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Convex Functionals in a Topological Vector Space

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A convex functional on a convex domain of a topological vector space is continuous if it is bounded above in an open subset, and then it becomes locally uniformly continuous [1]. W. Orlicz and Z. Ciesielski have shown [3] that any sequence of convex functionals on a convex domain of a Banach space is equicontinuous if it is bounded at each point of the domain.

In this paper a topological vector space E, locally convex or not, is called a t_0-space if it satisfies the following condition:

(t_0): *Any absorbing convex symmetric closed subset of E is a neighborhood of* 0 *in* E.

Any barrelled space and any topological vector Baire space belong to this type.

In section 1 we shall first prove that if a family of convex, continuous functionals on a convex domain of a t_0-space is bounded above at each point and is bounded at a point, it is equicontinuous. We then extend the theorem of W. Orlicz and Z. Ciesielski to a case of t_0-spaces. In section 2, with the aid of these results, we shall discuss the conditions sufficient for a separately continuous functional defined in a convex domain of a product space to be continuous. They also are extended to a family of functionals.

Throughout this paper a space is understood to be a topological real vector space and any functional is assumed to be real-ralued.

§1. We shall say that a functional f on a convex domain is convex if for any $x, y \in D$ the inequality $f(\lambda x+\mu y)\leq\lambda f(x)+\mu f(y)$ holds, where $\lambda+\mu=1$, $0\leq \lambda, \mu\leq 1$. A functional f is bounded in a set S if there exists a constant C such that $x \in S$ implies $|f(x)|\leq C$. f is locally bounded in a domain if there exists a neighbourhood of each point of the domain on which f is bounded. A family $\{f_\alpha\}_{\alpha\in A}$ of functionals is bounded at a point x if there exists a constant C such that $|f_\alpha(x)|\leq C$ holds for every $\alpha \in A$. It is uniformly bounded in a set S if there exists a constant C such that $x \in S$ implies $|f_\alpha(x)|\leq C$ for every $\alpha \in A$, where C does not depend on x. $\{f_\alpha\}_{\alpha\in A}$ is locally uniformly bounded in a domain if there exists a neighbourhood of each point of the domain in which $\{f_\alpha\}_{\alpha\in A}$ is uniformly bounded. The boundedness above (resp. below) of a functional or a family of functionals may be defined in an obvious manner. $\{f_\alpha\}_{\alpha\in A}$ is equicontinuous at a point x if, for any given $\varepsilon>0$, there exists a neighbourhood $K(x)$ of x such that $x' \in K(x)$ implies $|f_\alpha(x)-f_\alpha(x')|<\varepsilon$ for every $\alpha \in A$, where $K(x)$ depends on x but not on $\alpha \in A$. The family is simply called equicontinuous if it is equicontinuous at each point. $\{f_\alpha\}_{\alpha\in A}$ is uniformly equicontinuous in a set S if, for any given $\varepsilon>0$, there exists a neighbourhood U of the origin 0 such that $x, x' \in S$, $x-x' \in U$ imply $|f_\alpha(x)-f_\alpha(x')|<\varepsilon$ for every $\alpha \in A$.

If a space E is a t_0-space, any convex, symmetric subset of a convex domain D with 0 which is closed in D and absorbs every point of D is a neighbourhood of 0 in E.

Let f be a convex functional on a convex domain $D\subset E$. We note that if f is bounded above in an open subset $K\subset D$, then it is locally uniformly continuous in D ([1], [3]). In the same manner we can show that if a family of convex, continuous functionals on a convex domain $D \subset E$ is bounded in a neighbourhood of a point, then it is uniformly equicontinuous.

In this section we assume that E is a t_0-space, and that $\{f_\alpha\}_{\alpha\in A}$ is a family of convex, continuous functionals on a convex domain $D \subset E$. First we show.

PROPOSITION 1. (1) *If $\{f_\alpha\}_{\alpha\in A}$ is bounded above at each point of a neighbourhood U of a point $x_0 \in D$, then $\{f_\alpha\}$ is uniformly bounded above in a neighbourhood of x_0,*

(2) *Furthermore, if $\{f_\alpha\}_{\alpha\in A}$ is bounded at the point x_0, then it is uniformly bounded and uniformly equicontinuous in a neighbourhood of x_0 and bounded below at each point of D.*

(3) *If $\{f_\alpha\}_{\alpha\in A}$ is bounded at the point $x_0 \in D$, then $\{f_\alpha\}_{\alpha\in A}$ is equicontinuous in D if and only if it is bounded at each point of D.*

