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A Lattice Theoretic Treatment of Stochastic Independence

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In an earlier paper of the author [3], a binary relation "$\perp$" between elements of a lattice L with 0 was called a semi-orthogonal relation if it satisfies the following four axioms:

($\perp$1) $a \perp b$ *implies* $b \perp a$;

($\perp$2) $a \perp b$, $a_1 \leqq a$ *imply* $a_1 \perp b$;

($\perp$3) $a \perp b$, $a \cup b \perp c$ *imply* $a \perp b \cup c$;

($\perp$4) $a \perp a$ *implies* $a=0$; $0 \perp a$ *for every* $a \in L$.

This relation plays an important rôle in the dimension theory on lattices, stated in [5].

In this paper, a relation satisfying the three axioms ($\perp$1), ($\perp$2) and ($\perp$3) is called an independence relation, and firstly we shall show some properties of independent families.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $L(\mathscr{A})$ be the lattice formed by all sub σ-fields of $\mathscr{A}$. The main result of this paper is that the stochastic independence is an independence relation in $L(\mathscr{A})$. From this result, the argument of stochastically independent families of events or random functions can be stated by a lattice theoretic treatment.

In the last section, we shall show that the stochastic independence is a semi-orthogonal relation in some sublattice of $L(\mathscr{A})$, and give some other examples of semi-orthogonality.

§ **1. Independence relation.** Let L be a lattice. A binary relation "$\perp$" between elements of L is called an *independence relation* in L if it satisfies the three axioms ($\perp$1), ($\perp$2) and ($\perp$3). A set S of elements of L is called an *independent family* if $\bigcup(a;\, a \in F_1) \perp \bigcup(a;\, a \in F_2)$ holds for every pair of disjoint finite subsets F_1, F_2 of S. The following proposition can be easily proved.

Proposition 1.1. (i) *S is an independent family if every finite subset of S is an independent family.*

(ii) *Let S be a countable (finite or infinite) set and put $S=\{a_1, a_2, \cdots\}$. S is an independent family if $a_1 \cup \cdots \cup a_i \perp a_{i+1}$ for every i.*

(iii) *S is an independent family if $a_0 \perp \bigcup(a;\, a \in F)$ holds for every finite subset F of S and every element $a_0 \in S-F$.*

Proposition 1.2. *Let F_α be a finite independent family for every $\alpha \in I$.*

The union $\sum_\alpha F_\alpha$ is an independent family if and only if the set $\{\bigvee(a;\ a \in F_\alpha);\ \alpha \in I\}$ is an independent family.

PROOF. The "only if" part is obvious. It can be proved by Proposition 1.1 (iii) and ($\perp 3$) that the "if" part holds in the case $I=\{1, 2\}$, which implies that it holds in the case where I is finite. Hence, in the general case, it holds by Proposition 1.1 (i).

An independence relation in a complete lattice is called to be *complete* if it has the following property:

$$\text{If } a_\delta \uparrow a \text{ and } a_\delta \perp b \text{ for all } \delta \text{ then } a \perp b.$$

The following proposition can be easily proved.

PROPOSITION 1.3. *Let "$\perp$" be a complete independence relation in a complete lattice L.*

(i) *If S is an independent family, then $\bigvee(a;\ a \in S_1) \perp \bigvee(a;\ a \in S_2)$ holds for every pair of disjoint subsets S_1, S_2 of S.*

(ii) *Let S_α ($\alpha \in I$) be sets of elements of L. The set $\{\bigvee(a;\ a \in S_\alpha);\ \alpha \in I\}$ is an independent family if and only if, for arbitrary finite subset F_α of each S_α, $\{\bigvee(a;\ a \in F_\alpha);\ \alpha \in I\}$ is an independent family.*

(iii) *Let S_α be an independent family for every $\alpha \in I$. The union $\sum_\alpha S_\alpha$ is an independent family if and only if the set $\{\bigvee(a;\ a \in S_\alpha);\ \alpha \in I\}$ is an independent family.*

We remark that the argument of this section can be applied to the case where L is a semi-lattice.

