J. Sci. Hiroshima Univ. Ser. A-I
28 (1964), 39-51

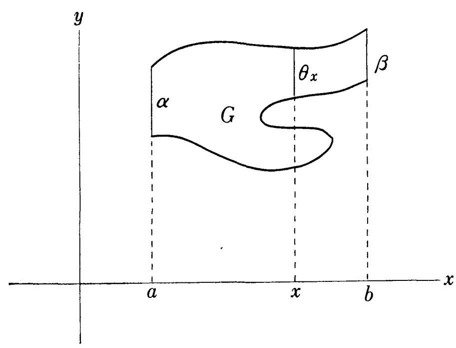
Extremal Length of Families of Parallel Segments

Makoto Ohtsuka

(Received March 18, 1964)

Introduction

In this paper we shall compute explicitly the extremal length of families of parallel segments and give some evaluations. For simplicity we limit ourselves to the (x, y)-plane R^2, although it is possible to generalize the results to the higher dimensional case.



Hersch [2] considered a simply-connected domain G whose boundary contains two vertical segments α and β with respective coordinates $x=a$ and $x=b$. Let θ_x, $a<x<b$, be a vertical crosscut of G separating α and β as in the figure, and let $\theta(x)$ be the length of θ_x. He remarked (p. 326, footnote 25)) that $M\{\theta_x; a<x<b\}=\int_a^b dx/\theta(x)$ and $\rho=1/\theta(x)$ is extremal.

We shall be interested in computing more generally the extremal length of a family of collections of vertical segments which do not necessarily form a domain. We shall discuss the case where the segments form a 2-dimensional Lebesgue measurable set in §1, and the non-measurable case in §2. In §3 we shall seek relations between the extremal length of a family of collections of vertical segments and that of a family of collections of curves, each collection of the latter family intersecting all members of the former. A part of the results in this paper is found in [3].

In a similar fashion we can treat families of collections of radial segments and families of collections of concentric circular arcs. It is quite easy to do so and we shall not state the results explicitly.

Now we shall define extremal length and state some properties. By a curve we mean a continuous image of an open interval or a circle. Furthermore we assume in this paper that each curve contains more than one point and that it is locally rectifiable. Namely, every closed subarc is rectifiable. We can represent it in terms of arc-length s. We shall use the notation c to denote a collection of curves. An integral along c is defined as the sum of the integrals along the components of c. Let $\Gamma=\{c\}$ be a family of collections of curves. We shall call a Lebesgue measurable function $\rho(z)\geqq 0$ in R^2 *admis-*

sible (in association with Γ) if $\rho(z(s))$ is measurable with respect to s and $\int_c \rho ds \geqq 1$ for each $c \in \Gamma$. Since no curve reduces to a point, $\rho \equiv \infty$ is always admissible. We define the *module* of Γ by

$$M(\Gamma) = \inf_{\rho} \iint \rho^2 dxdy,$$

where ρ is admissible, and call $1/M(\Gamma) = \lambda(\Gamma)$ the *extremal length* of Γ. While defining the integral we set $0 \cdot \infty = \infty \cdot 0 = 0$. We call an admissible function *extremal* if it attains the minimum. On account of Carathéodory-Vitali's theorem ([4], p. 75) we obtain the same value $M(\Gamma)$ if we restrict ρ to be lower semicontinuous. Naturally we may restrict ρ to be Borel.

The following properties are well known:

(1) $M(\Gamma) \leqq M(\Gamma')$ if $\Gamma \subset \Gamma'$.

(2) $M(\bigcup_n \Gamma_n) \leqq \sum_n M(\Gamma_n)$ for a countable family $\{\Gamma_n\}$. If $M(\Gamma')=0$, $M(\Gamma)= M(\Gamma \cup \Gamma')$.

(3) If $\{E_n\}$ is a sequence of mutually disjoint Lebesgue measurable sets and E_n contains all members of Γ_n for each n, then $M(\bigcup_n \Gamma_n) = \sum_n M(\Gamma_n)$.