PROOF. Without the loss of generality we may assume that $x_0=0$ and U is symmetric. The proof is carried out under these assumptions.
(1): We may suppose that $M=\sup_\alpha f_\alpha(0)\geq 0$. Let $C=\{x; f_\alpha(x), f_\alpha(-x)\leq M+1$ for every $\alpha \in A\}$. It is easy to verify that C is convex, symmetric and closed in D. To the end of the proof it is sufficient to show that C absorbs every point of D. Let x be any point in D. x may be supposed to be contained in U. Let λ be a positive number less than 1 such that $\lambda \sup_\alpha f_\alpha(x)$, $\lambda \sup_\alpha f_\alpha(-x)\leq 1$. Then we can show that $f_\alpha(\lambda x), f_\alpha(-\lambda x)\leq M+1$. In fact, for example,

$$f_\alpha(\lambda x)\leq(1-\lambda)f_\alpha(0)+\lambda f_\alpha(x)\leq M+1.$$

Thus C absorbs the point x.
(2): Let C be the same as above and $M'=\sup_{x\in C,\ \alpha\in A} f_\alpha(x)$. Then, for any $x \in C$, we have

$$f_\alpha(x)\geq 2f_\alpha(0)-f_\alpha(-x)\geq 2\inf_\alpha f_\alpha(0)-M'$$

which shows that $\{f_\alpha\}$ is uniformly bounded below in C. By the remark preceding Proposition 1, we can conclude that $\{f_\alpha\}$ is locally uniformly equicontinuous in C.

Now we show that $\{f_\alpha\}$ is bounded below at each point of D. Let x be an arbitrary point in D and let λ be such that $a=\lambda x \in C$, $0<\lambda<1$. We have

$$f_\alpha(x) \geq \frac{1}{\lambda} f_\alpha(a) - \frac{1-\lambda}{\lambda} f_\alpha(0) \geq \frac{1}{\lambda} \inf_\alpha f_\alpha(a) - M,$$

consequently $\{f_\alpha\}$ is bounded below at x.

(3): Necessity. We may assume that $f_\alpha(0)=0$ for every $\alpha \in A$. Since $\{f_\alpha\}$ is equicontinuous at 0, it is uniformly bounded in a neighbourhood of 0. (2) shows that $\{f_\alpha\}$ is bounded below at each point of D. We put $h_\alpha(x)=f_\alpha(x)-f_\alpha(x_1)$ for an $x_1 \in D$. In the same manner as above $\{h_\alpha\}$ will be equicontinuous, so that it is bounded below at each point of D. Since $h_\alpha(0)=-f_\alpha(x_1)$, we can conclude that $\{f_\alpha(x_1)\}_{\alpha \in A}$ is bounded above. Since $x_1 \in D$ may be arbitrarily chosen, we see that $\{f_\alpha\}$ is bounded at each point of D.

Sufficiency. It is an immediate consequence of (2).

Thus the proof is completed.

As an immediate consequence we have

Proposition 2. *If $\{f_\alpha\}_{\alpha \in A}$ is bounded above at each point of D, then it is locally uniformly bounded above in D and $f(x)=\sup_\alpha f_\alpha(x)$ is convex and locally uniformly continuous in D. Furthermore, if $\{f_\alpha\}$ is bounded at a point in D, then it is locally uniformly bounded and locally uniformly equicontinuous in D.*

Proof. It remains only to show that f is locally uniformly continuous in D. Clearly f is convex in D. Now $\{f_\alpha\}$ becomes locally uniformly bounded above in D by Proposition 1, so that f is locally bounded above in D, whence f is locally uniformly continuous in D.

The next theorem is an extension of a theorem of W. Orlicz and Z. Ciesielski ([3], Prop. 3) to a family of convex continuous functionals on a convex domain of a t_0-space.

Theorem 1. *If $\{f_\alpha\}_{\alpha \in A}$ is bounded above at each point of a dense subset H which contains a non-void open subset, and if it is bounded below at a point, then we have,*

(1) *$\{f_\alpha\}$ is locally uniformly bounded and locally uniformly equicontinuous;*

(2) *if $\{f_\alpha\}$ is a net converging on H, then it converges on the whole D and the limit functional $f(x)=\lim_\alpha f_\alpha(x)$ is convex and continuous in D.*

Proof. (1): Owing to Proposition 2 it is sufficient to show that $\{f_\alpha\}$ is bounded above at each point of D. H may be assumed to contain 0 as an interior point. For any point $x \in D$ we can take such a positive number λ that

$x' = (1+\lambda)x \in D$. Let $\{x'_\beta\}_{\beta \in B}$ be a net converging on H to x' and put $x''_\beta = \frac{1+\lambda}{\lambda}x - \frac{1}{\lambda}x'_\beta$. Then we see that there exists an $x''_{\beta_0} \in H$ since x''_β converges to 0. $x = \frac{1}{1+\lambda}x'_{\beta_0} + \frac{\lambda}{1+\lambda}x''_{\beta_0}$. Since $\{f_\alpha\}$ is bounded above at x'_{β_0} and x''_{β_0}, it is also bounded above at x.

