J. Sci. Hiroshima Univ. Ser. A-I
28 (1964), 271-299

An Elementary Introduction of Kuramochi Boundary

Makoto Ohtsuka

(Received September 21, 1964)

Introduction

Recently the importance of the ideal boundary, which was introduced by Kuramochi [4] in 1956 and is now called the Kuramochi boundary, has been well recognized. For instance, a large part of the book of Constantinescu and Cornea [3] is devoted to the study of the Kuramochi boundary.

Independently of them, a seminar to read Kuramochi's paper [4] was held in 1959-60 with Dr. Matsumoto, then an assistant at Hiroshima University. By the aid of the notes made by Dr. Matsumoto, the present author justified the whole part of pp. 145-162 of [4] and gave lectures based on it at Kyushu University in 1962. The theory of BLD functions (called Dirichlet functions in [3]) was used there. However, his results had been left unpublished because the theory was rather complicated and needed further improvements to be regarded as an accessible version of Kuramochi's theory. Meanwhile, it was informed that Constantinescu and Cornea succeeded in developing the theory of Kuramochi boundary rigorously and their book was under preparation. This made the present author more reluctant to publish his notes in spite of a kind suggestion by Constantinescu to publish them in a Roumanian journal. It is also to be noted that Kuramochi himself tried to make his theory clear in [5; 6] but there have been remaining still several obscure points.

Now in this paper the original method of Kuramochi is made elementary by avoiding the theory of BLD functions. Although there is nothing new in results concerning the Kuramochi boundary, it is hoped that this paper will help people become familiar with Kuramochi's theory on his boundary.

It is possible to introduce the Kuramochi boundary in higher dimensional spaces in a similar manner. It needs, however, more careful discussions in details and is not carried out in this paper. For the Kuramochi boundary for Green spaces of higher dimension, we refer to Maeda's papar [7] where it is discussed along the line of [3].

Except §1 in which we give a proof of the Dirichlet principle by the aid of the notion of harmonic subflows, the contents of this paper are quite parallel to those of the paper [8] of R. S. Martin. Readers must be very well acquainted with this paper [8] and so detailed explanation of each section is omitted here. We shall not be concerned with the values of potentials on the Kuramochi boundary in this paper.

The prerequiste knowledge is some fundamental notions about abstract Riemann surfaces such as exhaustion, double, normal family of harmonic functions, etc. and certain classical results in the theory of logarithmic potentials in the plane.

§ 1. Dirichlet principle

Let R be an open Riemann surface. We shall call a domain in R a parametric disk if a local variable z is defined on its closure and it is mapped onto $|z|<1$. The closure of a parametric disk will be called a closed parametric disk. In this paper we take a closed parametric disk K_0 once for all, and set $R'=R-K_0$. An exhaustion will mean an increasing sequence $\{R_n\}$, $n=1, 2, \cdots$, of relatively compact domains such that $K_0\subset R_1$, $R_n\cup\partial R_n\subset R_{n+1}$ $(n=1, 2, \cdots)$, each ∂R_n is analytic, i.e. it consists of a finite number of closed analytic curves, and no component of $R-R_n$ is compact. We shall write $R'_n=R_n-K_0$. We shall call a relatively compact open set or a compact set in R *regular* if its boundary consists of a finite number of analytic arcs.

Let a harmonic function $h(P)$ be given in an open set G on R such that it is not constant in any component. A curve γ will be called orthogonal (with respect to $h(P)$) if $\operatorname{grad} h\neq 0$ on γ and there is a neighborhood of γ in which a single-valued harmonic conjugate h^* of h can be defined so as to be constant on γ. A maximal orthogonal curve will be called an *orthogonal trajectory* (for $h(P)$). It is orthogonal to level curves of $h(P)$ at each point of intersection. Since $h(P)$ increases or decreases strictly on each orthogonal trajectory, no orthogonal trajectory is a closed curve. Each orthogonal trajectory tends to the boundary of G unless it terminates at a critical point where $\operatorname{grad} h=0$. Let c be an open analytic arc[1] on whose closure $\operatorname{grad} h\neq 0$ and h is constant. In a neighborhood of c we can find a single-valued harmonic conjugate h^* of h. We call the bundle of orthogonal trajectories passing through c a *harmonic flow* (for $h(P)$), and call a subbundle a *harmonic subflow* (for $h(P)$) if its intersection with c is measurable with respect to h^*. Let $h(P)=a$ on c. If $d>0$ is suitably chosen, h takes all values of $[a-d, a+d]$ on each orthogonal trajectory passing through c. The part of the harmonic flow on which $a-d<h(P)<a+d$ is called a *regular tube*. It is a domain and its closure is called a *regular compact tube*.

In the proof of the Dirichlet principle, we shall use

---

1) By an open analytic arc we mean an open analytic curve which does not oscillate, i.e. which has definite end-points.

LEMMA 1. *We can divide G minus all critical points into disjoint harmonic subflows.*

PROOF. We cover the remaining open set by a countable number of regular tubes. With each tube we associate the harmonic flow passing through it. If the harmonic flows are denoted by $F_1, F_2, \cdots$, the harmonic subflows $\Gamma_1=F_1$, $\Gamma_2=F_2-F_1$, $\Gamma_3=F_3-F_2-F_1, \cdots$ satisfy the condition.

Let K be a regular compact set in R'. A continuous function f on $R'-K$ will be called *piecewise smooth* if f is continuously differentiable in an open subset $G\subset R'-K$ such that $R'-K-G$ locally consists of a finite number of points and open analytic arcs. Given a continuous function φ on ∂K, we shall denote by $\mathscr{D}_{R'-K}(\varphi)$ the class of all piecewise smooth Dirichlet finite (meaning that the Dirichlet integral is finite) functions with boundary values φ on ∂K and 0 on ∂K_0.

Now we formulate the Dirichlet principle as follows:

THEOREM 1. *Let K be a regular compact set in R', and assume $\mathscr{D}_{R'-K}(\varphi)\neq\phi$. Then there is a unique $h\in\mathscr{D}_{R'-K}(\varphi)$ which has the minimum Dirichlet integral among the functions of $\mathscr{D}_{R'-K}(\varphi)$, and h is harmonic in $R'-K$.*

PROOF.[2] We may assume $K\subset R_1$. We shall denote the Dirichlet integral by $\|\ \|^2$. Take any $f\in\mathscr{D}_{R'-K}(\varphi)$. First we prove the theorem in R_n. Let h_n be the harmonic function in R'_n-K which has the boundary values φ on ∂K and 0 on ∂K_0 and whose normal derivative vanishes on ∂R_n. One way to find h_n is as follows: Consider the double $\hat{R}_n$ of R_n along ∂R_n (see p. 199 of [1]) and denote by $\hat{K}$ and $\hat{K}_0$ the symmetric extensions of K and K_0 respectively. The restriction to R'_n-K of the Dirichlet solution on $\hat{R}_n-\hat{K}-\hat{K}_0$ for the boundary function φ on $\partial\hat{K}$ and 0 on $\partial\hat{K}_0$ is equal to h_n.

Let G be an open subset of $R'-K$ such that f is continuously differentiable in G and $R'-K-G$ locally consists of a finite number of points and open analytic arcs. With the aid of Lemma 1 we divide all components of R'_n-K, in which h_n is not constant, into disjoint harmonic subflows $\Gamma_1, \Gamma_2, \cdots$ for h_n. In order to show $\|h_n\|_{R'_n-K}\leqq\|f\|_{R'_n-K}$, it suffices to show $\|h_n\|_{[\Gamma_k]}\leqq\|f\|_{[\Gamma_k]}$ for each k where the subscript $[\Gamma_k]$ indicates that the integrals are taken on Γ_k as a point set.

Let γ be an element of Γ_k which does not coincide with any open analytic arc contained in $R'-K-G$, does not terminate at any critical point and which does not pass through any point belonging to $R'-K-G$ minus the open analytic arcs. Then γ tends to the boundary of $R'-K$ in both directions and

2) This proof is found in [9] in the plane case.

every compact subarc of γ meets any one of these open analytic arcs at most a finite number of times. Take a conjugate h_n^* of h_n on the harmonic flow which contains Γ_k, and regard $h_n + ih_n^*$ as a local variable on the harmonic flow. Since the normal derivative $\partial h_n/\partial\nu$ vanishes on ∂R_n, each component of ∂R_n minus the critical points is orthogonal to level curves terminating on ∂R_n and hence γ does not terminate on ∂R_n. If γ terminates at two points P_1, P_2 on ∂K, then

$$\int_\gamma |\operatorname{grad} f|\, dh_n \geqq \int_\gamma |df| \geqq |\varphi(P_2)-\varphi(P_1)| = \int_\gamma dh_n,$$

where grad f is defined with respect to the local variable $h_n+ih_n^*$ and γ is oriented so that h_n increases. We obtain the same inequality $\int_\gamma |\operatorname{grad} f|\, dh_n \geqq \int_\gamma dh_n$ if one end-point of γ lies on ∂K_0 instead on ∂K; evidently it does not happen that both end-points of γ lie on ∂K_0. If γ oscillates, we take sequences of points on γ converging to some points of $\partial K \cup \partial K_0$ and conclude the same inequality.[3] We derive

$$\left(\int_\gamma dh_n\right)^2 \leqq \int_\gamma |\operatorname{grad} f|^2 dh_n \int_\gamma dh_n$$

and $\int_\gamma dh_n \leqq \int_\gamma |\operatorname{grad} f|^2 dh_n$. Thus we obtain

$$\|h_n\|^2_{[\Gamma_k]} = \int_{[\Gamma_k]\cap c_k} \int_\gamma dh_n dh_n^* \leqq \iint_{[\Gamma_k]} |\operatorname{grad} f|^2 dh_n dh_n^* = \|f\|^2_{[\Gamma_k]},$$

where c_k is an analytic arc which Γ_k intersects orthogonally.

We remark that the mixed Dirichlet integral $(f-h_n, h_n)_{R_n'-K}$ vanishes. Actually, for any $\varepsilon>0$, $\|h_n\|_{R_n'-K} \leqq \|h_n \pm \varepsilon(f-h_n)\|_{R_n'-K}$. Hence $0 \leqq \pm 2\varepsilon(h_n, f-h_n)_{R_n'-K} + \varepsilon^2\|f-h_n\|^2_{R_n'-K}$, so that $0 \leqq \pm 2(h_n, f-h_n)_{R_n'-K} + \varepsilon\|f-h_n\|^2_{R_n'-K}$. By letting $\varepsilon \to 0$ we conclude $(f-h_n, h_n)_{R_n'-K}=0$.