It is easy to prove them perhaps except for the inequality in (2). Let ρ_n be lower semicontinuous and admissible in association with Γ_n. We set $\rho(z) = \sup_n \rho_n(z)$. This is admissible in association with $\bigcup_n \Gamma_n$ and it follows that

$$M(\bigcup_n \Gamma_n) \leqq \iint \rho^2 dxdy \leqq \sum_n \iint \rho_n^2 dxdy.$$

It is easy to conclude the inequality in (2) from this relation.

§ 1. Measurable case

In this section we consider $\Gamma = \{c\}$ such that each c consists of mutually disjoint open segments of finite or infinite length contained in one line parallel to the y-axis. We assume further that $c \cap c' = \emptyset$ if c and c' are different. We denote by Γ_x the family of c supported by the vertical line with coordinate x, by $l(c)$ the length of c and by χ_c the characteristic function of c in R^2.

We prove first

THEOREM 1. *If $\sum_c l(c)\chi_c$ is Lebesgue measurable in R^2 and $l(c) < \infty$ for each $c \in \Gamma$, then*

(4) $$M(\Gamma)=\int_A \sum_{c\in\Gamma_x} \frac{1}{l(c)}dx,$$

where $A=\{x;\ \Gamma_x \neq \emptyset\}$.

PROOF. We consider the function

(5) $$\rho_0(x,y)=\sum_c l^{-1}(c)\chi_c=\begin{cases}\dfrac{1}{l(c)} & \text{at } (x,y)\in c\in\Gamma,\\ 0 & \text{on } R^2-E,\end{cases}$$

where E is the union of c as point sets. By our assumption $\sum_c l(c)\chi_c$ is Lebesgue measurable in R^2. It follows that E and hence $\rho_0=\chi_E(\sum_c l(c)\chi_c)^{-1}$ is measurable. Since

$$\int_c \rho_0 ds=\frac{1}{l(c)}\int_c ds=1$$

for each $c\in\Gamma$, ρ_0 is admissible. It follows that

$$M(\Gamma)\leqq\iint\rho_0^2dxdy=\int_A\Big(\sum_{c\in\Gamma_x}\int_c\frac{dy}{l^2(c)}\Big)dx=\int_A\sum_{c\in\Gamma_x}\frac{1}{l(c)}dx.$$

To prove the inverse inequality we take any admissible ρ. The inequality $\int_c \rho\, dy\geqq 1$ yields

$$1\leqq\int_c dy\int_c\rho^2dy=l(c)\int_c\rho^2dy$$

and

$$\int_A\sum_{c\in\Gamma_x}\frac{1}{l(c)}dx\leqq\iint\rho^2dydx.$$

Consequently

$$\int_A\sum_{c\in\Gamma_x}\frac{1}{l(c)}dx\leqq M(\Gamma).$$

Thus we obtain the equality.

COROLLARY. *$M(\Gamma)$ is invariant under any vertical translation of each segment so far as the measurability of $\sum_c l(c)\chi_c$ is preserved.*

We shall see in the next section that this is no longer true if the measurability condition is dropped.

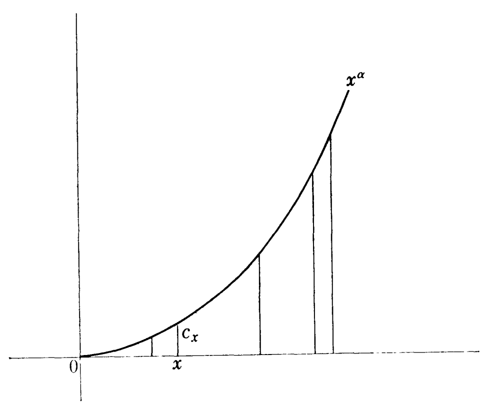
REMARK 1. $\rho_0(x, y)$ is an extremal function.

REMARK 2. In case Γ_x contains at most one $c=c_x$ for each x, the condition that $\sum_{c\in\Gamma} l(c)\chi_c$ is measurable is equivalent to the measurability of E. In fact, by Fubini's theorem

$$\iint \chi_E dxdy = \int\left(\int \chi_E dy\right)dx = \int l(c_x)dx$$

and $l(c_x)$ is a measurable function of x, where we set $l(c_x)=0$ if $\Gamma_x=\emptyset$. The function $f(x, y)$, which is defined to be $l(c_x)$ if (x, y) is on the vertical line with coordinate x, is measurable in R^2. Hence $\sum_c l(c)\chi_c = f\cdot\chi_E$ is measurable in R^2.