(2): $\{f_\alpha\}$ becomes equicontinuous by (1). Since $\{f_\alpha\}$ converges on a dense subset H, it follows that $\{f_\alpha\}$ converges on D to a continuous functional, which becomes convex.

REMARK. In the case (2) of the preceding theorem, if we assume that $\{f_\alpha\}$ is a sequence, the conditions of boundedness above at each point of H and of boundedness at a point are superfluous.

§2. This section is devoted to the study of the sufficient conditions under which a separately continuous functional on a product space turns out to be continuous.

Let E, F be spaces. Let D stand for a convex domain of $E \times F$ such that $D = D_1 \times D_2$, where D_1 and D_2 are the convex domains of E and F respectively. Let f be a functional on D. We shall use the notation f_x(resp. f_y) to indicate a functional $y \in F \to f(x, y)$ (resp. $x \in E \to f(x, y)$).

PROPOSITION 3. *If f is convex, then f is continuous if and only if it is separately continuous.*

PROOF. It is sufficient to show that if f is separately continuous, then it becomes continuous at any $(x, y) \in D$ which may be assumed to be (0, 0). $f(x, 0)$ and $f(0, y)$ are bounded in 0-neighbourhoods U and V respectively since f is separately continuous. Here we may assume that U and V are convex. Let W be the convex envelope of $U \times \{0\}$ and $\{0\} \times V$. It is clear that W is an 0-neighbourhood of $E \times F$. Any element of W is of the form

$$\lambda(x, 0) + \mu(0, y), \quad x \in U, \quad y \in V, \quad \lambda + \mu = 1, \quad \lambda, \mu \geq 0.$$

$$f(\lambda x, \mu y) \leq \lambda f(x, 0) + \mu f(0, y),$$

which shows that f is bounded above on W, and therefore continuous.

Next we show the following

THEOREM 2. *Suppose that E is a t_0-space and f is separately continuous. Then f is continuous if any of the following conditions is satisfied:*

(1) *f_x, f_y are convex for every $x \in D_1$ and $y \in D_2$. $\{f_y\}_{y \in V(b)}$, $V(b)$ being a neighbourhood of b in D_2, is bounded above at each point of a neighbourhood $U(a)$ of $a \in D_1$.*

(2) *F is finite-dimensional and f_y is convex for every $y \in D_2$:*

(3) *E, F are metrisable and f_y is convex for every $y \in D_2$.*

PROOF. (1): Since f is separately continuous, we may assume that $\{f(a, y)\}_{y \in V(b)}$ is bounded. Then it follows from Proposition 1 that the family $\{f_y\}_{y \in V(b)}$ is equicontinuous in a neighbourhood of a. This implies that f becomes continuous at (a, b).

Let (x_0, y_0) be any point of D. We shall show that (x_0, y_0) also shares the property stipulated for (a, b), which enables us to conclude that f is continuous at (x_0, y_0). Put $y_0 = \lambda b + (1-\lambda) y_0'$ for a λ, $0<\lambda<1$, $y_0' \in D$. Consider the set $C_2 = \{y; f_x(y) \leq \max[\sup_{y \in V(b)} f_x(y), f_y(y_0')]$ for every $x \in U(a)\}$. Clearly C_2 is convex and contains $V(b)$ and y_0' so that y_0 is an interior point of C_2. This shows that $\{f_y\}_{y \in V(y_0)}$ is bounded above at each point of $U(a)$, where $V(y_0)$ is a suitable neighbourhood of y_0. Similarly we choose a μ, $0<\mu<1$ in such a way that $x_0 = \mu a + (1-\mu) x_0'$, $x_0' \in D_1$. Since $f_{x_0'}$ is continuous at y_0, we may assume that $\{f_y\}_{y \in V(y_0)}$ is bounded at x_0'. If we consider the set $C_1 = \{x; \{f_y(x)\}_{y \in V(y_0)}$ is bounded above$\}$, then C_1 is convex and contains $U(a)$ and x_0'. This shows that x_0 is an interior point of C_1, that is, $\{f_y\}_{y \in V(y_0)}$ is bounded above at each point of a neighbourhood of x_0.