If $m>n$, $h_m \in \mathscr{D}_{R_n'-K}(\varphi)$ and hence $(h_m-h_n, h_n)_{R_n'-K}=0$. Therefore, $0 \leqq \|h_m-h_n\|^2_{R_n'-K} = \|h_m\|^2_{R_n'-K} - \|h_n\|^2_{R_n'-K} \leqq \|h_m\|^2_{R_m'-K} - \|h_n\|^2_{R_n'-K}$. Since $\|h_n\|^2_{R_n'-K} \leqq \|f\|^2_{R'-K} < \infty$ for all n, $\|h_n\|^2_{R_n'-K}$ increases to a finite limit. Hence $\{h_n\}$ form a Cauchy sequence and h_n tends to a harmonic function h on $R'-K$ both in norm and locally uniformly, because $h_n=0$ on ∂K_0. In order to show that h takes the boundary values φ on ∂K, set $M = \max_{\partial K} |\varphi|$ and consider the harmonic

3) We can actually prove that no γ oscillates.

function h' (h'' resp.) in R'_1-K which takes the boundary values φ on ∂K, 0 on ∂K_0 and M ($-M$ resp.) on ∂R_1. Evidently $h''\leqq h_n\leqq h'$ in R'_1-K for each n and hence $h''\leqq h\leqq h'$ in R'_1-K. Since both h' and h'' take the boundary values φ on ∂K and 0 on ∂K_0, so does h. The relation $\|h\|\leqq\lim_{n\to\infty}\|h_n\|\leqq\|f\|<\infty$ shows that $h\in\mathscr{D}_{R'-K}(\varphi)$ and also that h has the minimum Dirichlet integral among the functions of $\mathscr{D}_{R'-K}(\varphi)$.

Finally, let us prove the uniqueness. As in the case of R_n we derive $(g-h, h)=0$ for any $g\in\mathscr{D}_{R'-K}(\varphi)$. If there is another extremal h', $(h'-h, h)=(h-h', h')=0$. Hence $\|h-h'\|=0$. Since $h-h'$ vanishes on ∂K_0, $h\equiv h'$.

COROLLARY. *$(g-h, h)=0$ for any $g\in\mathscr{D}_{R'-K}(\varphi)$.*

We shall denote h by φ_K. We note that the maximum principle holds for φ_K: $\min(\min_{\partial K}\varphi, 0)\leqq\varphi_K\leqq\max(\max_{\partial K}\varphi, 0)$ in $R'-K$. If φ is constantly 1 on ∂K, we can find $f\in\mathscr{D}_{R'-K}(1)$ easily. This special φ_K will be denoted by ω_K. Furthermore, we remark that φ_K is a linear functional of φ. Namely, if c is a constant and if $\mathscr{D}_{R'-K}(\varphi)\neq\emptyset$ and $\mathscr{D}_{R'-K}(\psi)\neq\emptyset$, then $(c\varphi)_K=c\varphi_K$ and $(\varphi+\psi)_K=\varphi_K+\psi_K$.

We give a property of φ_K.

THEOREM 2. *Let K, K' be regular compact sets in R' such that $K\subset K'$. Then $(\varphi_K)_{K'}=\varphi_K$.*

PROOF. The function f in $R'-K$ which is equal to $(\varphi_K)_{K'}$ in $R'-K'$ and to φ_K in $K'-K$ belongs to $\mathscr{D}_{R'-K}(\varphi)$. Hence $\|\varphi_K\|_{R'-K}\leqq\|f\|_{R'-K}$ by Theorem 1. This gives $\|\varphi_K\|_{R'-K'}\leqq\|(\varphi_K)_{K'}\|_{R'-K'}$. Since φ_K belongs to $\mathscr{D}_{R'-K'}(\varphi_K)$, $\varphi_K=(\varphi_K)_{K'}$ on account of the uniqueness of $(\varphi_K)_{K'}$.

§ 2. Definition of φ_K for general φ

So far we have defined φ_K only for φ which is continuous on ∂K and for which $\mathscr{D}_{R'-K}(\varphi)\neq\emptyset$. We shall denote by CD_K the class of such functions on ∂K. If $\varphi\in CD_K$, $\max(\varphi_K(P), 0)$ as a function on $R'-K$ is piecewise smooth Dirichlet finite and has the boundary values $\max_{\partial K}(\varphi, 0)$ on ∂K and 0 on ∂K_0. It is easy to observe that the points of ∂K are separated by functions of CD_K. By means of Stone's theorem (see [2], p. 54) we can infer that CD_K is a dense subclass, with respect to the uniform convergence, of the class C_K of continuous functions on ∂K. Since $|\varphi_K(P)-\psi_K(P)|\leqq\max|\varphi-\psi|$ for any φ, $\psi\in CD_K$ by the maximum principle, $\varphi_K(P)$ is uniquely defined for any $\varphi\in C_K$ even if $\mathscr{D}_{R'-K}(\varphi)=\emptyset$. It is a harmonic function of P in $R'-K$ and continuous on $(R'-K)\cup\partial K\cup\partial K_0$. For each fixed P, it is a positive linear functional on C_K.

Hence there is a Radon measure μ_K^P supported by K such that

$$\varphi_K(P)=\int_{\partial K}\varphi(Q)d\mu_K^P(Q) \tag{1}$$

for every $\varphi\in C_K$.

For an arbitrary $\mu_K^{P_0}$-measurable $\varphi\geqq 0$, we define $\varphi_K(P_0)$ by $\int\varphi d\mu_K^{P_0}$. Suppose that $\varphi_K(P_0)$ is finite and let D be the component of $R'-K$ which contains P_0. Let $\{\varphi_j\}$ be an increasing sequence of upper semicontinuous functions on ∂K such that $\int\varphi_j d\mu_K^{P_0}$ increases to $\int\varphi d\mu_K^{P_0}$, and $\{\psi_j\}$ be a decreasing sequence of lower semicontinuous functions on ∂K such that $\int\psi_j d\mu_K^{P_0}$ decreases to $\int\varphi d\mu_K^{P_0}$. We shall write $\lim\varphi_j=\underline{\varphi}$ and $\lim\psi_j=\bar{\varphi}$. Since both $\int\underline{\varphi}d\mu_K^P$ and $\int\bar{\varphi}d\mu_K^P$ are harmonic functions in D and coincide with each other at $P_0\in D$, they are identical in D. Since $\underline{\varphi}\leqq\varphi\leqq\bar{\varphi}$, $\underline{\varphi}=\bar{\varphi}$ μ_K^P-a.e. for every $P\in D$. This implies that φ is μ_K^P-measurable. Thus the μ_K^P-measurability does not depend on the choice of P in a fixed component of $R'-K$. Furthermore $\varphi_K(P)$ is harmonic in a component once it is finite at some point of the component.

In case $\int\varphi d\mu_K^{P_0}=\infty$ we express φ by $\lim_{M\to\infty}\min(\varphi, M)$. Since $\int d\mu_K^P<\infty$, $\min(\varphi, M)$ is μ_K^P-measurable for every $P\in D$ and so is φ.

It is easily seen that $\min(\inf_{\partial K}\varphi, 0)\leqq\varphi_K\leqq\max(\sup_{\partial K}\varphi, 0)$ holds in $R'-K$ and that $\varphi_{K_1\cup K_2}\leqq\varphi_{K_1}+\varphi_{K_2}$ for any compact sets K_1 and K_2 in R' and for any Borel measurable function $\varphi\geqq 0$ given on $\partial K_1\cup\partial K_2$.

We shall prove a theorem analogous to Theorem 2.

THEOREM 3. *If φ is non-negative μ_K^P-measurable for $P\in R'-K'$ and if $K\subset K'$, then φ_K is $\mu_{K'}^P$-measurable as a function on $\partial K'$ and $(\varphi_K)_{K'}(P)=\varphi_K(P)$.*

PROOF. First we shall show that this is true if φ is continuous. Certainly φ_K is $\mu_{K'}^P$-measurable. We approximate φ by $\varphi_j\in CD_K$ uniformly on ∂K. For each j, $((\varphi_j)_K)_{K'}=(\varphi_j)_K$ by Theorem 2. We obtain $(\varphi_K)_{K'}=\varphi_K$ by letting $j\to\infty$. Next if $\{\psi_j\}$ is monotone, if $(\psi_j)_K$ is $\mu_{K'}^P$-measurable as a function on $\partial K'$ and if $((\psi_j)_K)_{K'}=(\psi_j)_K$ for each j, then $(\lim\psi_j)_K$ is $\mu_{K'}^P$-measurable as a function on $\partial K'$ and $((\lim\psi_j)_K)_{K'}=(\lim\psi_j)_K$. Therefore, for $\underline{\varphi}$ and $\bar{\varphi}$ defined above, $\underline{\varphi}_K$ and $\bar{\varphi}_K$ are $\mu_{K'}^P$-measurable, $(\underline{\varphi}_K)_{K'}(P)=\underline{\varphi}_K(P)$ and $(\bar{\varphi}_K)_{K'}(P)=\bar{\varphi}_K(P)$. It holds that $\int\underline{\varphi}_K d\mu_{K'}^P=\int\bar{\varphi}_K d\mu_{K'}^P$, which implies $\underline{\varphi}_K=\bar{\varphi}_K$ $\mu_{K'}^P$-a.e. Since $\underline{\varphi}_K\leqq$

$\underline{\varphi}_K \leqq \bar{\varphi}_K$ everywhere, φ_K is $\mu^P_{K'}$-measurable on $\partial K'$. Furthermore, $(\varphi_K)_{K'}(P)=\varphi_K(P)$ because $\underline{\varphi}_K(P)=\bar{\varphi}_K(P)=\varphi_K(P)$ and $\underline{\varphi}_K(P)=(\underline{\varphi}_K)_{K'}(P)\leqq(\varphi_K)_{K'}(P)\leqq(\bar{\varphi}_K)_{K'}(P)=\bar{\varphi}_K(P)$.

§ 3. Function N

Let Q be a point in R' and $\{R_n\}$ be an exhaustion such that $Q\in R_1$. Let $N_n(P, Q)$ be the positive harmonic function in $R'_n-\{Q\}$, which vanishes on ∂K_0, has a vanishing normal derivative on ∂R_n and has a logarithmic singularity with coefficient 1 at Q. We can show its existence by considering the double of R'_n.

Before proving a theorem we state a general remark. If a harmonic function h is defined in a ring subdomain D of R' partly bounded by ∂K_0 and if h has the boundary value 0 on ∂K_0, then h is harmonic on ∂K_0 and in a ring domain which is the reflexion of D along ∂K_0 on account of the symmetry principle.

We prove first

THEOREM 4. *$N_n(P, Q)$ converges to a function $N(P, Q)$ locally uniformly on $R-(K_0-\partial K_0)-\{Q\}$ and $\|N_n-N\|_{R'_n}$ tends to zero as $n\to\infty$. The function $N(P, Q)$ has a logarithmic singularity with coefficient* 1 *at Q and vanishes on ∂K_0.*

PROOF. Let z be a fixed local variable at Q such that $z(Q)=0$ and $|z|<1$ corresponds to a parametric disk in R'. We set $h_n(z)=N_n(P(z), Q)+\log|z|$. It is defined at $z=0$ so as to be harmonic there. We denote $h_n(0)$ by γ_n and have

$$\gamma_n=\lim_{P\to Q}\{N_n(P, Q)+\log|z(P)|\}.$$

We denote by D_r the image on R of the disk $|z|\leqq r<1$. By the aid of Green's formula it holds that

$$\|N_n\|^2_{R'_n-D_r}=\int_{\partial D_r} N_n \frac{\partial N_n}{\partial \nu}\, ds$$

and

$$\lim_{r\to 0}(\|N_n\|^2_{R'_n-D_r}+2\pi\log r)=2\pi\gamma_n.$$

Let $m>n$. It holds that

$$(N_n, N_m)_{R'_n - D_r} = \int_{\partial D_r} N_m \frac{\partial N_n}{\partial \nu} ds$$

and hence

$$\lim_{r \to 0} \{(N_n, N_m)_{R'_n - D_r} + 2\pi \log r\} = \lim_{r \to 0} \int_{\partial D_r} h_m \frac{\partial N_n}{\partial \nu} ds = 2\pi \gamma_m.$$

Therefore

$$0 \leqq \|N_n - N_m\|^2_{R'_n} = \lim_{r \to 0} \|N_n - N_m\|^2_{R'_n - D_r} = \lim_{r \to 0} [(\|N_n\|^2_{R'_n - D_r} + 2\pi \log r)$$
$$+ (\|N_m\|^2_{R'_n - D_r} + 2\pi \log r) - 2\{(N_n, N_m)_{R'_n - D_r} + 2\pi \log r\}]$$
$$\leqq 2\pi\gamma_n + 2\pi\gamma_m - 4\pi\gamma_m = 2\pi(\gamma_n - \gamma_m).$$

Accordingly, γ_n decreases as $n \to \infty$.