EXAMPLE. As an illustration we consider a very simple case in which $l(x)=x^\alpha$, $\alpha>0$. For each $x>0$, let c_x be the segment in the figure. By our theorem



$$M\{c_x;\, 0<x<1\} = \int_0^1 \frac{dx}{x^\alpha}\begin{cases} <\infty & \text{if } 0<\alpha<1, \\ =\infty & \text{if } \alpha\geqq 1 \end{cases}$$

and

$$M\{c_x;\, 1<x<\infty\} = \int_1^\infty \frac{dx}{x^\alpha}\begin{cases} =\infty & \text{if } 0<\alpha\leqq 1, \\ <\infty & \text{if } \alpha>1. \end{cases}$$

Let us turn to the case where $l(c)$ may be infinite for some $c\in\Gamma$. If the condition $l(c)=\infty$, assumed for all $c\in\Gamma$, implies $M(\Gamma)=0$, we obtain (4) easily under the same condition as in Theorem 1. However, we do not know whether or not this is true and we can prove only

THEOREM 2. *Suppose that Γ can be expressed as a mutually disjoint countable union $\bigcup_n \Gamma_n$ such that $(\Gamma_n)_x$ contains at most one $c=c_x^{(n)}\in\Gamma_x$ for every n and x and each $E_n=\bigcup_{c\in\Gamma_n} c$ as a point set is measurable in R^2. Then (4) is true.*

PROOF. First we consider the case where $l(c)=\infty$ for each $c\in\Gamma_n$ and prove $M(\Gamma_n)=0$. We may assume that the set $\{x; (\Gamma_n)_x\neq\emptyset\}$ is bounded. We shall write $\{-k<y<k\}$ for the strip domain $\{(x, y); -\infty<x<\infty, -k<y<k\}$. The set

$$E_n^{(k)}=E_n\cap\{-k<y<k\}$$

is measurable. By means of Fubini's theorem we observe that $l(c_x^{(n)}\cap\{-k<y<k\})$ is a measurable function of x, where l is defined to be zero if $(\Gamma_n)_x=\emptyset$ or $c_x^{(n)}\cap\{-k<y<k\}=\emptyset$. Hence

$$X_n^{(k)}=\{x; l(c_x^{(n)}\cap\{-k<y<k\})>1\}$$

is a measurable set on the x-axis. We set $\Gamma_n^{(k)}=\bigcup_{x\in X_n^{(k)}}\Gamma_x$. By considering

$$\rho_j=\sum_{x\in X_n^{(k)}}\frac{\chi_{c_x^{(n)}\cup\{-j<y<j\}}}{l(c_x^{(n)}\cap\{-j<y<j\})}$$

we have

$$M(\Gamma_n^{(k)})\leqq\iint\rho_j^2\,dxdy=\int_{X_n^{(k)}}\frac{1}{l(c_x^{(n)}\cap\{-j<y<j\})}dx.$$

By letting $j\to\infty$ we derive $M(\Gamma_n^{(k)})=0$. Since $\Gamma_n=\bigcup_k\Gamma_n^{(k)}$, $M(\Gamma_n)=0$.

If $l(c)<\infty$ for each $c\in\Gamma_n$, we have

$$M(\Gamma_n)=\int_{A_n}\frac{1}{l(c_x^{(n)})}dx$$

in virtue of Theorem 1, where $A_n=\{x; (\Gamma_n)_x\neq\emptyset\}$; see Remark 2 to Theorem 1. In the general case we denote by Γ_n' the family of members $\{c\}$ of Γ_n such that $l(c)<\infty$. Then both $\bigcup_{c\in\Gamma_n'} c$ and $\bigcup_{c\in\Gamma_n-\Gamma_n'} c$ as point sets are measurable. By (2), $M(\Gamma_n)=M(\Gamma_n')$ and it holds that

$$M(\Gamma_n')=\int_{A_n'}\frac{1}{l(c_x^{(n)})}dx=\int_{A_n}\frac{1}{l(c_x^{(n)})}dx,$$

where $A'_n=\{x;\,(\Gamma'_n)_x\neq\emptyset\}$. Thus (4) is valid for Γ_n.