(2): F is locally compact. Let (x_0, y_0) be any point of D. If we consider a compact neighbourhood $V(y_0)$, every f_x is bounded on $V(y_0)$. Then $\{f_y\}_{y \in V(y_0)}$ becomes bounded at each point of D_1, and in turn equicontinuous in D_1 by Proposition 2. This together with the separate continuity of f implies that f is continuous at (x_0, y_0).

(3): Let (x_0, y_0) be any point of D. Let $\{(x_n, y_n)\}$ be any sequence of D which converges to (x_0, y_0). It suffices to show that $f(x_n, y_n) \to f(x_0, y_0)$ as $n \to \infty$.

Put $f_n(x) = f(x, y_n)$. Since f is separately continuous, $\{f_n(x)\}$ converges to $f(x, y_0)$ for each $x \in D_1$. Applying Theorem 1 to the sequence $\{f_n\}$, we see that $\{f_n\}$ converges uniformly on the compact set $\{x_n\}_{n \geq 0}$, whence we can conclude that $f(x_n, y_n) \to f(x_0, y_0)$ as $n \to \infty$.

PROPOSITION 4. *Let E be a Baire space and F a metrisable space. If f is separately continuous and f_y is a convex for every $y \in D_2$, then f is continuous.*

PROOF. Let $(a, b) \in D$. We consider the sets $C_n = \left\{x; f(x, y) \leq n \text{ for every } y \in S\left(b, \frac{1}{n}\right)\right\}$, where $S\left(b, \frac{1}{n}\right)$ stands for a closed ball with center b and radius $\frac{1}{n}$. C_n is a convex closed subset of D_1 and $D_1 = \cup_n C_n$. Since D_1 is a Baire space, $D_1 = \cup_n C_n^0$, so that there exists an n such that $a \in C_n^0$, and in turn f is bounded above on $C_n^0 \times S\left(b, \frac{1}{n}\right)$. It follows from this and Theorem 2 that f is continuous. The proof is completed.

Finally we show

THEOREM 3. *Suppose that E is a metrisable t_0-space and F is metrisable.*

If a family $\{f_\alpha\}_{\alpha\in A}$ of functionals on D is bounded at each point of D, $\{(f_\alpha)_x\}_{\alpha\in A}$ is equicontinuous in D_2 for every $x\in D_1$, and $(f_\alpha)_y$ is convex and continuous in D_1 for every $y\in D_2$, then $\{f_\alpha\}$ is equicontinuous in D. In particular, if $\{f_\alpha\}$ is a convergent net, then $f=\lim_\alpha f_\alpha$ is continuous in D.

PROOF. Let (x_0, y_0) be any point of D. Consider a sequence $\{(x_n, y_n)\}$ converging in D to (x_0, y_0). Put $f_\alpha^{(n)}(x)=f_\alpha(x, y_n)$. The family $\{f_\alpha^{(n)}\}$ is bounded at each point of D_1. In fact, for any x_0' we can take a neighbourhood $V(y_0)$ on which $|f_\alpha(x_0', y)-f_\alpha(x_0', y_0)|<1$ for every $\alpha\in A$. Then it follows from Theorem 1 that $\{f_\alpha^{(n)}\}$ is equicontinuous, so that we can infer that $f_\alpha(x_n, y_n)\rightarrow f(x_0, y_0)$ uniformly with respect to α as $n\rightarrow\infty$. It follows from a lemma of Bourbaki ([2], p. 29) that $\{f_\alpha\}$ is equicontinuous. In particular, if $\{f_\alpha\}$ is a convergent net, $f=\lim_\alpha f_\alpha$ is continuous since $\{f_\alpha\}$ is equicontinuous. The proof is completed.

REMARK. In the preceding Theorem, if $\{(f_\alpha)_x\}_{\alpha\in A}$ and $\{(f_\alpha)_y\}_{\alpha\in A}$ are respectively families of equicontinuous convex functionals for every x and every y, then the same conclusions will also hold. The proof will be carried out by applying Theorem 3 and (3) of Proposition 1 to the family of functionals $h_\alpha(x, y)=f_\alpha(x, y)-f_\alpha(x_0, y_0)$.

Most of the results established in this section may be extended with necessary modifications to the case of a functional or a family of functionals on a convex domain in a product space of more than two spaces.

References

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