§ **2. Stochastic independence.** Let $(\Omega, \mathscr{A}, P)$ be a probability space ($\mathscr{A}$ is a σ-field of subsets of Ω and P is a probability measure on $\mathscr{A}$). The definitions of stochastic independence were given by Loève [2; Chap. V] as follows (the union and the intersection of events are denoted by the symbols $+$ and $\cdot$ respectively):

(I_1) *Events A_α ($\alpha \in I$) are independent if for every set of finite indices $\{\alpha_1, \cdots, \alpha_n\}$*

$$P(A_{\alpha_1}\cdot \cdots \cdot A_{\alpha_n}) = P(A_{\alpha_1}) \cdots P(A_{\alpha_n});$$

(I_2) *Classes $\mathscr{C}_\alpha$ ($\alpha \in I$) of events are independent if events selected arbitrary one from each class are independent.*

The set $L(\mathscr{A})$ of all (non-empty) sub σ-fields of $\mathscr{A}$ forms a complete lattice, ordered by set-inclusion, since the intersection of sub σ-fields is also a sub σ-field. (The zero element of $L(\mathscr{A})$ is $\{\phi, \Omega\}$ and its unity element is $\mathscr{A}$.) In $L(\mathscr{A})$, we define a relation "$\perp$" by (I_2), that is, for $\mathscr{B}_1, \mathscr{B}_2 \in L(\mathscr{A})$, we write $\mathscr{B}_1 \perp \mathscr{B}_2$ if $P(A_1 \cdot A_2) = P(A_1)P(A_2)$ for every $A_1 \in \mathscr{B}_1$, $A_2 \in \mathscr{B}_2$.

In order to prove that "$\perp$" is an independence relation, we shall prepare the following lemma, due to Dynkin ([1; Chap. I], Lemma 1.1).

LEMMA. *A class $\mathscr{P}$ of events is called a π-system if $A_1, A_2 \in \mathscr{P}$ imply $A_1 \cdot A_2 \in \mathscr{P}$. A class $\mathscr{C}$ is called a λ-system if* (1) $\Omega \in \mathscr{C}$, (2) $A_1, A_2 \in \mathscr{C}$, $A_1 \cdot A_2 = \phi$ *imply* $A_1 + A_2 \in \mathscr{C}$, (3) $A_1, A_2 \in \mathscr{C}$, $A_1 \supset A_2$ *imply* $A_1 - A_2 \in \mathscr{C}$, (4) $A_n \in \mathscr{C}$, $A_n \uparrow A$ *imply* $A \in \mathscr{C}$. *If a λ-system contains a π-system $\mathscr{P}$ then it contains the σ-field generated by $\mathscr{P}$.*

PROPOSITION 2.1. *The relation "$\perp$" in $L(\mathscr{A})$ is a complete independence relation.*

