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On Limits of BLD Functions along Curves

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In the preceding paper [4], F-Y. Maeda proved that almost every Green line converges to one point on the boundary obtained by a certain compactification of a Green space, notably for the Kuramochi boundary. We shall use the contents of [4] freely. In this note we shall prove that every curve on a space $\mathscr{E}$ has a similar property, except for those belonging to a family with infinite extremal length.

Consider a space $\mathscr{E}$ in the sense of Brelot and Choquet [1]; $\mathscr{E}$ may not be a Green space. We begin with the definition of extremal length of a family Γ of locally rectifiable non-degenerate curves on $\mathscr{E}$. Any measurable function $\rho \geqq 0$ on $\mathscr{E}$ with the property that $\int_c \rho ds$ is defined and $\geqq 1$ for each $c \in \Gamma$ is called *admissible* (in association with Γ) and the *module* $M(\Gamma)$ of Γ is defined by $\inf_\rho \int \rho^2 dv$, where ρ is admissible and dv is the volume element. The *extremal length* of Γ is defined by $1/M(\Gamma)$. We shall say that *almost every* curve on $\mathscr{E}$ has a certain property if the module of the exceptional family vanishes. The definitions of an admissible ρ and the module need obvious modifications in case the dimension of $\mathscr{E}$ is two. However, we shall use higher dimensional phrases in the sequel.

Let $\bar{\mathscr{E}}$ be a topological space containing $\mathscr{E}$ such that $\mathscr{E}$ is everywhere dense in $\bar{\mathscr{E}}$ and any two points of $\bar{\mathscr{E}}$ are separated by a continuous function on $\bar{\mathscr{E}}$; $\bar{\mathscr{E}}$ may not be compact. We set $\Delta = \bar{\mathscr{E}} - \mathscr{E}$ and denote by $C_{\mathscr{E}}(\bar{\mathscr{E}})$ the family of functions consisting of the restrictions to $\mathscr{E}$ of all the bounded continuous functions on $\bar{\mathscr{E}}$.

A family $\mathscr{Q}$ of real functions on $\mathscr{E}$ is said to separate points of $\bar{\mathscr{E}}$ (Δ resp.) if, for any different $P_1, P_2 \in \bar{\mathscr{E}}$ (Δ resp.), there is $f \in \mathscr{Q}$ such that

$$\varliminf_{\substack{P \to P_1 \\ P \in \mathscr{E}}} f(P) > \varlimsup_{\substack{P \to P_2 \\ P \in \mathscr{E}}} f(P).$$

We shall say that a function has a limit (a finite limit resp.) along an open curve on $\mathscr{E}$ if it has a limit (a finite limit resp.) as the point moves on the curve in each direction.

Using the well-known inequality $M(\bigcup_n \Gamma_n) \leqq \sum_n M(\Gamma_n)$, we can prove the following theorem in a fashion similar to the proof of Theorem 1 of F-Y.

Maeda [4]:

THEOREM 1. *If one of the following conditions is satisfied, then almost every open curve on $\mathscr{E}$ has at most one limit point in $\bar{\mathscr{E}}$ as the curve is traced in any direction*:

i) *There exists a countable family $\mathscr{Q}$ of functions on $\mathscr{E}$ such that each $f \in \mathscr{Q}$ has a limit along almost every open curve and $\mathscr{Q}$ separates points of $\bar{\mathscr{E}}$.*

ii) *$C_{\mathscr{E}}(\bar{\mathscr{E}})$ is separable in the uniform convergence topology and every function of $C_{\mathscr{E}}(\bar{\mathscr{E}})$ has a limit along almost every open curve.*

Let us be concerned with BLD functions. We shall obtain a generalization of Theorem 2. 28 of [5].

THEOREM 2. *Every BLD function f on $\mathscr{E}$ has a finite limit along almost every open curve.*

PROOF. Fuglede [2] proved that any BLD function in a Euclidean space is absolutely continuous along almost all curves. It follows easily that f has this property on $\mathscr{E}$. If f is absolutely continuous along an open curve c on $\mathscr{E}$ and if f does not have a finite limit along it, then

$$\int_c |\mathrm{grad}\, f|\, ds \geqq \int_c \left|\frac{\partial f}{\partial s}\right| ds = \int_c |df| = \infty.$$

Hence, in association with the family Γ' of all such c, $\rho = \varepsilon |\mathrm{grad}\, f|$ on $\mathscr{E}$ is admissible for arbitrary $\varepsilon > 0$. Consequently

$$M(\Gamma') \leqq \int \rho^2 dv = \varepsilon^2 \int |\mathrm{grad}\, f|^2 dv \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

Our assertion is concluded.

Combining this result with Theorem 1 we obtain

THEOREM 3. *Suppose that Δ is not void and there exists a countable family of BLD functions on $\mathscr{E}$ separating points of Δ. Then almost every curve on $\mathscr{E}$, whose starting point lies in $\mathscr{E}$ and which tends to the ideal boundary, has at most one limit point in Δ.*

REMARK. If $\bar{\mathscr{E}}$ is compact and metrizable and if $\{f \in C_{\mathscr{E}}(\bar{\mathscr{E}}); f \text{ is a BLD function on } \mathscr{E}\}$ separates points of Δ, then the above condition is satisfied.