Now the equality

$$M(\Gamma)=\sum_n M(\Gamma_n)=\sum_n\int_{A_n}\frac{1}{l(c_x^{(n)})}dx=\int_A\sum_{c\in\Gamma_x}\frac{1}{l(c)}dx$$

is concluded by (3).

§ 2. Non-measurable case

For a general Γ as defined in the beginning of §1 with no measurability condition, we give

THEOREM 3.

$$\overline{\int}_A\sum_{c\in\Gamma_x}\frac{1}{l(c)}dx\leqq M(\Gamma)\leqq\overline{\iint}\sum_{c\in\Gamma}\frac{\chi_c}{l^2(c)}dxdy, \tag{6}$$

the right inequality being valid provided $l(c)<\infty$ *for each* $c\in\Gamma$.

PROOF. If $l(c)<\infty$ for each $c\in\Gamma$, we consider ρ_0 defined in (5), and take any Borel $\rho\geqq\rho_0$. It is admissible and $M(\Gamma)\leqq\iint\rho^2dxdy$, whence

$$M(\Gamma)\leqq\overline{\iint}\sum_{c\in\Gamma}\frac{\chi_c}{l^2(c)}dxdy.$$

To obtain an evaluation from below we take any admissible Borel ρ. From $\int_c\rho dy\geqq 1$ it follows that

$$1\leqq\left(\int_c\rho^2dy\right)\cdot l(c).$$

Even if $l(c)=\infty$ for some c we have

$$\overline{\int}_A\sum_{c\in\Gamma_x}\frac{1}{l(c)}dx\leqq\iint\rho^2dy\,dx,$$

and derive the left inequality of (6).

We raise

QUESTION. *Is the right inequality in* (6) *true generally?*
This question remains open. However, we can see easily that the answer is affirmative if $M(\Gamma)=0$ for any Γ such that the length of each $c\in\Gamma$ is infinite.

F-Y. Maeda remarked orally that $M(\Gamma)=0$ if there are sequences $\{a_n\}$ and $\{b_n\}$ increasing to ∞ such that $l(c\cap\{-a_n<y<a_n\})>b_n$ and $a_n/b_n^2\to 0$ as $n\to\infty$. In fact, it is sufficient to prove $M(\bigcup_{x\in I}\Gamma_x)=0$ for an interval I of finite length d. Observing that $\rho=b_n^{-1}\chi_{\{x\in I,\ -a_n<y<a_n\}}$ is admissible, we have

$$M(\bigcup_{x\in I}\Gamma_x)\leqq\iint\rho^2 dx\,dy\leqq\frac{2a_n d}{b_n^2}\to 0 \quad \text{as } n\to\infty.$$

We note that no measurability condition is required.

Next we establish the equality of $M(\Gamma)$ to the left hand side of (6) in a special case. We begin with

LEMMA 1. *If $A=\{x;\,\Gamma_x\neq\emptyset\}$ is of linear measure zero, then $M(\Gamma)=0$.*

PROOF. Set $\rho(x, y)=\infty$ if $x\in A$ and $=0$ if $x\notin A$. This is admissible and $M(\Gamma)\leqq\iint\rho^2\,dx\,dy=0$.

THEOREM 4. *Let $f(x)<\infty$ be a measurable function of x, and A be any subset of the x-axis. If each Γ_x, $x\in A$, consists of the segment of the form $\{(x, y);\, x\in A,\, f(x)<y<g(x)\}$ with any function $g(x)$ on A satisfying $g(x)>f(x)$, then*