We want to see that γ_n does not tend to $-\infty$ as $n \to \infty$. The function $h_n(z) = N_n(P(z), Q) + \log|z|$ is a harmonic function of z on $|z| \leqq r$ and it is $\geqq \log r$ on $|z| = r$. By the minimum principle, it is $\geqq \log r$ on $|z| \leqq r$. Consequently, $\gamma_n = h_n(0) \geqq \log r > -\infty$. Now we have $\lim_{n,m \to \infty} \|N_n - N_m\|_{R'_n} = 0$. Fix k. Since $N_n - N_k$ vanishes on ∂K_0 and is harmonic in a neighborhood of ∂K_0, $N_n - N_k$ converges both in norm and uniformly on any compact set in $R_k - (K_0 - \partial K_0)$ as $n \to \infty$. Consequently N_n converges uniformly on any compact set in $R - (K_0 - \partial K_0) - \{Q\}$. We shall denote $\lim_n N_n$ by N. This function has a logarithmic singularity at Q and $N_n - N$ is harmonic in R'_n. Since

$$\|(N_n - N) - (N_m - N)\|_{R'_n} = \|N_n - N_m\|_{R'_n} \to 0 \text{ if } n < m \text{ and } n \to \infty,$$

we conlcude that $N_n - N$ tends to a harmonic function H in R' such that $\|(N_n - N) - H\|_{R'_n} \to 0$ as $n \to \infty$. Since $N = \lim_n N_n$, $H \equiv 0$. Thus $\|N_n - N\|_{R'_n} \to 0$ as $n \to \infty$. Our theorem is now completely proved.

Our function $N(P, Q)$ does not depend on the choice of exhaustion. As a function of P it is harmonic in $R' - \{Q\}$, vanishes on ∂K_0 and has a logarithmic singularity at Q. Outside any neighborhood of Q it has a finite Dirichlet integral. Since $N_n(P, Q) = N_n(Q, P)$, $N(P, Q) = N(Q, P)$.[4]

Let us see that it is a continuous function in the extended sense (i.e. admitting ∞) on $(R - (K_0 - \partial K_0)) \times (R - (K_0 - \partial K_0))$. Let P_0, $Q_0 \in R - (K_0 - \partial K_0)$ and $P_0 \neq Q_0$. Let $a > 1$ be given. By making use of Poisson's formula we observe that there is a neighborhood D of P_0 such that

4) N is denoted by $\tilde{g}$ in [3].

$$a^{-1}N(P_0, Q) \leqq N(P, Q) \leqq aN(P_0, Q) \qquad \text{for any } P \in D$$

so far as Q is kept away from P_0. Since $N(P_0, Q)$ is a continuous function of Q, we can find a neighborhood D' of Q_0 such that $N(P_0, Q)$ is close to $N(P_0, Q_0)$ if $Q \in D'$. Consequently, $N(P, Q)$ is close to $N(P_0, Q_0)$ if $P \in D$ and $Q \in D'$.

Next we consider the case when $P_0 = Q_0$. Let D_0 be a disk on which a local variable z is defined such that $z(P_0)=0$. The function $h(P, Q)=N(P, Q)+\log|z(P)-z(Q)|$ is a harmonic function of each variable while the other is fixed. In the same way as above we can show that $h(P, Q)$ is a continuous function of (P, Q) on $D_0 \times D_0$. Since it is finite-valued, it is bounded if both P and Q are restricted to some neighborood of P_0. We infer that $N(P, Q)= h(P, Q)-\log|z(P)-z(Q)|$ tends to ∞ as both P and Q approach P_0. Thus $N(P, Q)$ is a continuous function of (P, Q) in the extended sense.

Now we may take $N(P, Q)$ as a kernel of potential. For any non-negative measure μ in R such that $\mu(K_0)=0$ we can define the potential $\int N(P, Q)d\mu(Q)$. We shall write it as $N\mu(P)$ too. When we consider a potential $N\mu$, we always assume that $\mu(K_0)=0$ and that $N\mu$ is not identically equal to ∞. It is a superharmonic function in R'. We shall use the terminology that *a measure μ is on* a Borel set B if $\mu(B')=0$ for any Borel set B' disjoint from B. Thus we shall consider measures on R'; later we shall consider also measures on the Kuramochi boundary.

We shall prove

THEOREM 5. *For any regular compact set $K \subset R'$, $(N\mu)_K \leqq N\mu$ in $R'-K$. The equality holds if S_μ is included in the interior K^i of K.*[5)]

PROOF. First we prove $N_K = N$ in the case that the pole Q is an inner point of K. We know that N_K is equal to the limit of the following harmonic function h_n in R'_n-K: h_n is equal to N on $\partial K \cup \partial K_0$ and $\partial h_n/\partial\nu=0$ on ∂R_n. Since $N_n \to N$ uniformly on ∂K, $h_n - N_n \to 0$ in $R'-K$ as $n \to \infty$. Consequently, $N_K = \lim_n h_n = \lim_n N_n = N$ in $R'-K$.

In case $Q_0 \in R'-K$, $N(Q, Q_0)$ is a continuous function of Q on ∂K. Hence N_K is continuous on ∂K and $N-N_K=0$ there. Since $N-N_K$ is superharmonic in $R'-K$, we can show $N \geqq N_K$ by the same reasoning as in the first case. If $Q_0 \in \partial K$, we approximate $N(Q, Q_0)$ by an increasing sequence $\{f_j\}$ of continuous functions of Q on ∂K. By definition $N_K = \lim_{j\to\infty}(f_j)_K$. We infer $(f_j)_K \leqq N$ for each j in $R'-K$ and derive $N_K \leqq N$ there.

If μ is a measure with $S_\mu \subset K^i$,

5) Actually we can prove the equality if $S_\mu \subset K$ but omit the proof.

$$(N\mu)_K(P) = \iint N(Q, Q')d\mu(Q')d\mu_K^P(Q) = \iint N(Q, Q')d\mu_K^P(Q)d\mu(Q')$$

$$= \int N(P, Q')d\mu(Q') = N\mu(P).$$

In the general case we infer $(N\mu)_K(P) \leqq N\mu(P)$ by making use of the inequality $N_K \leqq N$.

§ 4. HS functions and SHS functions

First we define an SHS function. Let $V(P)$ be a positive lower semi-continuous function in R' which is not identically equal to ∞. If $V_K(P) \leqq V(P)$ in $R' - K$ for any regular compact set K in R', then $V(P)$ is called an *SHS function*[6] in R'. It follows that $V(P)$ is superharmonic in R'. If $V(P)$ is harmonic in R', it is called an *HS function.* Next let $V(P)$ be an SHS function in R' and $\{K_m\}$ be a sequence of concentric closed parametric disks in R strictly decreasing to K_0. If $V_{\partial K_m}(P)$ tends to zero as $m \to \infty$ in R', $V(P)$ will be called an *SHS_0 function.*[7] If, in addition, $V(P)$ is harmonic in R', it will be called an *HS_0 function.* If $\{V_n\}$ is a decreasing sequence of SHS (SHS_0 resp.) function and if the limiting function is lower semicontinuous, then it is an SHS (SHS_0 resp.) function. If $\{V_n\}$ is an increasing sequence of SHS (SHS_0 resp.) functions and the limiting function V is not identically equal to ∞ (is dominated by an SHS_0 function resp.), then V is an SHS (SHS_0 resp.) function.

First we prove

THEOREM 6. *Every potential $N\mu$ is an SHS_0 function.*

PROOF. On account of Theorem 5 it satisfies the inequality $(N\mu)_K \leqq N\mu$. Let $\{K_m\}$ be a sequence taken as above. We fix any point $P_0 \in R'$. Denoting by μ_m the restriction of μ to K_m, if m_0 is large, $N\mu_{m_0}(P_0)$ is smaller than any given $\varepsilon > 0$ because $\mu(K_0) = 0$. Naturally $(N\mu_{m_0})_{\partial K_m}(P_0) \leqq N\mu_{m_0}(P_0) < \varepsilon$ for any m. On the other hand, since $N(\mu - \mu_{m_0})$ has the vanishing boundary value on ∂K_0, $(N(\mu - \mu_{m_0}))_{\partial K_m}(P_0) \downarrow 0$ as $m \to \infty$. Thus $\lim_{m\to\infty} (N\mu)_{\partial K_m} < \varepsilon$ and hence $= 0$.

We shall use the following well-known facts in the theory of logarithmic potentials in the plane.

LEMMA 2. *If $v_1(z)$ and $v_2(z)$ are the logarithmic potentials of measures μ*

---

6) This is called $\overline{\text{s}}$uperharmonic by Kuramochi and "positiv vollsuperharmonisch" in [3].

7) This is called a function of potential type in [3].

and ν respectively and $v_1(z)=v_2(z)+a$ harmonic function in a domain D, then $\mu(B)=\nu(B)$ for every Borel subset B of D.

LEMMA 3. (*Riesz decomposition theorem*). *Any superharmonic function in a plane domain D is equal to the sum of a harmonic function in D and the logarithmic potential of a measure on D.*

Consequently, for any superharmonic function $V(P)$ in an open set $G\subset R'$, we can speak of the measure which gives locally and hence globally in G the potential part in the Riesz decomposition of V with respect to the kernel N.

We shall establish

THEOREM 7. *If $V(P)$ is an SHS function in R' and K is a regular compact set in R', then the function $v(P)$ equal to $V_K(P)$ in $R'-K$ and to $V(P)$ on K is equal to the potential of some measure supported by K.*

PROOF. Since $V_K(P)\leqq V(P)$ in $R'-K$, $v(P)$ has the mean value property (i.e. $v(P)\geqq$ the mean value of v on any sufficiently small disk around P) on ∂K and hence in R'. In order to show the lower semicontinuity of $v(P)$, we approximate it from below by an increasing sequence $\{f_j\}$ of continuous functions on ∂K. We extend $(f_j)_K$ to a function on R' by setting it equal to V in $K-\partial K$. This extension is lower semicontinuous in R' and increases to $v(P)$ in R' as $j\to\infty$. Thus $v(P)$ is lower semicontinuous and hence $v(P)$ is superharmonic in R'.