PROOF. It is obvious that the axioms ($\perp 1$) and ($\perp 2$) hold. We shall prove that ($\perp 3$) holds. Let $\mathscr{B}_1 \perp \mathscr{B}_2$ and $\mathscr{B}_1 \cup \mathscr{B}_2 \perp \mathscr{B}_3$, and put $\mathscr{C} = \{B \in \mathscr{A};\ P(B \cdot A_1) = P(B)P(A_1)$ for all $A_1 \in \mathscr{B}_1\}$ and $\mathscr{P} = \{A_2 \cdot A_3;\ A_2 \in \mathscr{B}_2, A_3 \in \mathscr{B}_3\}$. Then, it is easy to show that $\mathscr{C}$ is a λ-system (see [2; p. 224]) and that $\mathscr{P}$ is a π-system. Moreover, $\mathscr{C}$ contains $\mathscr{P}$, because, for $A_i \in \mathscr{B}_i$ $(i=1, 2, 3)$, it follows from $\mathscr{B}_1 \perp \mathscr{B}_2$, $\mathscr{B}_2 \perp \mathscr{B}_3$ and $\mathscr{B}_1 \cup \mathscr{B}_2 \perp \mathscr{B}_3$ that $P(A_1 \cdot A_2) = P(A_1)P(A_2)$, $P(A_2 \cdot A_3) = P(A_2)P(A_3)$ and $P(A_1 \cdot A_2 \cdot A_3) = P(A_1 \cdot A_2)P(A_3)$, which imply $P(A_1 \cdot A_2 \cdot A_3) = P(A_1)P(A_2 \cdot A_3)$, and hence $A_2 \cdot A_3 \in \mathscr{C}$. Since $\mathscr{P}$ contains $\mathscr{B}_2$ and $\mathscr{B}_3$, the σ-field generated by $\mathscr{P}$ contains $\mathscr{B}_2 \cup \mathscr{B}_3$. Hence, $\mathscr{C}$ contains $\mathscr{B}_2 \cup \mathscr{B}_3$ by Dynkin's lemma, and consequently $\mathscr{B}_1 \perp \mathscr{B}_2 \cup \mathscr{B}_3$. Finally we shall prove the completeness. Let $\mathscr{B}_\delta \uparrow \mathscr{B}$ and $\mathscr{B}_\delta \perp \mathscr{B}'$, and put $\mathscr{C} = \{B \in \mathscr{A};\ P(B \cdot A') = P(B)P(A')$ for all $A' \in \mathscr{B}'\}$ and $\mathscr{P}$ = the union of all $\mathscr{B}_\delta$. Then it is obvious that $\mathscr{C}$ is a λ-system, $\mathscr{P}$ is a π-system and that $\mathscr{C}$ contains $\mathscr{P}$. Hence, $\mathscr{C}$ contains $\bigvee_\delta \mathscr{B}_\delta = \mathscr{B}$ by Dynkin's lemma, and consequently $\mathscr{B} \perp \mathscr{B}'$.

PROPOSITION 2.2. *Sub σ-fields $\mathscr{B}_\alpha$ $(\alpha \in I)$ are independent in the sense of (I_2) if and only if they form an independent family in $L(\mathscr{A})$ in the sense of* §1.

PROOF. Let $\mathscr{B}_\alpha$ be independent in the sense of (I_2). To prove that $\{\mathscr{B}_\alpha\}$ is an independent family, it suffices to show that $\mathscr{B}_{\alpha_0} \perp \bigvee_{i=1}^n \mathscr{B}_{\alpha_i}$ for every set of finite indices $\{\alpha_0, \cdots, \alpha_n\}$ (Proposition 1.1 (iii)). Putting $\mathscr{C} = \{B \in \mathscr{A};\ P(B \cdot A_0) = P(B)P(A_0)$ for all $A_0 \in \mathscr{B}_{\alpha_0}\}$ and $\mathscr{P} = \{A_1 \cdot \cdots \cdot A_n;\ A_i \in \mathscr{B}_{\alpha_i}\}$, $\mathscr{C}$ contains $\mathscr{P}$ by the assumption. Hence, by Dynkin's lemma, $\mathscr{C}$ contains $\bigvee_{i=1}^n \mathscr{B}_{\alpha_i}$, which implies $\mathscr{B}_{\alpha_0} \perp \bigvee_{i=1}^n \mathscr{B}_{\alpha_i}$. Conversely, let $\{\mathscr{B}_\alpha\}$ be an independent family. For every set of finite indices $\{\alpha_1, \cdots, \alpha_n\}$ and for every $A_{\alpha_i} \in \mathscr{B}_{\alpha_i}$ $(1 \leqq i \leqq n)$, we have $P(A_{\alpha_1} \cdot \cdots \cdot A_{\alpha_{i+1}}) = P(A_{\alpha_1} \cdot \cdots \cdot A_{\alpha_i}) P(A_{\alpha_{i+1}})$ $(1 \leqq i \leqq n-1)$ since $A_{\alpha_1} \cdot \cdots \cdot A_{\alpha_i} \in \mathscr{B}_{\alpha_1} \cup \cdots \cup \mathscr{B}_{\alpha_i} \perp \mathscr{B}_{\alpha_{i+1}}$. Hence $P(A_{\alpha_1} \cdot \cdots \cdot A_{\alpha_n}) = P(A_{\alpha_1}) \cdots P(A_{\alpha_n})$. This completes the proof.