$$M(\Gamma)=\overline{\int_A}\frac{dx}{g(x)-f(x)}. \tag{7}$$

PROOF. If $x\notin A$, we may set $g(x)=\infty$ because both sides of (7) do not change for the new enlarged family; the invariance of M is inferred by the aid of (2) and the above Maeda's remark. Therefore we assume that A coincides with the whole x-axis. By (6), $M(\Gamma)=\overline{\int}(g(x)-f(x))^{-1}dx$ if $\overline{\int}(g(x)-f(x))^{-1}dx=\infty$. So we assume that $\overline{\int}(g(x)-f(x))^{-1}dx<\infty$. Given $\varepsilon>0$, let $h(x)$ be a measurable function such that $0\leqq h(x)\leqq g(x)-f(x)$ and $\int dx/h(x)\leqq\overline{\int}(g(x)-f(x))^{-1}dx+\varepsilon$. Then $h(x)>0$ for almost all x. Let $B=\{x;\,h(x)>0\}$ and let Γ_h be the family of segments $\{f(x)<y<f(x)+h(x);\, x\in B\}$. Since both $f(x)$ and $h(x)$ are measurable, the union of $c\in\Gamma_h$ as point sets is measurable in R^2. Hence we apply Lemma 1 and Theorem 2, and obtain

$$M(\Gamma)=M(\bigcup_{x\in B}\Gamma_x)\leqq M(\Gamma_h)=\int\frac{dx}{h(x)}\leqq\overline{\int}\frac{dx}{g(x)-f(x)}+\varepsilon,$$

whence $M(\Gamma)\leqq\overline{\int}dx/(g(x)-f(x))$. The inverse inequality being known, we obtain the equality.

By an example we shall show that it can happen that

$$\overline{\int}_A \sum_{c\in\Gamma_x} \frac{1}{l(c)}dx < M(\Gamma) < \overline{\iint} \sum_{c\in\Gamma} \frac{\chi_c}{l^2(c)}dxdy.$$

We consider the segment $S=\{0<x<1\}$ and a non-measurable subset X_1 such that the outer measure $\bar{m}X_1=1$ and the inner measure $\underline{m}X_1=0$; for the existence of such a set, see p. 70 of [1]. The complement $X_2=S-X_1$ has $\bar{m}X_2=1$ and $\underline{m}X_2=0$. If $x\in X_1$, we set $c_x=\{(x, y);\ 0<y<1\}$. If $x\in X_2$, we set $c_x=\{(x, y);\ -1/2<y<1/2\}$. Evidently $l(c_x)=1$ and hence $\int_S dx/l(c_x)=1$. We set $E_1=E\cap\{y>1/2\}$, $E_2=E\cap\{0\leqq y\leqq 1/2\}$ and $E_3=E\cap\{y<0\}$. We have

$$\overline{\iint}_E \frac{dxdy}{l^2(c_x)} = \left(\overline{\iint}_{E_1} + \overline{\iint}_{E_2} + \overline{\iint}_{E_3}\right)dxdy = \overline{\iint}_{E_1} dxdy + \frac{1}{2} + \overline{\iint}_{E_3} dxdy.$$

Let ρ, $0<\rho\leqq 1$, be a lower semicontinuous function which is equal to 1 on E_1. We observe that $\int_{1/2}^1 \rho(x, y)dy$ is lower semicontinuous on the x-axis. Hence $T=\left\{x\in S;\ \int_{1/2}^1 \rho dy=1/2\right\}$ is a Borel set on the x-axis. Since $T\supset X_1$ and $\bar{m}X_1=1$, $mT=1$ and $\iint \rho dxdy \geqq \int_T \left(\int \rho dy\right)dx \geqq 1/2$. Evidently $\overline{\iint}_{E_1} dxdy \leqq 1/2$ and $\overline{\iint}_{E_1} dxdy=1/2$ is concluded. Similarly $\overline{\iint}_{E_3} dxdy=1/2$ and consequently $\overline{\iint}_E dxdy/l^2(c_x)=3/2$.