Let μ be the measure which gives the potential part in the Riesz decomposition of $v(P)$. On account of Lemma 2 it is supported by K. The function $v(P)-N\mu(P)$ is harmonic in R'. Let K_1 be a regular compact set in R' containing K in its interior. From Theorems 3 and 5 it follows that

$$(v-N\mu)_{K_1}=v-N\mu \qquad \text{in } R'-K_1.$$

Consequently

$$\sup_{R'-K_1}|v-N\mu|=\max_{\partial K_1}|v-N\mu|.$$

Thus $\max\limits_{R'}|v-N\mu|$ is attained on ∂K_1. By the maximum principle $v-N\mu$ must be constant in R'. Since it vanishes on ∂K_0, $v\equiv N\mu$ in R'.

In virtue of Theorem 6 we have

COROLLARY. *V_K extended by V is equal to an SHS_0 function.*

We approximate R' by an increasing sequence $\{D_n\}$ of regular subdomains such that $D_n\cup\partial D_n\subset R'$. By the preceding theorem, on $D_n\cup\partial D_n$, $V(P)$ is equal to the potential of a measure μ_n supported by $D_n \cup \partial D_n$. We denote by μ_n' the restriction of μ_n to D_n and put $\mu_n'' = \mu_n - \mu_n'$. It holds that

$$(2) \qquad V = N\mu_n' + N\mu_n'' \qquad \text{on } D_n \cup \partial D_n.$$

We observe that μ_n' is identical to the measure which gives the potential part in the Riesz decomposition of $V(P)$ in D_n. As $n \to \infty$ $N\mu_n'$ increases and $N\mu_n''$ decreases to a harmonic function in R'. Consequently the function $U = \lim_{n\to\infty} N\mu_n''$ is an HS function in R. Thus we have the following decomposition theorem in R'.

THEOREM 8. *Every SHS function is equal to the sum of an HS function and a potential.*

§ 5. Definition of V_F for an SHS function V

We have considered φ_K for a regular compact set $K \subset R'$ and a μ_K^P-measurable function φ on ∂K. In this section, by F or by F' we shall mean a closed subset of R' whose boundary consists of a countable number of analytic curves clustering nowhere in R'. For an SHS function V in R', we define V_F by $\lim_{n\to\infty} V_{F_n}$ in $R' - F$, where F_n is defined by $F \cap (R_n \cup \partial R_n)$. The increasing limit exists because

$$(V_{F_n})_{F_m} = V_{F_n} \leqq V_{F_m} \leqq V \qquad \text{if } n < m$$

by Theorem 3. In the following we shall denote the extensions to R' by V of V_{F_n} and V_F again by V_{F_n} and V_F respectively. By the corollary of Theorem 7, V_{F_n} is an SHS_0 function in R'. We already noted that the limit of any increasing sequence of SHS functions is an SHS function if the limiting function is not identically equal to ∞. Consequently, V_F is an SHS function in R'. Furthermore, it is dominated in a ring domain $D \subset R'$, partially bounded by ∂K_0 and disjoint from F, by $V_{R'-D}$ which has the vanishing boundary value on ∂K_0. Therefore V_F is an SHS_0 function in R'. We remark that $(U+V)_F = U_F + V_F$ if U and V are SHS functions.

We shall prove an analogue of Theorem 3.

THEOREM 9. *Let $F \subset F'$, and V be an SHS function in R'. Then $(V_F)_{F'} = V_F$ in R'.*

PROOF. Let $n < m$. Since $F_n \subset F_m'$, $(V_{F_n})_{F_m'} = V_{F_n}$ in R' by Theorem 3. By definition, $V_{F_n} \leqq V_F$ in R'. Therefore, $V_{F_n} \leqq (V_F)_{F_m'}$ in R'. By letting $m \to \infty$

first and then $n\to\infty$, we obtain $V_F\leqq(V_F)_{F'}$ in R'. Conversely, since V_F is an SHS_0 function, $(V_F)_{F'}\leqq V_F$ in R'. Thus we obtain the equality.

Next we prove

THEOREM 10. *Let V be a piecewise smooth Dirichlet finite SHS function in R'. Then $\|V_{F_n}-V_F\|_{R'-F_n}$ tends to zero as $n\to\infty$.*

PROOF. Let $m>n$. By Theorem 1, $(V_{F_m}-V_{F_n}, V_{F_n})_{R'-F_n}=0$ and hence

$$0\leqq\|V_{F_m}-V_{F_n}\|^2_{R'-F_n}=\|V_{F_m}\|^2_{R'-F_m}+\|V\|^2_{F_m-F_n}-\|V_{F_n}\|^2_{R'-F_n}.$$

Therefore

$$\|V_{F_n}\|^2_{R'-F_n}\leqq\varliminf_{m\to\infty}\|V_{F_m}\|^2_{R'-F_m}+\|V\|^2_{F-F_n}.$$

We let $n\to\infty$ and have $\varlimsup_{n\to\infty}\|V_{F_n}\|^2_{R'-F_n}\leqq\varliminf_{n\to\infty}\|V_{F_n}\|^2_{R'-F_n}$. The existence of $\lim_{n\to\infty}\|V_{F_n}\|_{R'-F_n}$ is inferred and it follows that $\{V_{F_n}\}$ form a Cauchy sequence. The pointwise convergence $\lim_{n\to\infty}V_{F_n}=V_F$ being known, we conclude that $\|V_{F_n}-V_F\|_{R'-F_n}\to 0$ as $n\to\infty$.

In virtue of this theorem it is rather easy to prove

THEOREM 11. *Let V be a piecewise smooth Dirichlet finite SHS function in R', and f be a piecewise smooth Dirichlet finite function in R' which takes the values V on F and 0 on ∂K_0. Then $(f-V_F, V_F)_{R'-F}=0$, and V_F is the unique function which gives the smallest norm among the functions like f.*

PROOF. We apply Theorem 1 and obtain $(f-V_{F_n}, V_{F_n})_{R'-F_n}=0$. On account of Theorem 10 we conclude $(f-V_F, V_F)_{R'-F}=0$. The proof can be completed in the customary way.

COROLLARY 1. *Let $F\subset F'$. For the above V,*

$$\|V_{F'}-V_F\|^2_{R'-F}=\|V_{F'}\|^2_{R'-F'}-\|V_F\|^2_{R'-F}+\|V\|^2_{F'-F}.$$

COROLLARY 2. *Suppose that the above V is bounded. The function V_n which is harmonic in R'_n-F, which takes the boundary values V on $\partial F\cap R_n$ and 0 on ∂K_0 and which has the vanishing normal derivative on ∂R_n-F, converges to V_F as $n\to\infty$.*

For, extracting a subsequence $\{V_{n_k}\}$ converging to V_0 in $R'-F$, we have by Fatou's lemma

$$\|V_0\| \leqq \varliminf_{k\to\infty} \|V_{n_k}\|_{R'_{n_k}-F} \leqq \|V_F\|.$$

As in the proof of Theorem 1 we can show that V_0 takes the same boundary values as V on $\partial F \cup \partial K_0$. Therefore $V_0 = V_F$ by the above theorem. It follows that $\lim_{n\to\infty} V_n = V_F$ because every convergent subsequence of $\{V_n\}$ converges to V_F.

§ 6. Definition of Kuramochi boundary

We observed already that $N_n(P, Q)$ tends to $N(P, Q)$ uniformly in a neighborhood of ∂K_0 for any fixed $Q \in R'$ on account of the symmetry principle. Therefore

$$\int_{\partial K_0} \frac{\partial N}{\partial \nu} ds = \int_{\partial K_0} \lim_{n\to\infty} \frac{\partial N_n}{\partial \nu} ds = \lim_{n\to\infty} \int_{\partial K_0} \frac{\partial N_n}{\partial \nu} ds = 2\pi.$$

Let $\{Q_j\}$ be a sequence of points tending to the boundary of R. We can see that no subsequence of $\{N(P, Q_j)\}$ tends to the constant ∞ as $j\to\infty$. Actually, let D be a ring subdomain of R' whose one boundary component is ∂K_0, and h be the harmonic measure function of $C = \partial D - \partial K_0$. It holds that $\min \partial h/\partial \nu > 0$ on ∂K_0 and $N(P) \geqq (\min_C N) h(P)$ in D. If $N(P) \to \infty$ in R', then $\partial N/\partial \nu \geqq \min_C N \cdot \min_{\partial K_0} \partial h/\partial \nu \to \infty$ uniformly on ∂K_0. This contradicts the relation $\int_{\partial K_0} \partial N/\partial \nu ds = 2\pi$. Consequently, $\{N(P, Q_j)\}$ form a normal family and every converging subsequence converges uniformly on any compact subset of $R - (K_0 - \partial K_0)$.

If $N(P, Q_j)$ converges, $\{Q_j\}$ will be called a *fundamental sequence*. If the limiting functions of two converging sequences $\{N(P, Q_j)\}$ and $\{N(P, Q'_j)\}$ are equal to each other, we say that $\{Q_j\}$ and $\{Q'_j\}$ are equivalent and call an equivalence class a *Kuramochi boundary point*. We call the set of all Kuramochi boundary points the *Kuramochi boundary* of R and denote it by Δ_N. If $P \in R'$, $Q \in \Delta_N$ and $\{Q_j\}$ in R' determines Q, then we set

$$N(P, Q) = \lim_{j\to\infty} N(P, Q_j);$$

this value does not depend on the choice of fundamental sequence. We note that $N(P, Q) = 0$ for $P \in \partial K_0$ and $\int_{\partial K_0} \partial N/\partial \nu ds = 2\pi$. We introduce a metric on $R' \cup \Delta_N$ by

$$d(Q_1, Q_2)=\sup_{P\in R_1'}\left|\frac{N(P, Q_1)}{1+N(P, Q_1)}-\frac{N(P, Q_2)}{1+N(P, Q_2)}\right|$$

for any Q_1, Q_2 on $R'\cup\varDelta_N$, where we recall that $R_1'=R_1-K_0$. The topology induced by this metric on R' coincides with the original topology. With this metric, $(R-R_1)\cup\varDelta_N$ is a compact metric space because $\{N(P, Q_j)\}$ form a normal family for any $\{Q_j\}$ tending to $\varDelta_N$. The space $R'\cup\partial K_0\cup\varDelta_N$ has a countable base of open sets. It is easy to prove that, for any compact set $K_1\subset R'$ and any compact set $K_2\subset R'\cup\varDelta_N$ disjoint from K_1, $N(P, Q)$ is continuous as a function on $K_1\times K_2$.

