N}_k$ is said to be ϵ-optimal strategies of Players 1 and 2 of the dummy game at the position k, when

$$h_k(\mu_k^\epsilon, \nu_k, \vec{v}^*) \geqq v_k^* - \epsilon \quad \text{for any} \quad \nu_k \in \mathfrak{N}_k,$$

and

$$h_k(\mu_k, \nu_k^\epsilon, \vec{v}^*) \leqq v_k^* + \epsilon \quad \text{for any} \quad \mu_k \in \mathfrak{M}_k.$$

In the case where $\epsilon = 0$, ϵ-optimal strategies are no more than optimal strategies. Then we have

LEMMA 3. Let $0 \leqq \epsilon < 1$. Any complete set of ϵ-optimal strategies of Players 1 and 2 of dummy games $\Gamma_k^d(\vec{v}^*)$ are ϵ/c-optimal strategies of the original infinite game $\vec{\Gamma}_k$ for every $k = 1, 2, \cdots, N$.

PROOF. Denote by $G_k(\vec{\mu}, \vec{\nu})$ the expected value of the gains of Player 1. Putting

$$v_l = g_l p_{l0}(\mu_l, \nu_l) + \sum_{l_1=1}^N p_{ll_1}(\mu_l, \nu_l) v_{l_1}, \quad l = 1, 2, \cdots, N,$$

we have

$$\begin{aligned} G_k(\vec{\mu}, \vec{\nu}) - v_k^* &= g_k p_{k0}(\mu_k, \nu_k) + \sum_{l=1}^N p_{kl_1}(\mu_k, \nu_k) v_{l_1} - v_k^* \\ = h_k(\mu_k, \nu_k, \vec{v}^*) &- \sum_{l_1=1}^N p_{kl_1}(\mu_k, \nu_k) v_{l_1}^* + \sum_{l_1=1}^N p_{kl_1}(\mu_k, \nu_k) v_{l_1} - v_k^* \end{aligned}$$

$$= h_k(\mu_k, \nu_k, \vec{v}^*) - v_k^* + \sum_{l_1=1}^{N} p_{kl_1}(\mu_k, \nu_k)(v_{l_1} - v_{l_1}^*)$$

$$= h_k(\mu_k, \nu_k, \vec{v}^*) - v_k^* + \sum_{l_1=1}^{N} p_{kl_1}(\mu_k, \nu_k)\big(h_{l_1}(\mu_{l_1}, \nu_{l_1}, \vec{v}^*) - v_{l_1}^*\big)$$

$$+ \sum_{l_1=1}^{N}\sum_{l_2=1}^{N} p_{kl_1}(\mu_k, \nu_k) p_{l_1 l_2}(\mu_{l_1}, \nu_{l_1})\big(h_{l_2}(\mu_{l_2}, \nu_{l_2}, \vec{v}^*) - v_{l_2}^*\big) + \cdots.$$

Consequently, since

$$h_k(\mu_k, \nu_k^\epsilon, \vec{v}) - v_k^* \leqq \epsilon,$$

we have

$$G_k(\vec{\mu}, \vec{\nu}^\epsilon) - v_k^* \leqq \epsilon + \epsilon \sum_{l_1=1}^{N} p_{kl_1} + \epsilon \sum_{l_1=1}^{N}\sum_{l_2=1}^{N} p_{kl_1} p_{l_1 l_2} + \cdots$$
$$\leqq \epsilon/c.$$

Similarly we can show that

$$G_k(\vec{\mu}^\epsilon, \vec{\nu}) \geqq v_k^* - \epsilon/c,$$

and our lemma is proved.

LEMMA 4. For any positive ϵ, there exist ϵ-optimal strategies of the dummy games.