We shall prove that $M\{c_x\}$ is equal to the module of the following family Γ'. We denote by $\gamma_x^{(1)}$ the segment $\{(x, y);\ 0<y<1\}$ and by $\gamma_x^{(2)}$ the segment $\{(x, y);\ -1/2<y<1/2\}$, and define Γ' as the union of two families $\Gamma_1=\{\gamma_x^{(1)};\ x\in S\}$ and $\Gamma_2=\{\gamma_x^{(2)};\ x\in S\}$. We note that Γ_1 and Γ_2 overlap. By (1) $M(\Gamma)\leqq M(\Gamma')$. In order to establish the inverse inequality, we take any lower semicontinuous admissible function ρ in association with Γ. As a function of x, $\int_0^1 \rho(x, y)dy$ is lower semicontinuous. Therefore

$$T_1=\left\{x\in S;\ \int_0^1 \rho(x, y)dy \geqq 1\right\}$$

is a Borel set on the x-axis. Since $\bar{m}X_1=1$ and $T_1\supset X_1$, $mT_1=1$. Similarly $T_2=\left\{x\in S;\ \int_{-1/2}^{1/2} \rho(x, y)\geqq 1\right\}$ is a Borel set and $mT_2=1$. Now ρ is admissible in association with $\{\gamma_x^{(1)};\ x\in T_1\cap T_2\}\cup\{\gamma_x^{(2)};\ x\in T_1\cap T_2\}$. We define ρ^* by ∞ at points with x-coordinate in $S-T_1\cap T_2$ and by ρ elsewhere. This ρ^* is admissible in association with Γ' and it holds that

$$M(\Gamma')\leqq\iint\rho^{*2}dxdy=\iint\rho^2 dxdy,$$

whence $M(\Gamma')\leqq M(\Gamma)$. Thus $M(\Gamma')=M(\Gamma)$.

Next let us prove $M(\Gamma')=4/3$. We consider

$$\rho_0(x, y)=\begin{cases}\dfrac{2}{3} & \text{if } x\in[0, 1], y\in\left(-\dfrac{1}{2}, 0\right)\cup\left(\dfrac{1}{2}, 1\right), \\ \dfrac{4}{3} & \text{if } x\in[0, 1], y\in\left[0, \dfrac{1}{2}\right], \\ 0 & \text{elsewhere.}\end{cases}$$

This is admissible in association with Γ' and

$$M(\Gamma')\leqq\iint\rho_0^2 dxdy=\frac{4}{3}.$$

Let ρ be any function admissible in association with Γ'. To prove $M(\Gamma')\geqq 4/3$ it is sufficient to show $\int_{-1/2}^{1}\rho^2(x, y)dy\geqq 4/3$ for every $x\in S$. Under the condition that $\int_{1/2}^{1}\rho dy$ is constant, $\int_{1/2}^{1}\rho^2 dy$ is minimum if ρ is constant on $1/2<y<1$. The same is true for $\int_{0}^{1/2}\rho^2 dy$ and $\int_{-1/2}^{0}\rho^2 dy$. Suppose that $\rho=a$ on $0<y<1/2$. We may assume that $\rho=2-a$ both on $-1/2<y\leqq 0$ and $1/2\leqq y<1$. We can observe easily that $\int_{-1/2}^{1}\rho^2 dy=a^2/2+(2-a)^2$ is minimum when $a=4/3$ and the minimum value is $4/3$. Thus $M(\Gamma')=4/3$. As a remark we observe that ρ_0 is an extremal function for Γ'.

§ 3. Family of collections of curves intersecting Γ

In the rest of the paper we shall consider a problem of a somewhat different nature. Let $\Gamma=\{c\}$ be a family of mutually disjoint collections of vertical segments and $\Gamma'=\{c'\}$ be a family of collections of locally rectifiable curves such that each $c'\in\Gamma'$ intersects all members of Γ. We are interested in relations between $\lambda(\Gamma)$ and $\lambda(\Gamma')$.

First we prove

LEMMA 2. *Let c' be a locally rectifiable curve and $f(z)$ be a non-negative Borel measurable function on c'. Then it holds that*

$$\int_{c'}f(z(s))ds\geqq\int_{p(c')}\sum_{s\in S(x)}f(z(s))dx,$$

where $p(c')$ is the projection of c' into the x-axis and $S(x)$ is the set of values s such that $x(s)=x$.

