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Normal Derivatives on an Ideal Boundary

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Introduction. Let Ω be a Green space introduced by Brelot-Choquet [2]. In general, there is no notion of "smooth" boundary of Ω and we cannot define normal derivatives on "the boundary" in the usual way. Still, there are notions of Laplacian and Dirichlet integrals on Ω. Therefore, we may define "generalized" normal derivatives so as to make Green's formula valid. To be more precise, let Ω^* be a compactification of Ω so that "an ideal boundary" $\Delta=\Omega^*-\Omega$ is realized. If u is an *HD*-function (i.e., a harmonic function with finite Dirichlet integral) and if a function φ on Δ is to be a normal derivative of u, then Green's formula

$$\int_{\Omega}(\text{grad } u, \text{grad } f)dv = -\int_{\Delta}\varphi f\, d\sigma$$

will be satisfied for any function f with finite Dirichlet integral on Ω which is "properly" extended over Ω^*. Here, we have two points to be cleared: i) What is the measure σ, which is the surface element in the classical case? ii) What is the "proper" extension of f?

Constantinescu-Cornea [3] defined a normal derivative on the Kuramochi boundary as a measure, which corresponds to $\varphi\cdot d\sigma$ in the above argument. If we are to define a normal derivative as a function, we must specify the measure σ. Following Doob [4], we try with the harmonic measure μ. In order to assure its existence, we shall suppose that the compactification is resolutive (§2).

As for the second point, Constantinescu-Cornea [3] defined a "quasi-continuous" extension of BLD-functions over the Kuramochi compactification. The definition requires a potential theory on the compactification and is applicable only to the Kuramochi boundary. On the other hand, Doob [4] used "fine" boundary functions of *HD*-functions, which required a theory of fine limits. It also looks impossible to generalize the theory to an arbitrary compactification.

Studying these two cases closely, however, it becomes clear that we don't need such sophisticated tools as "quasi-continuity" or "fine limits". If we consider a function f on Δ which has the Dirichlet solution H_f on Ω, then the pair (f, H_f) plays the role of a "properly" extended function. Thus, our definition will be: Given an HD-function u on Ω, we say that u has a normal derivative φ on Δ if

$$\int_{\check{\Omega}} (\text{grad } u, \text{grad } H_f) dv = -\int_{\Delta} \varphi f \, d\mu$$

for any function f on Δ such that it has the Dirichlet solution $H_f \in HD$.

We can extend our definition to functions harmonic only near the boundary (§3). With this definition of normal derivatives, it is possible to discuss boundary value problems involving normal derivatives (§4 and §5). We shall obtain solutions of these problems under certain conditions, mostly applying the methods given in Doob [4]. In order to obtain reasonable results, it is often necessary that the compactification is large enough to ensure that every HD-function is a Dirichlet solution. Such a compactification will be called D-normal. Since the Martin and the Kuramochi compactifications are both D-normal, our theory includes some of Doob's results [4] as well as Constantinescu-Cornea's [3].

Finally, one should remark that we may consider any locally compact space on which there are notions of harmonic and superharmonic functions and of Dirichlet integrals and we may treat the theory axiomatically. In this treatise, however, we shall restrict ourselves to a Green space.

§ 1. Preliminaries

1.1 *BLD-functions and Dirichlet integrals.*

We refer to [1] for the definition and properties of BLD-functions on a Green space Ω. If ω is an open set in Ω and if f_1, f_2 are BLD-functions on ω, then the mutual Dirichlet integral

$$<f_1, f_2>_\omega = \int_{\check{\omega}} (\text{grad } f_1, \text{ grad } f_2) dv$$

is defined, where $\check{\omega} = \omega - \{\text{points of infinity}\}$ and dv is the volume element on $\check{\Omega}$. We denote $\|f\|_\omega^2 = <f, f>_\omega$. The subscript will be omitted if $\omega = \Omega$.