PROOF. Let ϵ be any positive number. We put $\epsilon' = \epsilon c/2c+1$. We divide A_k (resp. B_k) into non-empty measurable subsets $A_{k,1}, A_{k,2}, \cdots, A_{k,m_k}$ (resp. $B_{k,1}, B_{k,2}, \cdots, B_{k,n_k}$), where $A_{k,i}$ (resp. $B_{k,j}$) are smaller than ϵ' in diameter in the metric $\delta^{h_k(a,b,\vec{v}^*)}$. Let $\alpha_k = (a_{k,1}, a_{k,2}, \cdots, a_{k,m_k})$ (resp. $\beta_k = (b_{k,1}, b_{k,2}, \cdots, b_{k,n_k})$) denote the finite subset of A_k (resp. B_k) where $a_{k,i}$ (resp. $b_{k,j}$) is any point chosen from $A_{k,i}$ (resp. $B_{k,j}$). Let $\mathfrak{M}_k'$ (resp. $\mathfrak{N}_k'$) be the set of probability measures concentrated on α_k (resp. β_k). For any $\mu \in \mathfrak{M}_k$ (resp. $\nu \in \mathfrak{N}_k$) we define $\bar{\mu} \in \mathfrak{M}_k'$ (resp. $\bar{\nu} \in \mathfrak{N}_k'$) as follows:

$$\bar{\mu}(a_{k,i}) = \mu(A_{k,i}) \text{ (resp. } \bar{\nu}(b_{k,j}) = \nu(B_{k,j})\text{)}.$$

Then we have

$$(3) \qquad |h_k(\mu, \nu, \vec{v}^*) - h_k(\bar{\mu}, \bar{\nu}, \vec{v}^*)| \leqq \sum_{i,j} \int_{A_{k,i}} \int_{B_{k,j}} |h_k(a, b, \vec{v}^*) - h_k(a_{k,i}, b_{k,j}, \vec{v}^*)|\, d\mu_k d\nu_k < \epsilon'$$

If we denote by T' the value transformation of

$$\Gamma_k'^d = (A_k, B_k, h_k(a, b, \vec{v}), \mathfrak{M}_k', \mathfrak{N}_k'),$$

then it follows from (3) that

$$(4) \qquad \|\vec{v}^* - T'\vec{v}^*\| \leqq \epsilon'.$$

Let $\vec{v}'^*$ be the principal value vector of Γ'^d_k for $k=1, 2, \cdots, N$. Then by (2) and (4) we have

$$\begin{aligned}\|T'^j\vec{v}^* - T'^{j+1}\vec{v}^*\| &\leqq (1-c)\,\|T'^{j-1}\vec{v}^* - T'^j\vec{v}^*\| \\ &\leqq (1-c)^j\,\|\vec{v}^* - T'\vec{v}^*\| \\ &\leqq \epsilon'(1-c)^j.\end{aligned}$$

Consequently

$$\begin{aligned}\|\vec{v}^* - \vec{v}'^*\| &= \lim_{n\to\infty}\|\vec{v}^* - T'^n\vec{v}^*\| \\ &\leqq \epsilon' + \epsilon'(1-c) + \epsilon'(1-c)^2 + \cdots \\ &= \epsilon'/c.\end{aligned}$$

Since α_k and β_k are finite there exists optimal strategies $\bar{\mu}^\epsilon$, $\bar{\nu}^\epsilon$ of the game $(A_k, B_k, h_k(a, b, \vec{v}'^*), \mathfrak{M}'_k, \mathfrak{N}'_k)$.

Therefore

$$h_k(\bar{\mu}^\epsilon, \bar{\nu}, \vec{v}'^*) \geqq v_k'^* \geqq v_k^* - \epsilon'/c, \tag{5}$$

and

$$h_k(\bar{\mu}, \bar{\nu}^\epsilon, \vec{v}'^*) \leqq v_k'^* \leqq v_k^* + \epsilon'/c. \tag{6}$$

Then for any $\nu \in \mathfrak{N}_k$, by using the equation

$$h_k(\mu, \nu, \vec{v}) = g_k p_{k0}(\mu, \nu) + \sum_{l_1=1}^{N} p_{kl_1}(\mu_k, \nu_k)v_{l_1},$$

we have

$$\begin{aligned}h_k(\bar{\mu}^\epsilon, \nu, \vec{v}^*) &= h_k(\bar{\mu}^\epsilon, \bar{\nu}, \vec{v}'^*) + h_k(\bar{\mu}^\epsilon, \nu - \bar{\nu}, \vec{v}^*) \\ &\quad + \sum_{l_1=1}^{N}(v^*_{l_1} - v'^*_{l_1})p_{kl_1}(\bar{\mu}^\epsilon, \bar{\nu}),\end{aligned} \tag{7}$$

where

$$|h_k(\bar{\mu}, \nu - \bar{\nu}, \vec{v}^*)| < \epsilon', \tag{8}$$

and

$$\left|\sum_{l_1=1}^{N}(v^*_{l_1} - v'^*_{l_1})p_{kl}(\bar{\mu}, \bar{\nu})\right| < \epsilon'. \tag{9}$$