PROOF. We observe that

$$s(B)\geqq\int_B|dx(s)|=\int_{p(B)}n_B(x)dx$$

for any Borel subset B of c', where $p(B)$ is the projection of B into the x-axis and $n_B(x)$ is the number of the points of intersection of B with the vertical line with coordinate x. The last equality can be justified first for any open subarc of c', for any countable union of open subarcs of c', then for any compact subset of c' and finally for any Borel set $B\subset c'$. We decompose c' into mutually disjoint Borel sets $e_1, e_2, \cdots$ and have

$$\sum_i\Big(\sup_{z\in e_i}f(z)\Big)s(e_i)\geqq\sum_i\Big(\sup_{z\in e_i}f(z)\Big)\int_{p(e_i)}n_{e_i}(x)dx$$

$$\geqq\sum_i\int_{p(e_i)}\sum_{\substack{s\in S(x)\\ z(s)\in e_i}}f(z(s))dx=\int_{p(c')}\sum_{s\in S(x)}f(z(s))dx.$$

Because of the arbitrariness of the decomposition of c' we obtain the desired inequality.

We shall establish

THEOREM 5. *Under the same condition as in Theorem* 2,

$$M(\Gamma')\leqq\lambda(\Gamma). \tag{8}$$

PROOF. By Remark 2 of Theorem 1 $\sum_{c\in\Gamma}l(c)\chi_c$ is measurable in R^2. First we assume $0<\lambda(\Gamma)<\infty$, and consider $\rho_0=\lambda(\Gamma)\sum_{c\in\Gamma}l^{-1}(c)\chi_c$. As we observed in the proof of Theorem 1, it is measurable in R^2. Any Borel $\rho\geqq\rho_0$ is admissible in association with Γ' because

$$\int_{c'}\rho ds\geqq\int_{p(c')}\sum_{s\in S(x)}\rho(z(s))dx\geqq\lambda(\Gamma)\int_A\sum_{c\in\Gamma_x}\frac{1}{l(c)}dx=\lambda(\Gamma)M(\Gamma)=1$$

by Lemma 2 and Theorem 2, where $A=\{x;\Gamma_x\neq\emptyset\}$. It follows that

$$M(\Gamma')\leqq\iint\rho_0^2dxdy=\lambda^2(\Gamma)\iint\sum_{c\in\Gamma}\frac{\chi_c}{l^2(c)}dxdy$$

$$=\lambda^2(\Gamma)\int_A\sum_{c\in\Gamma_x}\frac{1}{l(c)}dx=\lambda^2(\Gamma)M(\Gamma)=\lambda(\Gamma).$$

If $\lambda(\Gamma)=\infty$ then (8) holds trivially.

Next consider the case $\lambda(\Gamma)=0$. For each m we set

$$\Gamma_n^{(m)}=\left\{c\in\bigcup_{-m<x<m}(\Gamma_n)_x;\ l(c\cap\{-m<y<m\})>\frac{1}{m}\right\}.$$

We can see easily that $\sum_{c\in\Gamma_n^{(m)}} l(c)\chi_c$ is measurable in R^2. We set

$$A_n^{(m)}=\{x;\ (\Gamma_n^{(m)})_x\neq\emptyset\}.$$

As $m\to\infty$

$$M(\bigcup_{n=1}^{m}\Gamma_n^{(m)})=\sum_{n=1}^{m}\int_{A_n^{(m)}}\frac{1}{l(c_x)}dx\nearrow\int_A\sum_{c\in\Gamma_x}\frac{1}{l(c)}dx=M(\Gamma)=\infty.$$

Since

$$M(\bigcup_{n=1}^{m}\Gamma_n^{(m)})\leqq\sum_{n=1}^{m}\int_{-m}^{m}mdx=2m^3<\infty,$$

we can apply (8) to $\bigcup_{n=1}^{m}\Gamma_n^{(m)}$ and obtain

$$M(\Gamma')\leqq\lambda(\bigcup_{n=1}^{m}\Gamma_n^{(m)})\searrow 0.$$

Thus $M(\Gamma')=0$ and (8) is true in all cases.

Corollary. *$\lambda(\Gamma')=0$ implies $\lambda(\Gamma)=\infty$.*

The last question is as to whether the converse of this corollary is true or not. In Theorem 5, Γ' is *a family*. However, in Theorem 6 we shall consider *the family* Γ' of all collections $\{c'\}$ of locally rectifiable curves such that each c' intersects all $c\in\Gamma$.