If f is a BLD-function on ω, it is decomposed into $f = h + f_0$ on ω, where h is harmonic on ω and f_0 is a BLD-function of potential type on ω (cf. [1], [4] and [5]). This decomposition will be referred to as the Royden decomposition on ω.

1.2 *Classes of harmonic functions.*

Let HP be the space of all harmonic functions expressed as a difference of two non-negative harmonic functions on Ω. The bounded harmonic functions on Ω form a subspace HB and the BLD-harmonic functions form another subspace HD. We denote $HBD=HB\cap HD$.

A family $Y\subseteq HP$ is called monotone if, for any monotone sequence in Y having its limit $f\in HP$, we have $f\in Y$. Given a family $Y\subseteq HP$, there is the smallest monotone family $M(Y)$ containing Y. It is known that $HD\subseteq M(HB)$ and $M(HBD)=M(HD)$. (Cf. [3], §2 and §7).

If ω is an open set in Ω, let $HD(\omega)$ be the space of all harmonic functions u on ω with finite Dirichlet norm $\|u\|_{\omega}$.

1.3 *Compactifications.*

If Ω^* is a Hausdorff compact space and if there is a homeomorphism of Ω into Ω^* such that the image of Ω is open and dense in Ω^*, then Ω^* is called a compactification of Ω and $\Delta=\Omega^*-\Omega$ is called an ideal boundary of Ω.

Let Q be a family of bounded continuous functions on Ω. If a compactification Ω^* satisfies:

a) every $f\in Q$ can be continuously extended over Ω^*,
b) Q separates points of $\Delta=\Omega^*-\Omega$,

then Ω^* is called a Q-compactification of Ω. It is known (cf. [3]) that a Q-compactification always exists and is unique up to a homeomorphism. Thus, it will be denoted by Ω^*_Q and its ideal boundary by Δ_Q.

Given a compactification Ω^*, let $C(\Delta)$ be the space of all continuous (bounded) functions on Δ.

§ 2. Dirichlet problem

2.1 *Resolutive functions.*

Let Ω^* be a compactification of Ω and let $\Delta=\Omega^*-\Omega$. Given a function f (extended real valued) on Δ, we consider the following classes:

$$\overline{\mathfrak{S}}_f=\left\{\begin{array}{l} s;\ \text{superharmonic, bounded below on } \Omega, \\ \varliminf_{a\to\xi} s(a)\geq f(\xi) \quad \text{for any} \quad \xi\in\Delta \end{array}\right\}\cup\{\infty\}$$

$$\underline{\mathfrak{S}}_f=\{s;\ -s\in\overline{\mathfrak{S}}_{-f}\}.$$

Let $\overline{H}_f(a)=\inf\{s(a);\ s\in\overline{\mathfrak{S}}_f\}$ and $\underline{H}_f(a)=\sup\{s(a);\ s\in\underline{\mathfrak{S}}_f\}$. It is known (Perron-Brelot) that $\overline{H}_f$ (resp. $\underline{H}_f$) is either harmonic, $\equiv+\infty$ or $\equiv-\infty$. If $\overline{H}_f=\underline{H}_f$ and are harmonic, then we say that f is resolutive and $H_f=\overline{H}_f=\underline{H}_f$ is called the Dirichlet solution of f (with respect to Ω^*).

LEMMA 1. *If f is resolutive, then there exists a positive superharmonic function s such that*

$$H_f + \varepsilon s \in \overline{\mathfrak{S}}_f \qquad \text{and} \qquad H_f - \varepsilon s \in \underline{\mathfrak{S}}_f$$

for any $\varepsilon > 0$.
(Cf. [3], Hilfssatz 8.1)