On account of (5), (7), (8), and (9), we have

$$\begin{aligned}h_k(\bar{\mu}^\epsilon, \nu, \vec{v}^*) &\geqq v_k'^* - \epsilon' - \epsilon' \\ &\geqq v_k^* - \epsilon'/c - 2\epsilon' \\ &= v_k^* - \epsilon.\end{aligned}$$

Similarly we can show that

$$h_k(\mu, \bar{\nu}^\epsilon, \vec{v}^*) \leqq v_k^* + \epsilon,$$

and our lemma is proved.

THEOREM 1. If $p_{k0} \geqq c > 0$, $k = 1, 2, \cdots, N$, then the recursive game $\Gamma = \{\vec{\Gamma}_k, k=1, 2, \cdots, N\}$ is strictly determined, and the value of $\vec{\Gamma}_k$ is equal to v_k^*, the k-th component of the principal value vector of the associated dummy games.

PROOF. By Lemmas 3 and 4, we see that there exist ϵ-optimal strategies $\vec{\mu}^\epsilon$, $\vec{\nu}^\epsilon$ of $\vec{\Gamma}_k$ for any positive ϵ, that is,

$$G_k(\vec{\mu}^\epsilon, \vec{\nu}) \geqq v_k^* - \epsilon \quad \text{and} \quad G_k(\vec{\mu}, \vec{\nu}^\epsilon) \leqq v_k^* + \epsilon.$$

Then

$$\inf_{\nu} G_k(\vec{\mu}^\epsilon, \vec{\nu}) \geqq v_k^* - \epsilon.$$

As ϵ is any positive number, so we have

$$\sup_{\mu} \inf_{\nu} G_k(\vec{\mu}, \vec{\nu}) \geqq v_k^*. \tag{10}$$

Similarly we have

$$\inf_{\nu} \sup_{\mu} G_k(\vec{\mu}, \vec{\nu}) \leqq v_k^*. \tag{11}$$

On the other hand it is easy to see that

$$\sup_{\mu} \inf_{\nu} G_k(\vec{\mu}, \vec{\nu}) \leqq \inf_{\nu} \sup_{\mu} G_k(\vec{\mu}, \vec{\nu}). \tag{12}$$

Therefore the inequalities (10), (11), together with the inequality (12) impliy

$$v_k^* = \sup_{\mu} \inf_{\nu} G_k(\vec{\mu}, \vec{\nu}) = \inf_{\nu} \sup_{\mu} G_k(\vec{\mu}, \vec{\nu}).$$

Thus the proof is completed.

Now we turn to

CASE 2. We assume that there exist positive numbers c, c' such that

$$p_{i0} = 0 \quad \text{and} \quad \sum_{l=1}^{h} p_{il} < c' < 1 \quad \text{for} \quad i = 1, 2, \cdots, h; \tag{13}$$
$$p_{i0} \geqq c > 0 \quad \text{for} \quad i = h+1, h+2, \cdots, N.$$

We shall show that Theorem 1 remains valid also when we assume (13) instead of $p_{k0} \geqq c > 0$, $k = 1, 2, \cdots, N$. Let for any $\vec{v}$ we put

$$v_j' = \text{value of } \Gamma_j^d(\vec{v}) \quad \text{for} \quad j = 1, 2, \cdots, N,$$

then by using (2) and (13) we have

$$(14)\qquad |v_j'-w_j'| \leqq \begin{cases} c' \max\limits_{1\leqq j\leqq h} |v_j-w_j| + \max\limits_{h<l} |v_l-w_l| & \text{for } j=1, 2, \cdots, h, \\ (1-c)\max\limits_{h<l} |v_l-w_l| & \text{for } j=h+1, h+2, \cdots, N. \end{cases}$$

Choose a positive number α in such a way that $\alpha<1-c'$ and define the norm of $\vec{v}$ by

$$\|\vec{v}\|' = \max\{\alpha|v_1|, \alpha|v_2|, \cdots, \alpha|v_h|, |v_{h+1}|, |v_{h+2}|, \cdots, |v_N|\}.$$