Next we prove

THEOREM 12. *The definition of boundary points of R does not depend on the choice of K_0 in the sense that every equivalence class of sequences of points near the boundary of R is the same.*

PROOF. It will be sufficient to consider two closed parametric disks K_0 and $\tilde{K}_0$ such that $K_0\subset\tilde{K}_0-\partial\tilde{K}_0$. Let $\{Q_j\}$ be a fundamental sequence with respect to K_0 converging to $Q\in\varDelta_N$. We choose $\{R_n\}$ such that $R_1\supset\tilde{K}_0$. We consider $\tilde{N}_n(P, Q_j)$ and $\tilde{N}(P, Q_j)$ defined in $\tilde{R}_n'=R_n-\tilde{K}_0$ and $\tilde{R}'=R-\tilde{K}_0$ respectively. We set

$$H_n(P, Q_j)=N_n(P, Q_j)-\tilde{N}_n(P, Q_j)$$

and

$$H(P, Q_j)=N(P, Q_j)-\tilde{N}(P, Q_j).$$

Since N_n and $\tilde{N}_n$ tend to N and $\tilde{N}$ respectively as $n\to\infty$, $H_n(P, Q_j)$ tends to $H(P, Q_j)$. We want to verify that $\lim H(P, Q_j)$ exists as $j\to\infty$. We have

$$\begin{aligned}(3)\qquad |H(P, Q_j)-H(P, Q_k)| &= \lim_{n\to\infty}|H_n(P, Q_j)-H_n(P, Q_k)|\\ &\leqq \lim_{n\to\infty}\max_{P'\in\partial\tilde{K}_0}|H_n(P', Q_j)-H_n(P', Q_k)|\\ &= \lim_{n\to\infty}\max_{P'\in\partial\tilde{K}_0}|N_n(P', Q_j)-N_n(P', Q_k)|\\ &= \max_{P'\in\partial\tilde{K}_0}|N(P', Q_j)-N(P', Q_k)|\end{aligned}$$

on $\tilde{R}'$. Since the last side is small if j and k are large, the convergence of $H(P, Q_j)$ is concluded. Hence $\tilde{N}(P, Q_j)$ tends to a harmonic function in $\tilde{R}'$ as

$j \to \infty$. It implies that if $\{Q_j\}$ and $\{Q'_j\}$ are equivalent fundamental sequences with respect to K_0, they are so with respect to $\tilde{K}_0$.

Next we shall prove the converse. Let $\{Q_j\}$ and $\{Q'_j\}$ be sequences which determine different boundary points Q and Q' with respect to K_0 but the same boundary point $\tilde{Q}$ with respect to $\tilde{K}_0$. Since $|\tilde{N}(P, Q_j) - \tilde{N}(P, Q'_j)|$ tend to 0 on $\tilde{R}'$ as $j \to \infty$, it follows from (3) that

$$|N(P, Q) - N(P, Q')| \leqq \max_{P' \in \partial \tilde{K}_0} |N(P', Q) - N(P', Q')|$$

on $\tilde{R}'$. On the other hand it is evident that

$$\sup_{\tilde{K}_0 - K_0} |N(P, Q) - N(P, Q')| \leqq \max_{P' \in \partial \tilde{K}_0} |N(P', Q) - N(P', Q')|.$$

Thus $|N(P, Q) - N(P, Q')|$ takes its maximum at an interior point of R'. Therefore it is a constant in R'. Now it follows that $N(P, Q) \equiv N(P, Q')$.

§ 7. Integral representation of $\mathbf{HS}_0$ and $\mathbf{SHS}_0$ functions

First we prove

LEMMA 4. *Every HS_0 function $U(P)$ takes the vanishing boundary value on ∂K_0.*

PROOF. We consider a sequence $\{K_m\}$ of concentric closed parametric disks strictly decreasing to K_0. By Theorem 7 U is equal on $K_1 - (K_m - \partial K_m)$ $(m \geqq 2)$ to the potential of a measure μ_m supported by $\partial K_1 \cup \partial K_m$. We denote the restrictions of μ_m to ∂K_1 and ∂K_m by μ'_m and μ''_m respectively. If $m < m'$, Theorem 5 yields

$$N\mu''_m = (N\mu''_m)_{K_{m-1} - (K_{m'} - \partial K_{m'})} = (N\mu''_m)_{\partial K_{m-1}} \leqq U_{\partial K_{m-1}} \qquad \text{on } R - K_{m-1}.$$

Since U is an HS_0 function, $U_{\partial K_{m-1}}$ tends to zero as $m \to \infty$. Hence $\lim_{m \to \infty} N\mu''_m = 0$ and $U = \lim_{m \to \infty} N\mu'_m$ in $K_1 - K_0$. The total mass of μ'_m, $m = 1, 2, \cdots$, is bounded because for any fixed $P \in K_1 - K_2$, $\min_{Q \in \partial K_1} N(P, Q)\mu'_m(R') \leqq N\mu'_m(P) \leqq U(P) < \infty$. Therefore we can choose a subsequence of $\{\mu'_m\}$ converging vaguely to a measure on ∂K_1. Then U is equal in $K_1 - K_0$ to the potential of the measure. This shows that U has the boundary value zero on ∂K_0.

THEOREM 13. *Every SHS_0 function $V(P)$ in R' can be expressed by the potential of a measure on $\partial R' \cup \Delta_N$, and vice versa.*

PROOF. By Theorem 8 V is equal to $U+N\mu$, where U is an HS function. By Theorem 6 $N\mu$ is an SHS_0 function and hence U is an HS_0 function. It suffices to prove our theorem for U. Consider an exhaustion $\{R_n\}$ of R. On account of the above lemma $U_{\partial R_n}$ is equal to U on R_n'. We denote by μ_n the measure on ∂R_n which gives $N\mu_n = U_{\partial R_n}$. The total mass of μ_n is equal to $(2\pi)^{-1}\int \partial N\mu_n/\partial\nu ds$ where the integral is taken along $\partial K_0 \cup \partial R_p$ for any $p>n$. Since

$$\int_{\partial R_p} \frac{\partial N\mu_n}{\partial\nu} ds = \lim_{q\to\infty}\int_{\partial R_p} \frac{\partial N_q\mu_n}{\partial\nu} ds = \lim_{q\to\infty}\int_{\partial R_q} \frac{\partial N_q\mu_n}{\partial\nu} ds = 0,$$

the mass is equal to

$$\frac{1}{2\pi}\int_{\partial K_0} \frac{\partial N\mu_n}{\partial\nu} ds = \frac{1}{2\pi}\int_{\partial K_0} \frac{\partial U}{\partial\nu} ds < \infty.$$

We extract a subsequence of $\{\mu_n\}$ vaguely convergent to a measure μ on Δ_N. The equality $U=N\mu$ follows.

To prove the converse, we shall show first that, if $Q\in\Delta_N$, then $(N(\cdot, Q))_K(P) \leqq N(P, Q)$ for any regular compact set $K\subset R'$. Let $Q_j\in R'-K$ tend to Q. Since $(N(\cdot, Q_j))_K(P)\leqq N(P, Q_j)$ and both sides tend to $(N(\cdot, Q))_K$ and $N(P, Q)$ respectively, the required inequality follows. Next we consider any μ with $S_\mu\subset\Delta_N$. Certainly $N\mu$ has the boundary value 0 on ∂K_0 and

$$(N\mu)_K(P) = \int N\mu(Q)d\mu_K^P(Q) = \iint N(Q, Q')d\mu_K^P(Q)d\mu(Q')$$

$$\leqq \int N(P, Q')d\mu(Q') = N\mu(P).$$

Thus $N\mu$ is an SHS_0 function.

Let F be a closed set with analytic boundary in R' as considered in §5, and let $V(P)$ be an SHS function in R'. We shall prove

THEOREM 14. *Denote the closure of F in $R'\cup\Delta_N$ by F^a. There esists μ with $S_\mu\subset F^a$ such that $N\mu = V_F$ in R'.*

PROOF. Let $F_n = F\cap(R_n\cup\partial R_n)$. By Theorem 7 there is μ_n supported by F_n such that $N\mu_n = V_{F_n}$ in R'. The total mass of μ_n is equal to

$$\frac{1}{2\pi}\int_{\partial K_0} \frac{\partial N\mu_n}{\partial\nu} ds = \frac{1}{2\pi}\int_{\partial K_0} \frac{\partial V_{F_n}}{\partial\nu} ds \leqq \frac{1}{2\pi}\int_{\partial K_0} \frac{\partial V_{\partial K}}{\partial\nu} ds < \infty,$$

where K is a closed parametric disk which contains K_0 in its inside and is disjoint from F. There is a subsequence $\{\mu_{n_k}\}$ converging vaguely to a measure μ supported by F^a, because F^a is a compact set in $R' \cup \varDelta_N$. Since $N(P, Q)$ is a continuous function of $Q \in R' \cup \varDelta_N - \{P\}$ for any fixed $P \in R'$, $V_{F_{n_k}}(P) = N\mu_{n_k}(P)$ tends to $N\mu(P)$ for any $P \in R' - F$. Consequently,

$$V_F(P) = \lim_{n\to\infty} V_{F_n}(P) = N\mu(P).$$

We recall that the restriction of μ_n to the interior F_n^i is equal to the measure which gives the potential part of the Riesz decomposition of V in F_n^i. We shall denote it by ν_n. Hence the restriction ν of μ to F^i has the same property and ν_n increases to ν as $n \to \infty$. It follows that $\mu_{n_k} - \nu_{n_k}$ converges vaguely to $\mu - \nu$. At any $P \in F^i$

$$\int N d\mu = \int N d\nu + \int N d(\mu - \nu) = \lim_{k\to\infty} \int N d\nu_{n_k} + \lim_{k\to\infty} \int N d(\mu_{n_k} - \nu_{n_k})$$

$$= \lim_{n\to\infty} \int N d\mu_n = V.$$

What remains is to prove $N\mu(P) = V_F(P)$ for every $P \in \partial F$. Since the function V_F is superharmonic in R', its mean value on a disk around P tends to $V_F(P)$ as the disk diminishes. The mean value of V_F equals the mean value of $N\mu$ and hence $V_F(P) = N\mu(P)$.

Another theorem concerning V_F is

THEOREM 15. *Let μ be a measure on $R' \cup \varDelta_N$, and F be as above. Then*

$$(N\mu)_F = N_F\mu \qquad \text{in } R'.$$

PROOF. If K is a regular compact set in R',

$$(N\mu)_K = \iint N d\mu d\mu_K = \iint N d\mu_K d\mu = N_K\mu.$$

Therefore

$$(N\mu)_F = \lim_{n\to\infty} (N\mu)_{F_n} = \lim_{n\to\infty} N_{F_n}\mu = N_F\mu.$$

Let A be a closed subset of $\varDelta_N$. Define

$$A(m) = \left\{P \in R';\ d(P, A) \leqq \frac{1}{m}\right\}.$$

We cover $\partial A(m)$ by a countable number of closed disks $D_1, D_2, \cdots$ with centers on $\partial A(m)$ such that each D_j has a positive distance from $\partial A(m-1)$ and no compact set in R' intersects an infinite number of D_j's. We set $A'(m)=A(m)\cup(\cup_j D_j)$. Next we consider the harmonic measure of $\partial A'(m)$ with respect to the open set $A'(m-1)-A'(m)-\partial A'(m-1)$. There exists a level curve $h=\varepsilon$, $0<\varepsilon<1$, consisting of a countable number of analytic curves. We set

$$A^{(m)}=A'(m)\cup\{P;\, h(P)\geqq\varepsilon\}.$$

This is a closed set with analytic boundary in R' and its closure in $R'\cup\Delta_N$ is a neighborhood of A.