2.2 *Resolutive compactification.*

If any $f \in C(\Delta)$ is resolutive, then we say that Ω^* is a resolutive compactification. In this case, we introduce the following classes of functions:

$$R(\Delta) = \text{all resolutive functions on } \Delta,$$

$$R_D(\Delta) = \{f \in R(\Delta);\ H_f \in HD\},$$

$$C_D(\Delta) = R_D(\Delta) \cap C(\Delta).$$

$$H(\Omega^*) = \{H_f;\ f \in R(\Delta)\},$$

$$H_D(\Omega^*) = \{H_f;\ f \in R_D(\Delta)\} = H(\Omega^*) \cap HD,$$

$$H_C(\Omega^*) = \{H_f;\ f \in C(\Delta)\}.$$

It is known that $H(\Omega^*) = M(H_C(\Omega^*))$ (cf. [3]). Therefore, if Ω_1^* and Ω_2^* are two resolutive compactifications such that $H_C(\Omega_1^*) = H_C(\Omega_2^*)$, then $H(\Omega_1^*) = H(\Omega_2^*)$ and $H_D(\Omega_1^*) = H_D(\Omega_2^*)$.

2.3 *D-normal compactification and regular compactification.*

DEFINITION. A resolutive compactification Ω^* is said to be *D-normal* if $H_D(\Omega^*) = HD$.

DEFINITION. A resolutive compactification Ω^* is said to be *regular* if $C_D(\Delta)$ is dense in $C(\Delta)$ in the uniform convergence topology.

Examples.

1. Wiener's compactification (cf. [3], §9) is the largest resolutive compactification. It is D-normal.
2. Royden's compactification (cf. [3], §9) is D-normal and regular.
3. If Q consists of BLD-functions, then the Q-compactification is regular.
4. The HB-compactification is D-normal; the HBD-compactification is D-normal and regular.
5. The Martin compactification is D-normal. It is not known if it is

regular or not.

6. The Kuramochi compactification is D-normal and regular.

7. The Alexandroff compactification is D-normal if and only if there are no non-constant functions in HD.

1, 2, 3, 4, 7 and the regularity of the Kuramochi compactification are immediate from the definitions. (Wiener's and Royden's compactifications were defined for Riemann surfaces in [3]. Analogous definitions can be given for Green spaces.) The D-normalities of the Martin and Kuramochi compactifications are less obvious.

By Doob [4], we see that every $u \in HB$ has a "fine" boundary value f on the Martin boundary and that u is the Dirichlet solution of f with respect to the Martin compactification Ω_M^*. It follows then that $H(\Omega_M^*) = M(HB)$. Hence the Martin compactification is D-normal.

Constantinescu-Cornea proved (Hilfssats 16.1 in [3]) that $H(\Omega_N^*) = M(HD)$ for the Kuramochi compactification Ω_N^* of a Riemann surface Ω. The same proof is applicable for a Green space. Thus, the Kuramochi compactification is also D-normal.

Remark. It can be seen by examples that the notions of D-normality and regularity are independent.

2.4 *Harmonic measure.*

If Ω^* is a resolutive compactification, then $f \to H_f(a)$ is a positive linear form on $C(\Delta)$ for each $a \in \Omega$. Therefore, there is a Radon measure μ_a on Δ such that $H_f(a) = \int f d\mu_a$ for all $f \in C(\Delta)$. μ_a is called the harmonic measure on Δ with respect to a. μ_a-measurability and μ_a-summability are independent of a. We know that

$$\underline{H}_f(a) \leq \underline{\int} f \, d\mu_a \leq \overline{\int} f \, d\mu_a \leq \overline{H}_f(a)$$

for any f on Δ. The functions $a \to \underline{\int} f \, d\mu_a$ and $a \to \overline{\int} f \, d\mu_a$ are harmonic whenever they are finite. If Ω^* is metrizable, then $\underline{H}_f(a) = \underline{\int} f \, d\mu_a$ and $\overline{H}_f(a) = \overline{\int} f \, d\mu_a$. From these facts, we see that $R(\Delta) \subseteq L^1(\mu)$; the equality holds if Ω^* is metrizable; $H(\Omega^*) = \{u(a) = \int f \, d\mu_a ; f \in L^1(\mu)\}$.

2.5 *Lemmas* (cf. Doob [4]).