Then by (14) we have

$$\begin{aligned}\|\vec{v}'-\vec{w}'\|' &= \max\{\alpha|v_1'-w_1'|, \alpha|v_2'-w_2'|, \cdots, \alpha|v_h'-w_h'|, \\ &\qquad |v_{h+1}'-w_{h+1}'|, |v_{h+2}'-w_{h+2}'|, \cdots, |v_N'-w_N'|\} \\ &\leqq \max\{\alpha c' \max_{1\leqq j\leqq h}|v_j-w_j| + \max_{h<l}|v_l-w_l|, (1-c)\max_{h<l}|v_l-w_l|\} \\ &\leqq \max(c'+\alpha, 1-c)\|\vec{v}-\vec{w}\|.\end{aligned}$$

Consequently by the principle of contraction there exists a unique principal value vector $\vec{v}^*$ of dummy games $\Gamma_k^d(\vec{v}^*)$ ($k=1, 2, \cdots, N$). Regarding ϵ-optimal strategies we have

LEMMA 5. Any complete set of ϵ-optimal strategies of Players 1 and 2 of dummy games $\Gamma_k^d(\vec{v}^*)$ is $\epsilon(1+1/c)$-optimal strategies of each original infinite game $\vec{\Gamma}_k$.

PROOF. Denote by $G_k(\vec{\mu}, \vec{\nu})$ the expected value of the gains of Player 1. Putting

$$v_l = g_l p_{l0} + \sum_{l_1=1}^{N} p_{ll_1} v_{l_1}, \quad l=1, 2, \cdots, N,$$

where

$$p_{k0}=0, \quad k=1, 2, \cdots, h,$$

we have for $k=1, 2, \cdots, h$,

$$\begin{aligned}G_k(\vec{\mu}, \vec{\nu}) - v_k^* &= \sum_{l_1=1}^{N} p_{kl_1}(\mu_k, \nu_k) v_{l_1} - v_k^* \\ &= h_k(\mu_k, \nu_k, \vec{v}) - v_k^* \\ &= h_k(\mu_k, \nu_k, \vec{v}^*) - v_k^* + h_k(\mu_k, \nu_k, \vec{v}) - h_k(\mu_k, \nu_k, \vec{v}^*) \\ &= h_k(\mu_k, \nu_k, \vec{v}^*) - v_k^* + \sum_{l_1=1}^{N} p_{kl_1}(\mu_k, \nu_k)(v_{l_1} - v_{l_1}^*) \\ &= h_k(\mu_k, \nu_k, \vec{v}^*) - v_k^* + \sum_{l_1=1}^{N} p_{kl_1}(\mu_k, \nu_k)\left(h_{l_1}(\mu_{l_1}, \nu_{l_1}, \vec{v}^*) - v_{l_1}^*\right) \\ &\quad + \sum_{l_1}\sum_{l_2} p_{kl_1}(\mu_k, \nu_k) p_{l_1 l_2}(\mu_{l_1}, \nu_{l_2})\left(h_{l_2}(\mu_{l_2}, \nu_{l_2}, \vec{v}^*) - v_{l_2}^*\right) + \cdots.\end{aligned}$$

Consequently we have

$$\begin{aligned} G_k(\vec{\mu}, \vec{\nu}^\epsilon) - v_k^* &\leqq \epsilon + \epsilon \sum_{l_1} p_{kl_1} + \epsilon \sum_{l_1} \sum_{l_2} p_{kl_1} p_{l_1 l_2} + \cdots \\ &\leqq \epsilon + \epsilon + \epsilon(1-c) + \epsilon(1-c)^2 + \cdots \\ &= \epsilon(1 + 1/c). \end{aligned}$$

And by Lemma 3 we have

$$\begin{aligned} G_k(\vec{\mu}, \vec{\nu}^\epsilon) - v_k^* &\leqq \epsilon/c \\ &\leqq \epsilon(1+1/c) \quad \text{for} \quad k = h+1, h+2, \cdots, N. \end{aligned}$$

Similarly we can show that

$$G_k(\vec{\mu}^\epsilon, \vec{\nu}) \geqq v_k^* - \epsilon(1 + 1/c),$$

and our lemma is proved.

On account of Lemmas 4 and 5 we can reach the same conclusion as Theorem 1.

Reference

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