For an SHS function V in R' we consider $V_{A^{(m)}}$. This decreases as $m\to\infty$ to an HS_0 function. We shall denote the limit by V_A. We note that $V_{\Delta_n}=V$ for any HS_0 function V. Let us prove

THEOREM 16. *Let V be an SHS function in R' and A be a closed subset of Δ_N. Then there exists μ supported by A such that*

$$V_A(P)=\int_A N(P,Q)d\mu(Q) \qquad on\ \ R'.$$

PROOF. Take above $\{A^{(m)}\}$. By Theorem 14 there is a measure μ_m supported by $(A^{(m)})^a$ and satisfying $N\mu_m=V_{A^{(m)}}$ in R'. Its total mass is not greater than $(2\pi)^{-1}\int_{\partial K_0}\partial V_{A^{(1)}}/\partial\nu ds$. We extract a vaguely convergent subsequence of $\{\mu_m\}$ and denote by μ the vague limit. We have

$$V_A(P)=\lim_{m\to\infty}V_{A^{(m)}}(P)=N\mu(P) \qquad \text{for any}\ \ P\in R'.$$

Finally we prove

THEOREM 17. *Let μ be a measure on $R'\cup\Delta_N$, and A be a closed subset of Δ_N. Then*

$$(N\mu)_A=N_A\mu.$$

PROOF. Take $\{A^{(m)}\}$. By Theorem 15

$$(N\mu)_A=\lim_{m\to\infty}(N\mu)_{A^{(m)}}=\lim_{m\to\infty}N_{A^{(m)}}\mu=N_A\mu.$$

§ 8. Classification of boundary points

First we aim at proving $(V_A)_A=V_A$ for any piecewise smooth Dirichlet finite SHS function V in R'.

THEOREM 18. *For such* V

$$\|V_{A^{(m)}} - V_A\| \to 0 \qquad as\ m \to \infty.$$

PROOF. Let $p<m$. By Corollary 1 of Theorem 11, it holds that

$$0 \leqq \|V_{A^{(m)}} - V_{A^{(p)}}\|^2_{R'-A^{(m)}} = \|V_{A^{(p)}}\|^2_{R'-A^{(p)}} - \|V_{A^{(m)}}\|^2_{R'-A^{(m)}} + \|V\|^2_{A^{(p)}-A^{(m)}}.$$

It follows that $\|V_{A^{(m)}}\|_{R'-A^{(m)}}$ has a limit as $m\to\infty$ and that $\{V_{A^{(m)}}\}$ form a Cauchy sequence. We know the existence of $\lim_{m\to\infty} V_{A^{(m)}} = V_A$ and derive $\lim_{m\to\infty} \|V_{A^{(m)}} - V_A\|_{R'-A^{(m)}} = 0$.

THEOREM 19. *For the above* V,

$$(V_A)_A = V_A.$$

PROOF. Let $p<m$. By Theorem 11

$$\|(V_A - V_{A^{(m)}})_{A^{(p)}}\|_{R'-A^{(p)}} \leqq \|V_A - V_{A^{(m)}}\|_{R'-A^{(p)}} \leqq \|V_A - V_{A^{(m)}}\|_{R'-A^{(m)}}.$$

The last quantity tends to zero as $m\to\infty$ by the preceding theorem. Therefore, by Theorem 9 and then by the preceding theorem again,

$$\begin{aligned} 0 &= \lim_{m\to\infty} \|(V_A)_{A^{(p)}} - (V_{A^{(m)}})_{A^{(p)}}\|_{R'-A^{(p)}} \\ &= \lim_{m\to\infty} \|(V_A)_{A^{(p)}} - V_{A^{(m)}}\|_{R'-A^{(p)}} = \|(V_A)_{A^{(p)}} - V_A\|. \end{aligned}$$

Hence $V_A = (V_A)_{A^{(p)}}$ for each p. The equality $(V_A)_A = V_A$ follows from this.

COROLLARY. $(\omega_A)_A = \omega_A$.

THEOREM 20. *Let* V *be an SHS function in* R', *and* A *be a closed subset of* Δ_N *with* $\omega_A = 0$. *Then* $(V_A)_A = V_A$.

PROOF. Take $\{A^{(m)}\}$ as above. Since V_A is an HS_0 function, $(V_A)_A \leqq (V_A)_{A^{(m)}} \leqq V_A$.

To prove $V_A \leqq (V_A)_A$ we use the decomposition $V = U + \int_{R'} N d\mu$ obtained in Theorem 8. By Theorem 17 $V_A = U_A + \int_{R'} N_A d\mu$. Similarly we obtain $(V_A)_A$

$=(U_A)_A+\int_{R'}(N_A)_A d\mu$. If $Q\in R'$ and $M>0$ is large, the function $N_M(P, Q)=\min(N(P, Q), M)$ is a piecewise smooth Dirichlet finite SHS function and $(N_A)_A=((N_M)_A)_A$. By Theorem 19 we see that $((N_M)_A)_A=(N_M)_A=N_A$. Therefore it suffices to prove $U_A\leqq(U_A)_A$ for any HS function U.

First we prove

$$(4)\qquad (U_{A^{(m)}}-U_A)_K\leqq U_{A^{(m)}}-U_A \qquad \text{in } R'-K$$

for any regular compact set K in R'. We shall set $A_n^{(m)}=A^{(m)}\cap(R_n\cup\partial R_n)$. Let p $(p>m)$ be a large number so that $A^{(p)}\cap K=\emptyset$. Set $M=\max U$. We consider

$$U_{A_n^{(m)}}-U_{A_n^{(p)}}+M\omega_{A_n^{(p)}}-(U_{A_n^{(m)}}-U_{A_n^{(p)}})_K$$

as a function in $R'-K-A_n^{(p)}$. This is bounded superharmonic and takes the non-negative boundary values. Therefore, the function is non-negative in $R'-K-A_n^{(p)}$ so that

$$(U_{A_n^{(m)}}-U_{A_n^{(p)}})_K\leqq U_{A_n^{(m)}}-U_{A_n^{(p)}}+M\omega_{A_n^{(p)}}.$$

We let $n\to\infty$ and have

$$(U_{A^{(m)}}-U_{A^{(p)}})_K\leqq U_{A^{(m)}}-U_{A^{(p)}}+M\omega_{A^{(p)}}$$

in $R'-K-A^{(p)}$. Next we let $p\to\infty$ and derive (4).

We use (4) for $K=A_j^{(k)}$ with $k<m$, and obtain

$$(U_{A^{(m)}}-U_A)_{A_j^{(k)}}\leqq U_{A^{(m)}}-U_A \qquad \text{in } R'-A^{(k)}.$$

As $j\to\infty$, $(U_{A^{(m)}})_{A_j^{(k)}}\to(U_{A^{(m)}})_{A^{(k)}}$ and $(U_A)_{A_j^{(k)}}\to(U_A)_{A^{(k)}}$. Therefore, $(U_{A^{(m)}})_{A^{(k)}}-(U_A)_{A^{(k)}}\leqq U_{A^{(m)}}-U_A$ in $R'-A^{(k)}$. We apply Theorem 9 and have $(U_{A^{(m)}})_{A^{(k)}}=U_{A^{(m)}}$. Thus $U_A\leqq(U_A)_{A^{(k)}}$ in $R'-A^{(k)}$, whence $U_A\leqq(U_A)_A$.

Now we proceed to classify the points of Δ_N. We set

$$\alpha(Q)=\frac{1}{2\pi}\int_{\partial K_0}\frac{\partial N_{\{Q\}}(P, Q)}{\partial\nu}ds \qquad \text{for } Q\in\Delta_N.$$

THEOREM 21. *For $Q\in\Delta_N$, $\alpha(Q)=0$ or 1.*

PROOF. By Theorem 16, $N_{\{Q\}}=\alpha(Q)N$. If $\omega_{\{Q\}}=0$,

$$N_{\{Q\}} = (N_{\{Q\}})_{\{Q\}} = \alpha(Q) N_{\{Q\}} = \alpha^2(Q) N$$

on account of Theorem 20. Thus $\alpha(Q)(1-\alpha(Q))N=0$. Therefore $\alpha(Q)=0$ or 1.

If $\omega_{\{Q\}}>0$, $\omega_{\{Q\}}=cN$ with $c>0$ by Theorem 16. Using the Corollary of Theorem 19 we have

$$N_{\{Q\}} = c^{-1}(\omega_{\{Q\}})_{\{Q\}} = c^{-1}\omega_{\{Q\}} = N.$$

Since $N_{\{Q\}}=\alpha(Q)N$, $N=\alpha(Q)N$ follows. Thus $\alpha(Q)=1$.

COROLLARY. *According as* $\alpha(Q)=0$ *or* 1, $(N(\cdot, Q))_{\{Q\}}(P)=0$ *or* $N(P, Q)$.

A point $Q\in\Delta_N$ with $\omega_{\{Q\}}>0$ is called *singular* by Kuramochi. The above proof shows that $\alpha(Q)=1$ for every singular point Q.

Let us establish

THEOREM 22. *The set* $\Delta_0=\{Q\in\Delta_N;\ \alpha(Q)=0\}$ *is an* F_σ*-set.*

PROOF. Let m be a positive integer. Suppose that, for every closed set F in R' with analytic boundary whose closure F^a contains a neighborhood of $Q\in\Delta_N$ in $R'\cup\Delta_N$ and is contained in the m^{-1}-neighborhood of Q, it holds that

$$\alpha_F(Q) = \frac{1}{2\pi}\int_{\partial K_0} \frac{\partial(N(\cdot, Q))_F}{\partial\nu} ds \leqq \frac{1}{2}.$$

We shall show that the set δ_m of all such points Q is closed. This will prove the theorem because $\Delta_0=\bigcup_m \delta_m$.

Let $Q_j\in\delta_m$ and $Q_j\to Q_0$. Take F at Q_0 such that the closure F^a is contained in the m^{-1}-neighborhood of Q_0. There is j_0 with the property that, for every $j\geqq j_0$, F^a is contained in the m^{-1}-neighborhood of Q_j. By assumption $\alpha_F(Q_j)\leqq 1/2$ for every $j\geqq j_0$. Given $\varepsilon>0$, we can take a large n such that

$$\frac{1}{2\pi}\int_{\partial K_0}\frac{\partial(N(\cdot, Q_0))_F}{\partial\nu} ds - \varepsilon < \frac{1}{2\pi}\int_{\partial K_0}\frac{\partial(N(\cdot, Q_0))_{F_n}}{\partial\nu} ds,$$

where $F_n=F\cap(R_n\cup\partial R_n)$. It follows that

$$\frac{1}{2} \geqq \overline{\lim_{j\to\infty}} \frac{1}{2\pi}\int_{\partial K_0}\frac{\partial(N(\cdot, Q_j))_F}{\partial\nu} ds \geqq \frac{1}{2\pi}\int_{\partial K_0} \overline{\lim_{j\to\infty}} \frac{\partial(N(\cdot, Q_j))_F}{\partial\nu} ds$$

$$\geqq \frac{1}{2\pi}\int_{\partial K_0}\lim_{j\to\infty}\frac{\partial(N(\cdot, Q_j))_{F_n}}{\partial\nu} ds = \frac{1}{2\pi}\int_{\partial K_0}\frac{\partial(N(\cdot, Q_0))_{F_n}}{\partial\nu} ds$$

$$> \frac{1}{2\pi}\int_{\partial K_0}\frac{\partial(N(\cdot, Q_0))_F}{\partial\nu} ds - \varepsilon.$$

Since ε can be arbitrarily small, we have $\alpha_F(Q_0) \leqq 1/2$. This implies that $Q_0 \in \delta_m$. Thus δ_m is closed.