Lemma 2. *Let* $f \in R(\Delta)$, $u = H_f$ *and* $p \geq 1$. $f \in L^p(\mu)$ *if and only if* $|u|^p$ *is* *majorized by a harmonic function. In this case,* $\bar{H}_{|f|^p}(a)=\int |f|^p d\mu_a$ *is the least harmonic majorant of* $|u|^p$. (If Ω^* is metrizable, then $|f|^p \in R(\Delta)$ and $H_{|f|^p}(a)=\int |f|^p d\mu_a$.)

PROOF. "Only if": Since $f \in L^p(\mu)$, $h(a)=\int |f|^p d\mu_a$ is a harmonic function. By Hölder's inequality, we have

$$|u(a)|^p = \left|\int f d\mu_a\right|^p \leq \int |f|^p d\mu_a = h(a).$$

"If": Since f is μ-summable, $|f|^p$ is μ-measurable. Hence, it is enough to show that $\bar{\int} |f|^p d\mu < \infty$ or that $\bar{H}_{|f|^p}$ is finite.

By Lemma 1, there is a positive superharmonic function s such that

$$u+\varepsilon s \in \bar{\mathfrak{S}}_f \qquad \text{and} \qquad u-\varepsilon s \in \underline{\mathfrak{S}}_f$$

for any $\varepsilon>0$. Let h_0 be the least harmonic majorant of $|u|^p$. We shall show that $h_0+\varepsilon s \in \bar{\mathfrak{S}}_{|f|^p}$ for any $\varepsilon>0$. Then $\bar{H}_{|f|^p}$ is finite and $\bar{H}_{|f|^p} \leq h_0$.

Suppose $h_0+\varepsilon_0 s \notin \bar{\mathfrak{S}}_{|f|^p}$ for some $\varepsilon_0>0$. Then there is $\xi \in \Delta$ such that

$$\varliminf_{a\to\xi} [h_0(a)+\varepsilon_0 s(a)] < |f(\xi)|^p.$$

Since $h_0+\varepsilon_0 s \geq 0$, $f(\xi) \neq 0$. Hence, we can choose a positive number β such that

$$\varliminf_{a\to\xi} [h_0(a)+\varepsilon_0 s(a)] < \beta < |f(\xi)|^p.$$

Then, there is a net $\{a_\alpha\}$ of points in Ω such that $a_\alpha \to \xi$ and $h_0(a_\alpha)+\varepsilon_0 s(a_\alpha)<\beta$. Let $\eta=p^{-1}\beta^{(1/p)-1}$ and let $\varepsilon_1=\eta\cdot\varepsilon_0$.

If $f(\xi)>0$, then we consider $u+\varepsilon_1 s$. We have

$$\begin{aligned} u(a_\alpha)+\varepsilon_1 s(a_\alpha) &= u(a_\alpha)+\eta\varepsilon_0 s(a_\alpha) \\ &< u(a_\alpha)+\eta\big(\beta-h_0(a_\alpha)\big) \\ &\leq u(a_\alpha)+\eta\big(\beta-|u(a_\alpha)|^p\big). \end{aligned}$$

The function $F(x)=-\eta(|x|^p-\beta)+x$ assumes its maximum at $x=\beta^{1/p}$, where $F(x)=\beta^{1/p}$. Hence

$$u(a_\alpha)+\varepsilon_1 s(a_\alpha)<\beta^{1/p} \qquad \text{for all } \alpha, \text{ or}$$

$$\varlimsup_{a\to\xi}\,[u(a)+\varepsilon_1 s(a)]\leq\beta^{1/p}<f(\xi),$$

which contradicts the choice of s.

If $f(\xi)<0$, then we consider $u-\varepsilon_1 s$ and we obtain a similar contradiction.

In the proof of the "only if" part, we have shown that $h(a)=\int|f|^p d\mu_a$ is a harmonic majorant of $|u|^p$. Hence $h_0\leq h$, so that we have

$$\int|f|^p d\mu_a\leq\bar{H}_{|f|^p}(a)\leq h_0(a)\leq\int|f|^p d\mu_a.$$

Therefore, $h_0(a)=\bar{H}_{|f|^p}(a)=\int|f|^p d\mu_a$.