From this proposition and the extension theorem of Loève ([2; p. 225]), the concept of independent classes of events is reduced to that of the independent family in $L(\mathscr{A})$. Hence, Propositions 1.1 and 1.3 can be applied for independent classes of events. For example, Proposition 1.3 (ii) implies the compounds theorem of Loève ([2; p. 225]), since the compound σ-field of σ-fields $\mathscr{B}_\alpha$ is equal to the join $\bigvee_\alpha \mathscr{B}_\alpha$.

The definition of independence of random functions was given by Loève as follows:

(I_3) *Random functions* X_α $(\alpha \in I)$ *are independent if they induce independent* σ-*fields* $\mathscr{B}(X_\alpha)$. (Any random function X taking values from a measurable space $(S, \mathscr{A}(S))$ induces a sub σ-field $X^{-1}(\mathscr{A}(S))$ of $\mathscr{A}$, which we denote by $\mathscr{B}(X)$.) It follows from Proposition 2.2 that random functions X_α are independent if and only if $\{\mathscr{B}(X_\alpha)\}$ is an independent family in $L(\mathscr{A})$. The following lemma is useful for the lattice theoretic treatment of independent families of random functions.

LEMMA. (i) *Let X be an S-valued random function and F be a measurable mapping of S into another measurable space $(T, \mathscr{A}(T))$. Then, $F(X)$ is a T-valued random function and $\mathscr{B}(F(X)) \leqq \mathscr{B}(X)$ in $L(\mathscr{A})$.*

(ii) *Let $X = (X_\alpha)$ be a family of S_α-valued random functions $X_\alpha(\alpha \in I)$. Then X may be a random function taking values in the product space of S_α and $\mathscr{B}(X) = \bigvee_\alpha \mathscr{B}(X_\alpha)$ in $L(\mathscr{A})$.*

(iii) *Let $(X_1, \cdots, X_n)$ be a family of S_i-valued random functions X_i and F be a measurable mapping of the product space of S_i into T. Then $F(X_1, \cdots, X_n)$ is a T-valued random function and*

$$\mathscr{B}(F(X_1, \cdots, X_n)) \leqq \mathscr{B}(X_1) \cup \cdots \cup \mathscr{B}(X_n).$$

PROOF. (i) holds since $\mathscr{A}(S)$ contains $F^{-1}(\mathscr{A}(T))$. (ii) holds since $\mathscr{B}(X)$ is generated by all finite intersections of events $A_\alpha \in \mathscr{B}(X_\alpha)$, $\alpha \in I$. (iii) is a consequence of (i) and (ii).

The Borel functions theorem of Loève ([2; p. 224]) follows from (iii) of this lemma, and the families theorem ([2; p. 225]) follows from (ii) of this lemma and Proposition 1.1 (i). Some other properties of independent random functions can be proved as consequences of Propositions 1.1 and 1.3. For example,

PROPOSITION 2.3. (i) *Countable random functions $X_1, X_2, \cdots$ are independent if the family $(X_1, \cdots, X_i)$ and X_{i+1} are independent for every i.*

(ii) *Let $X_\alpha = (X_\alpha^\beta; \beta \in J_\alpha)$ is a family of independent random functions for each $\alpha \in I$. Families $\{X_\alpha; \alpha \in I\}$ are independent if and only if all component random functions $\{X_\alpha^\beta; \beta \in J_\alpha, \alpha \in I\}$ are independent.*

§ **3. Semi-orthogonal relation.** Let $(\Omega, \mathscr{A}, P)$ be a probability space and $L(\mathscr{A})$ be the complete lattice formed by all sub σ-fields of $\mathscr{A}$. It is obvious that $\mathscr{A}_0 = \{A \in \mathscr{A}; P(A) = 0 \text{ or } 1\}$ is an element of $L(\mathscr{A})$.

PROPOSITION 3.1. (i) *If $\mathscr{B} \perp \mathscr{B}$ in $L(\mathscr{A})$ then $\mathscr{B} \leqq \mathscr{A}_0$; (ii) $\mathscr{A}_0 \perp \mathscr{B}$ for all $\mathscr{B} \in L(\mathscr{A})$. Hence, the stochastic independence is a semi-orthogonal relation in the sublattice $\{\mathscr{B} \in L(\mathscr{A}); \mathscr{B} \geqq \mathscr{A}_0\}$.*

PROOF. If $\mathscr{B} \perp \mathscr{B}$ then $P(A)^2 = P(A)$ for every $A \in \mathscr{B}$, which implies $\mathscr{B} \leqq \mathscr{A}_0$. (ii) is obvious.