§ 9. Canonical representation

Throughout this section $V(P)$ will mean an SHS function and F a closed set in R' with analytic boundary. We set $\Delta_1 = \Delta_N - \Delta_0$. It is equal to $\{Q \in \Delta_N; \alpha(Q)=1\}$.

First we prove

LEMMA 5. *Let $\{A_p\}$ be a sequence of closed subsets of Δ_N increasing to a closed set A, and assume $V_{A_p}=0$ for each p. Then $V_A=0$.*

PROOF. Let $P_0 \in R'$ and take $\varepsilon>0$. For each p we choose a closed set F_p with analytic boundary in R' such that its closure F_p^a in $R' \cup \Delta_N$ is a closed neighboorhood of A_p, $P_0 \notin F_p$ and $V_{F_p}(P_0) < \varepsilon/2^p$. It holds that $\bigcup_p F_p^a \supset \bigcup_p A_p = A$. There are $F_1, \cdots, F_q$ such that $\bigcup_{p=1}^{q} F_p^a$ is a closed neighobrhood of A. We take $\{A^{(m)}\}$ as in §7. If m is large, $A^{(m)} \subset \bigcup_{p=1}^{q} F_p$. We obtain

$$V_{A^{(m)} \cap (R_n \cup \partial R_n)} \leqq V_{(\bigcup_{p=1}^{q} F_p) \cap (R_n \cup \partial R_n)} \leqq \sum_{p=1}^{q} V_{F_p \cap (R_n \cup \partial R_n)}$$

for each n. It follows that $V_A \leqq V_{A^{(m)}} \leqq \sum_{p=1}^{q} V_{F_p}$ and

$$V_A(P_0) \leqq \sum_{p=1}^{q} V_{F_p}(P_0) \leqq \sum_{p=1}^{\infty} \frac{\varepsilon}{2^p} = \varepsilon.$$

Since ε can be arbitrarily small, $V_A(P_0)=0$.

THEOREM 23. *Let $V(P)$ be an SHS function. Then $V_E(P)=0$ for any closed subset E of Δ_0.*

PROOF. First we consider the case where $(V_E)_E = V_E$ for any closed subset E of Δ_0. Consider δ_m defined in the proof of Theorem 22, and let A be a closed subset of δ_m with diameter less than $(2m)^{-1}$. Let F be a closed set with analytic boundary in R' such that the closure F^a of F in $R' \cup \Delta_N$ is a closed neighborhood of A and F^a is contained in the $(2m)^{-1}$-neighborhood of A. Since the diameter of A is less than $(2m)^{-1}$ and F^a is contained in the $(2m)^{-1}$-neighborhood of A, F^a is contained in the m^{-1}-neighborhood of any point of A. Hence $\alpha_F(Q) \leqq 1/2$ for every $Q \in A \subset \delta_m$. By Theorem 16 V_A is represented as the potential $N\mu$ of a measure μ supported by A. Using Theorem 15 we have

$$\mu(A)=\frac{1}{2\pi}\int_{\partial K_0}\frac{\partial N\mu}{\partial\nu}ds=\frac{1}{2\pi}\int_{\partial K_0}\frac{\partial V_A}{\partial\nu}ds=\frac{1}{2\pi}\int_{\partial K_0}\frac{\partial(V_A)_A}{\partial\nu}ds$$

$$=\frac{1}{2\pi}\int_{\partial K_0}\frac{\partial(N\mu)_A}{\partial\nu}ds\leqq\frac{1}{2\pi}\int_{\partial K_0}\frac{\partial(N\mu)_F}{\partial\nu}ds=\frac{1}{2\pi}\int_{\partial K_0}\frac{\partial(N_F\mu)}{\partial\nu}ds$$

$$=\frac{1}{2\pi}\int_A d\mu\int_{\partial K_0}\frac{\partial N_F}{\partial\nu}ds=\int_A\alpha_F(Q)d\mu(Q)\leqq\mu(A)/2.$$

There follows $\mu(A)=0$ and hence $V_A\equiv 0$.

Since δ_m can be divided into a finite number of closed sets with diameter less than $(2m)^{-1}$, $V_{\delta_m}=0$ by Lemma 5. Let E be any closed subset of $\varDelta_0$. Each $E\frown\delta_m$ is closed and $E=\bigcup_m(E\frown\delta_m)$. On account of Lemma 5 $V_E=0$.

Now we recall that $(\omega_A)_A=\omega_A$ for any closed subset A of $\varDelta_N$ (Corollary of Theorem 19), and derive $\omega_E=0$ for $E\subset\varDelta_0$. Consequently we can apply Theorem 20 and conclude $V_E=0$ for any SHS function V.

We need some preparations for Theorem 24. Let F be a closed set with analytic boundary in R' and A be a closed subset of $\varDelta_N$. We take $\{A^{(m)}\}$ as before. The closure $(A^{(m)})^a$ in $R'\cup\varDelta_N$ decreases to A as $m\to\infty$. By Theorem 16 we represent $V_{A^{(m)}}$ by $N\mu^{(m)}=\int_{(A^{(m)})^a}Nd\mu^{(m)}$ in $R'-A^{(m)}$. We may assume that $\mu^{(m)}$ converges vaguely to a measure μ whose potential $N\mu=\int_A Nd\mu$ is equal to V_A in R'. We set $A_n^{(m)}=A^{(m)}\frown(R_n\cup\partial R_n)$. We know that $A_n^{(m)}$ supports a measure $\mu_n^{(m)}$ such that $V_{A_n^{(m)}}=N\mu_n^{(m)}$. We may assume that $\mu_n^{(m)}$ converges vaguely to $\mu^{(m)}$ as $n\to\infty$ for each m. Let $\nu_n^{(m)}$ be the restriction of $\mu_n^{(m)}$ to F. We may assume that $\nu_n^{(m)}$ converges vaguely to a measure $\nu^{(m)}$ as $n\to\infty$ for each m, and furthermore that $\nu^{(m)}$ converges vaguely to a measure ν as $m\to\infty$.

Let us prove

LEMMA 6. *Let F' be a similar closed set in R' such that $F\subset F'$ and $F\frown\partial F'=\emptyset$. Then $N\nu\leqq V_{F'}$ in $R'-F'$.*

PROOF. It holds that $N\nu_n^{(m)}\leqq N\mu_n^{(m)}=V_{A_n^{(m)}}$. Since F' contains $S_{\nu_n^{(m)}}$ in its interior, $(N\nu_n^{(m)})_{F'}=\lim_{p\to\infty}(N\nu_n^{(m)})_{F'\cap(R_p\cup\partial R_p)}=N\nu_n^{(m)}$ by Theorem 5. Therefore $N\nu_n^{(m)}\leqq(V_{A_n^{(m)}})_{F'}\leqq V_{F'}$ in $R'-F'$. By letting $n\to\infty$ and then $m\to\infty$ we conclude the lemma.

Now we can establish

THEOREM 24. *Let V be an SHS function and A be a closed subset of Δ_N. Then V_A can be represented in the form $\int_{A\cap\Delta_1} Nd\mu$.*

PROOF. Take $\{A^{(m)}\}$ and $\{\mu^{(m)}\}$ as above. We shall show that $\mu(\Delta_0)=0$. Let $\varepsilon>0$ be given and fix $P_0\in R'$. Since $V_{\delta_k}=0$ by Theorem 23, there is a closed neighborhood E' of δ_k in $R'\cup\Delta_N$ such that $F'=E'\cap R'$ is a closed set in R' with analytic boundary and satisfying $V_{F'}(P_0)<\varepsilon$. Let E be a similar closed set such that $F=E\cap R'\subset F'$ and $F\cap\partial F'=\emptyset$. We take $\{\nu_n^{(m)}\}$, $\{\nu^{(m)}\}$ and ν as above. Let us see $\mu(\delta_k)=\nu(\delta_k)$. Let f be a continuous function in $R'\cup\Delta_N$ such that $0\leqq f\leqq 1$, $f=1$ on δ_k, its support $S_f\subset E$ and $\int fd\mu$ and $\int fd\nu$ are close to $\mu(\delta_k)$ and $\nu(\delta_k)$ respectively. We have $\int fd\mu_n^{(m)}=\int fd\nu_n^{(m)}$. As $n\to\infty$ they tend to $\int fd\mu^{(m)}$ and $\int fd\nu^{(m)}$. Then by letting $m\to\infty$ we obtain $\int fd\mu=\int fd\nu$. It follows that $\mu(\delta_k)=\nu(\delta_k)$.

We apply Lemma 6 and have

$$\int_{\delta_k} N(P_0, Q)d\nu(Q)\leqq N\nu(P_0)\leqq V_{F'}(P_0)<\varepsilon \qquad \text{in } R'-F',$$

whence $\int_{\delta_k} Nd\nu=0$. This shows $\nu(\delta_k)=\mu(\delta_k)=0$. This is true for each k and accordingly $\mu(\Delta_0)=\mu(\bigcup_k \delta_k)=0$.

Since $V_{\Delta_N}=V$ for any HS_0 function V, we obtain

COROLLARY. *Any HS_0 function is represented as $\int_{\Delta_1} Nd\mu$.*

We shall call this measure μ *canonical*, and the representation a *canonical representation.* The uniqueness will be shown later.

We shall apply Theorem 24 to obtain a result which generalizes Theorems 19 and 20.

THEOREM 25. *For any SHS function V in R' and closed subsets A, A' of Δ_N such that $A\subset A'$, it holds that $(V_A)_{A'}=V_A$. If F' is a closed set with analytic boundary in R' and the closure of F' in $R'\cup\Delta_N$ is a closed neighborhood of A, then $(V_A)_{F'}=V_A$.*

PROOF. By Theorem 24 there is a measure μ such that $V_A=N\mu=\int_{A\cap\Delta_1} Nd\mu$. Theorem 17 implies that $(V_A)_{A'}=(N\mu)_{A'}=N_{A'}\mu$. If $Q\in A\cap\Delta_1$, $(N(\cdot, Q))_{A'}(Q)=N(P, Q)$ because $(N(\cdot, Q))_{\{Q\}}(P)\leqq(N(\cdot, Q))_{A'}(P)\leqq N(P, Q)$ and $(N(\cdot, Q))_{\{Q\}}(P)=N(P, Q)$ by the Corollary of Theorem 21. Thus $(V_A)_{A'}=N_{A'}\mu$

$=N\mu=V_A$. The equality $(V_A)_{F'}=V_A$ is established similarly.

§ 10. Minimal functions

Let U be an HS_0 function. It will be called *minimal* if $V=cU$ whenever V and $U-V$ are HS_0 functions.

We prove a lemma which will be used below.