We do not know the answer in the general case and can prove only

Theorem 6. *Suppose that each $c\in\Gamma$ consists of a finite number of segments. If $\lambda(\Gamma)=\infty$, $\lambda(\Gamma')=0$.*

Proof. Take any lower semicontinuous $\rho\geqq0$ with $\iint\rho^2dxdy<\infty$. For any given $\varepsilon>0$, we shall find $c'\in\Gamma'$ such that $\int_{c'}\rho ds<\varepsilon$. We may assume that $A=\{x;\ \Gamma_x\neq\emptyset\}$ is contained in an interval $-\infty<a<x<b<\infty$. Set $\Gamma_1=\{c\in\Gamma;\ l(c)<\infty\}$ and $A_1=\{x;\ (\Gamma_1)_x\neq\emptyset\}$. By Theorem 3 the linear measure of A_1 is zero.

First we shall find a collection c_1' of curves, having $\int_{c_1'}\rho ds<\varepsilon/2$ and intersecting all elements c of $\Gamma-\Gamma_1$. We set

$$m_k=\inf_{k\leqq y<k+1}\int_a^b\rho(x,y)dx,$$

where $k=0, \pm 1, \pm 2, \dots$ and have

$$m_k^2\leqq\left(\int_a^b\rho(x,y)dx\right)^2\leqq(b-a)\int_a^b\rho^2dx \qquad \text{if } k\leqq y<k+1.$$

It follows that

$$m_k^2\leqq(b-a)\int_k^{k+1}\int_a^b\rho^2 dxdy$$

and

$$\sum_{-\infty<k<\infty} m_k^2\leqq(b-a)\int_{-\infty}^{\infty}\int_a^b\rho^2 dxdy.$$

This shows that there are sequences $y_1<y_2<\cdots\to\infty$ and $y_{-1}>y_{-2}>\cdots\to-\infty$ such that

$$\sum_{k=1}^{\infty}\left\{\int_a^b\rho(x,y_k)dx+\int_a^b\rho(x,y_{-k})dx\right\}<\varepsilon.$$

The collection of horizontal segments $\{y=y_k, a<x<b\}$ and $\{y=y_{-k}, a<x<b\}$, $k=1, 2, \cdots$, intersects eventually all c and hence can be taken for c_1'.

Next we want to show the existence of a collection c_2' of curves, having $\int_{c_2'}\rho ds<\varepsilon/2$ and intersecting all elements of Γ_1. Take any y_0 and y_0' $(y_0<y_0')$. We have

$$\int_{y_0}^{y_0'}\left(\int_a^b\rho(x,y)dx\right)^2dy\leqq(b-a)\int_{y_0}^{y_0'}\int_a^b\rho^2(x,y)dxdy<\infty.$$

Therefore the values $y\in(y_0, y_0')$ for which $\int_a^b\rho dx<\infty$ are dense in (y_0, y_0'). We choose $y_1, y_2, \cdots$ such that they are dense on the y-axis and $\int_a^b\rho(x,y_p)dx$ is finite for each p. We find a linear open set c_p on each horizontal line $y=y_p$ such that $\int_{c_p}\rho(x,y_p)dx<\varepsilon/2^{p+1}$ and c_p intersects all vertical lines passing through A_1. Since $\{y_p\}$ are dense on the y-axis, each $c\in\Gamma_1$ intersects at least one of $\{c_p\}$. It holds that $\int_{\bigcup_p c_p} \rho\, dy < \varepsilon/2$ and hence $\bigcup_p c_p$ can be taken for c_2'. Now $c_1' \cup c_2'$ intersects all members of Γ and

$$\int_{c_1' \cup c_2'} \rho\, ds < \varepsilon.$$

This shows that $M(\Gamma') = \infty$.

References

[1] P. Halmos: Measure theory, New York, 1950.

[2] J. Hersch: Longueurs extrémales et théorie des fonctions, Comment. Math. Helv., 29 (1955), pp. 301-337.

[3] M. Ohtsuka: Dirichlet problem, extremal length and prime ends, Lecture Notes, Washington University, St. Louis, 1962-63.

[4] S. Saks: Theory of the integral, Warszawa-Lwow, 1937.

Department of Mathematics,
Faculty of Science,
Hiroshima University.