A lattice L with 0 is called a *semi-orthogonal lattice* if L has a semi-orthogonal relation. The above proposition gives an example of the semi-orthogonal lattice. Next, we shall gives some other examples.

EXAMPLE 1. Let L be a symmetric lattice defined by Wilcox [6], that is, $(a, b)M$ and $a\cap b=0$ imply $(b, a)M$ ($(a, b)M$ means that $(c\cup a)\cap b=c\cup(a\cap b)$ when $c\leqq b$). We write $a\perp b$ when $(a, b)M$ and $a\cap b=0$. Then, it is obvious that $(\perp 1)$ and $(\perp 4)$ hold. And, it follows from [6], Theorem 1.1 and Lemma 1.3 that $(\perp 2)$ and $(\perp 3)$ hold. Hence, any symmetric lattice (especially any semi-modular lattice of finite length or any modular lattice with 0) is a semi-orthogonal lattice.

EXAMPLE 2. In an orthocomplemented lattice, the orthogonal relation is a semi-orthogonal relation having the following special property: $a\perp b$, $a\perp c$ imply $a\perp b\cup c$ (see [3; §1]).

EXAMPLE 3. Let $\mathfrak{A}$ be a Rickart ring defined by [4], and L be the lattice formed by all principal right ideals generated by idempotents of $\mathfrak{A}$. Then, it is proved in [4; §1] that a semi-orthogonal relation in L can be defined by the ring structure of $\mathfrak{A}$. (Moreover, L is relatively semi-orthocomplemented.)

EXAMPLE 4. Let $\mathscr{A}$ be a field of subsets of a space Ω, and let P be a non-negative, finitely additive set function on $\mathscr{A}$ such that $P(\Omega)=1$. The set $L(\mathscr{A})$ of all (non-empty) subfields of $\mathscr{A}$ forms a complete lattice, ordered by set-inclusion. We write $\mathscr{B}_1\perp\mathscr{B}_2(\mathscr{B}_1, \mathscr{B}_2\in L(\mathscr{A}))$ if $P(A_1\cdot A_2)=P(A_1)P(A_2)$ for every $A_1\in\mathscr{B}_1$, $A_2\in\mathscr{B}_2$. Then, we can prove in the same way as Proposition 2.1 that "$\perp$" in $L(\mathscr{A})$ is a complete independence relation. (Dynkin's lemma can be modified as follows: If a class $\mathscr{C}$ satisfies the conditions (1), (2), (3) in the definition of λ-system and if $\mathscr{C}$ contains a π-system $\mathscr{P}$, then it contains the field generated by $\mathscr{P}$.) It is obvious that $\mathscr{A}_0=\{A\in\mathscr{A};\, P(A)=0 \text{ or } 1\}$ is an element of $L(\mathscr{A})$, and we can prove in the same way as Proposition 3.1 that the sublattice $\{\mathscr{B}\in L(\mathscr{A});\, \mathscr{B}\geqq\mathscr{A}_0\}$ is a semi-orthogonal lattice. (In the case where $\mathscr{A}$ is a σ-field, we remark that the lattice of sub σ-fields of $\mathscr{A}$ is not always a sublattice of the lattice of subfields of $\mathscr{A}$.)

References

[1] E. B. Dynkin, *Theory of Markov processes* (translated from Russian), London, 1960.
[2] M. Loève, *Probability theory* (2nd ed.), Princeton, 1960.
[3] S. Maeda, *On relatively semi-orthocomplemented lattices*, this Journal, **24** (1960), 155–161.
[4] ———, *On a ring whose principal right ideals generated by idempotents form a lattice*, ibid., **24** (1960), 509–525.
[5] ———, *Dimension theory on relatively semi-orthocomplemented complete lattices*, ibid., **25** (1961), 369–404.
[6] L. R. Wilcox, *Modularity in the theory of lattices*, Ann. of Math., **40** (1939), 490–505.

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