LEMMA 7. *Let a minimal HS_0 function $U(P)$ be expressed by $\int_B N(P, Q)\, d\mu(Q)$ with a Borel subset B of $\varDelta_N$. Then μ is a point measure at some $Q_0\in B$ with mass $c=(2\pi)^{-1}\int_{\partial K_0}\partial U/\partial\nu\, ds$.*

PROOF. Let A_1 be a closed subset of B with diameter less than 1 such that $\mu(A_1)>0$. Next let A_2 be a closed subset of A_1 with diameter less than 1/2 such that $\mu(A_2)>0$. In this way we obtain a sequence $\{A_j\}$ of closed subsets of B such that $\mu(A_j)>0$ for each j and $\bigcap_j A_j$ is a point $Q_0\in B$. Since U is minimal and $U-\int_{A_j}Nd\mu=\int_{B-A_j}Nd\mu$ is an HS_0 function, there is $c_j\geqq 1$ satisfying $U=c_j\int_{A_j}Nd\mu$. Thus U can be written as $\int Nd\mu_j$, where $S_{\mu_j}\subset A_j$. We see that the total mass of μ_j is equal to $c=(2\pi)^{-1}\int_{\partial K_0}\partial U/\partial\nu\, ds$. Let μ_0 be the vague limit of a subsequence of $\{\mu_j\}$. It follows that μ_0 is a point measure at Q_0 and

$$U(P)=\int N(P, Q)\, d\mu_0(Q)=cN(P, Q_0) \qquad \text{in } R'.$$

If μ is not the point measure at Q_0, there is a closed set $A\subset B$ such that $Q_0\notin A$ and $\mu(A)>0$. By the above reasoning we find a point $Q_1\in A$ such that $U(P)=cN(P, Q_1)$. This is equal to $cN(P, Q_0)$, and there follows $Q_0=Q_1$ which is a contradiction.

We shall establish

THEOREM 26. 1) *Let U be minimal and A a closed subset of $\varDelta_N$. If $U_A>0$ and $U-U_A$ is an HS_0 function, there is a point $Q_0\in A\cap\varDelta_1$ such that*

$$U(P)=cN(P, Q_0) \qquad \textit{with } c=\frac{1}{2\pi}\int_{\partial K_0}\frac{\partial U}{\partial\nu}ds.$$

2) *Any minimal function is a constant multiple of $N(P, Q_0)$ for some*

$Q_0 \in \varDelta_1$.

3) *$N(P, Q_0)$ is minimal if and only if $Q_0 \in \varDelta_1$.*

Proof. 1) We express U_A as a potential $\int_{A \cap \varDelta_1} N d\mu$ by Theorem 24. Since U is minimal and both U_A and $U - U_A$ are HS_0 functions by assumption, $U_A = c'U$. The condition $U_A > 0$ implies that $c' > 0$. Thus $U = c'^{-1} U_A = c'^{-1} \int_{A \cap \varDelta_1} N d\mu$. By Lemma 7 we find a point $Q_0 \in A \cap \varDelta_1$ such that $U(P) = cN(P, Q_0)$.

2) Take $\varDelta_N$ for A in 1).

3) Let $Q_0 \in \varDelta_1$ and suppose that both $V(P)$ and $W(P) = N(P, Q_0) - V(P)$ are HS_0 functions. By the Corollary of Theorem 21 $(N(\cdot, Q_0))_{\{Q_0\}}(P) = N(P, Q_0)$. Hence $V_{\{Q_0\}} + W_{\{Q_0\}} = N_{\{Q_0\}} = N = V + W$. Since $V_{\{Q_0\}} \leqq V$ and $W_{\{Q_0\}} \leqq W$, $V_{\{Q\}} = V$ and $W_{\{Q_0\}} = W$. By Theorem 16 again we have $V(P) = V_{\{Q_0\}}(P) = cN(P, Q_0)$.

Conversely, if $N(P, Q_0)$ is minimal, there are $Q_1 \in \varDelta_1$ and a constant c by 2) such that $N(P, Q_0) = cN(P, Q_1)$. It holds that $N(P, Q_0) = N(P, Q_1)$. Thus $Q_0 = Q_1 \in \varDelta_1$.

§ 11. Uniqueness of canonical representation

For the sake of completeness we shall prove the uniqueness, although the method is entirely due to Constantinescu-Cornea [3], 12. First let us see that any $f \in C^3$ (i.e. f is three times continuously differentiable) with compact support S_f in R' is equal to $(2\pi)^{-1} \iint N \Delta_z f \, dxdy$, where Δ_z is the Laplacian with respect to a local variable $z = x + iy$ and $(2\pi)^{-1} \Delta_z f \, dxdy$ is conformally invariant, thus defining a measure σ of general sign on R'.

We assume that $S_f \subset R_n$. Let N_n be the function on R'_n as defined in §3. We know that $N - N_n$ tends to 0 locally uniformly in $R - (K_0 - \partial K_0)$ as $n \to \infty$. It is a classical result that $\Delta_\zeta (\iint \log |z - \zeta| \Delta_z f dxdy) = -2\pi \Delta_\zeta f$. It follows that the Laplacian of $N_n \sigma - f$ vanishes everywhere in R'_n and hence $N_n \sigma - f$ is harmonic in R'_n. It vanishes on ∂K_0 and its normal derivative vanishes on ∂R_n. Therefore, it is equal to zero identically and $N_n \sigma = f$ follows in R'_n. By letting $n \to \infty$ we conclude $N\sigma = f$.

Let K be a regular compact subset of R' and Q be a point of $\varDelta_N$. By Theorem 7 $(N(\cdot, Q))_K(P)$ extended by $N(P, Q)$ over K is equal to the potential of a measure on ∂K. We shall denote this measure by $\mu_{Q,K}$. We shall show that $\int f d\mu_{Q,K}$ is a continuous function of $Q \in \varDelta_N$ for any continuous function f on K. We may suppose that f is defined in R' and has a compact support.

First assume that f belongs to C^3. We express it by $N\sigma = N\nu_f' - N\nu_f''$, where ν_f' and ν_f'' are both non-negative. It holds that

$$\int f d\mu_{Q,K} = \iint N(P, Q')\, d\mu_{Q,K}(P) d\nu_f'(Q') - \iint N(P, Q') d\mu_{Q,K}(P) d\nu_f''(Q')$$

$$= \int (N(\cdot, Q))_K(Q') d\nu_f'(Q') - \int (N(\cdot, Q))_K(Q') d\nu_f''(Q').$$

From the inequality

$$\sup_{Q' \in R'-K} |(N(\cdot, Q_2))_K(Q') - (N(\cdot, Q_1))_K(Q')| \leqq \max_{Q' \in \partial K} |N(Q', Q_2) - N(Q', Q_1)|$$

valid for any $Q_1, Q_2 \in \Delta_N$, it follows that $\int f d\mu_{Q,K}$ is a continuous function of $Q \in \Delta_N$. Next, any continuous function f with compact support in R' can be approximated uniformly by a sequence of functions of C^3 which vanish outside a fixed compact set in R'. We infer that $\int f d\mu_{Q,K}$ is a continuous function of $Q \in \Delta_N$ generally.

We give

LEMMA 8. *If $Q \in \Delta_1$, $\mu_{Q, \partial R_n}$ converges vaguely to the unit measure at Q as $n \to \infty$.*

PROOF. For simplicity we shall write μ_n^Q for $\mu_{Q, \partial R_n}$. We note that the total mass of μ_n^Q is one. Let μ_0 be the vague limit of a subsequence $\{\mu_{n_k}^Q\}$. On R', $N\mu_{n_k}^Q$ tends to $N\mu_0$. Since $N(P, Q) = \int N(P, Q') d\mu_n^Q(Q')$ in R_n', $N(P, Q) = \int N(P, Q') d\mu_0(Q')$ in R'. By Lemma 7 there is a point $Q_0 \in \Delta_N$ such that $N(P, Q) = N(P, Q_0)$ which implies $Q = Q_0$. Thus μ_0 is the unit measure at Q and our lemma is proved.

Now we prove

THEOREM 27. *Canonical representation of any HS_0 function is unique.*

PROOF. Suppose that $N\mu = \int_{\Delta_1} N d\mu = \int_{\Delta_1} N d\nu = N\nu$ in R'. We define a measure μ_n on R' by $\int f d\mu_n = \iint f d\mu_n^Q d\mu(Q)$. This is possible because $\int f d\mu_n^Q$ is a continuous function of Q on Δ_N. Similarly we define ν_n. We have

$$\int N(P, Q')d\mu_n(Q') = \iint N(P, Q')\,d\mu_n^Q(Q')\,d\mu(Q) = \iint N(Q, Q')d\mu_{\partial R_n}^P(Q')d\mu(Q)$$
$$= (N\mu)_{\partial R_n}(P) \qquad \text{if } P \in R' - \partial R_n$$

and

$$\int N(P, Q')d\mu_n(Q') = \iint N(P, Q')d\mu_n^Q(Q')d\mu(Q) = \int N(P, Q)d\mu(Q) = N\mu(P) \text{ if } P \in \partial R_n.$$

Similarly $N\nu_n = (N\nu)_{\partial R_n}$ in $R' - \partial R_n$ and $= N\nu$ on ∂R_n. Hence $N\mu_n = N\nu_n$ in R'. By the aid of Lemma 2 we conclude $\mu_n \equiv \nu_n$.

Let f be a continuous function on $R' \cup \Delta_N$. Since $\int f d\mu_n^Q$ is bounded and tends to $f(Q)$ as $n \to \infty$ by Lemma 8,

$$\lim_{n\to\infty} \int f d\mu_n = \lim_{n\to\infty} \int \left(\int f d\mu_n^Q\right) d\mu(Q) = \int f d\mu.$$

Thus μ_n converges vaguely to μ. Similarly ν_n converges vaguely to ν and the identity $\mu \equiv \nu$ is concluded.

Taking Theorem 24 into consideration we derive

COROLLARY. *Let A be a closed subset of Δ_N. The canonical measure for V_A is supported by A.*

References

[1] L. Ahlfors and L. Sario: Riemann surfaces, Princeton, 1960.
[2] N. Bourbaki: Topologie générale, Chapitre X, Paris, 1949.
[3] C. Constantinescu and A. Cornea: Ideale Ränder Riemannscher Flächen, Berlin-Göttingen-Heidelberg, 1963.
[4] Z. Kuramochi: Mass distributions on the ideal boundaries of abstract Riemann surfaces, II, Osaka Math. J., 8 (1956), pp. 145-186.
[5] Z. Kuramochi: Mass distributions on the ideal boundaries, Proc. Japan Acad., 36 (1960), pp. 118-122.
[6] Z. Kuramochi: Potentials on Riemann surfaces, J. Fac. Sci. Hokkaido Univ. Ser. I, 16 (1962), pp. 5-79.
[7] F-Y. Maeda: Notes on Green lines and Kuramochi boundary of a Green space, J. Sci. Hiroshima Univ. Ser. A-I Math., 28 (1964), pp. 59-66.
[8] R. S. Martin: Minimal positive harmonic functions, Trans. Amer. Math. Soc., 49 (1941), pp. 137-172.
[9] M. Ohtsuka: Dirichlet problem, extremal length and prime ends, Lecture Notes, Washington University, St. Louis, 1962-63.

Department of Mathematics,
Faculy of Science,
Hiroshima University.