COROLLARY. *Suppose that $\mathscr{E}$ is a Green space. Then almost every curve, whose starting point lies in $\mathscr{E}$ and which tends to the ideal boundary, converges to one point of the Kuramochi boundary of $\mathscr{E}$.*

Next we are interested in Green lines on a Green space $\mathscr{E}$ defined with respect to the Green function $G(P, P_0)$ with pole at P_0.

THEOREM 4. *Let Γ be a family of Green lines issuing from the pole and having a positive Green measure. Let Γ' be the family consisting of the parts of the members of Γ outside a small Green sphere $\Sigma_0=\{P; G(P, P_0)=t_0\}$ around the pole P_0.*[1] *Then $M(\Gamma')>0$.*

PROOF. We shall denote by γ the Green measure. It is defined on the family Λ_0 of all Green lines issuing from the pole and $\varphi_\tau\gamma(\Lambda)$ is equal to $\int_{\Sigma_0\cap\Lambda}\partial G/\partial n dS$, where φ_τ is a constant, Λ is any γ-measurable subfamily of Λ_0, $\partial G/\partial n$ is the normal derivative and dS is the surface element on the boundary ∂B_0. If ρ is admissible in association with Γ', then $\int_c \rho ds\geqq 1$ for each $c\in\Gamma'$ and

$$1\leqq\int_c \rho^2|\mathrm{grad}\ G|^{-1}ds\int_c |\mathrm{grad}\ G|ds=\int_c \rho^2\left|\frac{\partial G}{\partial s}\right|^{-1}ds\int_c |dG|.$$

It follows that

$$\frac{\gamma(\Gamma)}{t_0}\leqq\int_\Gamma\frac{d\gamma}{\int_c|dG|}\leqq\int_\Gamma\int_c\frac{\rho^2}{\left|\frac{\partial G}{\partial s}\right|}dsd\gamma=\frac{1}{\varphi_\tau}\iint_{[\Gamma']}\frac{\rho^2}{\left|\frac{\partial G}{\partial s}\right|}ds\left|\frac{\partial G}{\partial n}\right|dS$$

$$=\frac{1}{\varphi_\tau}\iint_{[\Gamma']}\rho^2 dsdS=\frac{1}{\varphi_\tau}\int_{[\Gamma']}\rho^2 dv,$$

where $[\Gamma']$ means the set of points on Γ' and dS is the surface element on a level surface $\{P; G(P, P_0)=\mathrm{const.}\}$. Consequently,

$$M(\Gamma')\geqq\frac{\varphi_\tau\gamma(\Gamma)}{t_0}>0.$$

In order to show that our Theorem 2 is an extension of Godefroid's theorem in [3] which asserts that every BLD function on any Green space has a finite limit along almost every regular Green line, we prove

THEOREM 5. *Let f be any BLD function on a Green space $\mathscr{E}$. Then the set of regular Green lines which issue from the fixed pole P_0 and along each of which* $\lim f$ *exists is measurable with respect to the Green measure γ.*

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1) In case P_0 is a point at infinity, by a "small" Green sphere we mean actually a large Green sphere.

PROOF. As a point set the family of all Green lines issuing from P_0 forms a domain D. We denote by $\mathscr{B}'$ the family of subsets of D such that, for every $B'\in\mathscr{B}'$, there exists a Borel set $B\supset B'$ with the property that $B-B'$ is a polar set. We observe that f is $\mathscr{B}'$-measurable and hence

$$A_t(\alpha)=\{P\in D;\ G(P,P_0)<t,\ f(P)>\alpha\}$$

belongs to $\mathscr{B}'$ for any $t>0$ and α. Given a small Green sphere Σ_0 around P_0, we call the intersection of Σ_0 with a Green line issuing from P_0 the *projection* on Σ_0 of any point of the Green line. We can speak of the projection on Σ_0 of any subset E of D too, and denote it by $p(E)$. Denote by $d(P_1, P_2)$ the Euclidean distance considered locally. Then $d(p(P_1), p(P_2))/d(P_1, P_2)$ is locally bounded, so that any polar set in D is projected to a polar set on Σ_0. Consequently $p(A_t(\alpha))$ differs from an analytic set at most by a polar set and hence is measurable with respect to γ.

Let Σ_1 be the Borel set on Σ_0 where the regular Green lines intersect Σ_0, and denote by c_P the regular Green line passing through $P\in\Sigma_1$. Since

$$\{P\in\Sigma_1;\ \overline{\lim_{\substack{G(Q,P_0)\to 0\\ Q\in c_P}}} f(Q)>\alpha\}=\bigcup_n\bigcap_k p\Big(A_{1/k}\Big(\alpha+\frac{1}{n}\Big)\Big),$$

$\overline{\lim}_{c_P} f$ is a γ-measurable function of P on Σ_1. Similarly $\underline{\lim}_{c_P} f$ is γ-measurable and the conclusion in the theorem follows immediately.

Now, suppose that a BLD function does not have a finite limit along any curve of a family Γ of regular Green lines with positive Green measure. The parts of the curves of Γ outside a small Green sphere around the pole form a family with finite extremal length by Theorem 4. This contradicts Theorem 2. Thus Godefroid's theorem is derived. We observe further that our Theorem 3 together with its remark generalizes Theorem 2 of F-Y. Maeda [4].

References

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