LEMMA 3. *If $f\in R_D(\Delta)$, then $f\in L^2(\mu)$ and*

$$\int f^2 d\mu_a\leq M_a(\|H_f\|+|H_f(a)|)^2,$$

where $M_a>0$ is independent of f.

PROOF. We can apply the methods of the proofs of Theorem 3.1, 4.1 and 4.2 in Doob [4] and we see that u^2 is majorized by a harmonic function, so that $f\in L^2(\mu)$ by the above lemma, and also that

$$\int(f-H_f(a))^2 d\mu_a\leq M'_a\|H_f\|^2$$

for some $M'_a>0$. Then, taking $M_a=\max(M'_a, 1)$, we have the inequality in the lemma.

2.6 *Closedness of $H_D(\Omega^*)$.*

Let $a_0\in\Omega$ be fixed. The space HD is a Banach space with respect to the norm $\|u\|+|u(a_0)|$. Lemma 3 shows that the mapping $H_f\to f$ is continuous from $H_D(\Omega^*)$ into $L^2(\mu_{a_0})$. We see furthermore:

THEOREM 1. *$H_D(\Omega^*)$ is closed in HD.*

PROOF. It is enough to show that $u_n\in H_D(\Omega^*)$, $u\in HD$, $\|u_n-u\|\to 0(n\to\infty)$ and $u_n(a_0)=u(a_0)=0$ imply $u\in H_D(\Omega^*)$. Let $u_n=H_{f_n}$ with $f_n\in R_D(\Delta)$. By Lemma 3, $\{f_n\}$ is a Cauchy sequence in $L^2(\mu)$. Since $L^2(\mu)$ is complete, there is $f_0 \in L^2(\mu)$ such that $f_n \to f_0$ in $L^2(\mu)$. We may assume f_0 is resolutive. Hence, for each $a \in \Omega$,

$$H_{f_0}(a) = \int f_0 \, d\mu_a = \lim_{n\to\infty} \int f_n \, d\mu_a = \lim_{n\to\infty} H_{f_n}(a) = \lim_{n\to\infty} u_n(a) = u(a).$$

Thus, $f_0 \in R_D(\Delta)$ and $u \in H_D(\Omega^*)$.

2.7. *Dirichlet problem on a subdomain.*

Let Ω^* be a resolutive compactification of Ω and let ω be an open set in Ω. We denote the boundary of ω in Ω^* by $\partial^*\omega$ and the relative boundary of ω by $\partial\omega$. Given a real function f on $\partial^*\omega$, we can discuss the Dirichlet problem. If f is resolutive with respect to ω, then we denote the solution by $H_f^{\omega*}$. If g is a real function on the relative boundary $\partial\omega$, then let g^* be equal to g on $\partial\omega$ and to zero on $\partial^*\omega \cap \Delta$. If g^* is resolutive then we say that g on $\partial\omega$ is resolutive and we denote the solution by $H_g^{\omega} = H_{g^*}^{\omega*}$.

LEMMA 4. *If $f \in R(\Delta)$ and g is a resolutive function on $\partial\omega$, then*

$$f_1 = \begin{cases} f & \text{on} \quad \partial^*\omega \cap \Delta \\ g & \text{on} \quad \partial\omega \end{cases}$$

is resolutive.

PROOF. Let $g_1 = g - H_f$ on $\partial\omega$. Since H_f is continuous, it is resolutive as a function on $\partial\omega$. Now it is easy to see that $H_{g_1}^{\omega} + H_f$ is the Dirichlet solution of f_1 with respect to ω.

LEMMA 5. *Let K be a compact set in Ω. For a real function f on Δ, let*

$$f_K = \begin{cases} f & \text{on} \quad \Delta \\ 0 & \text{on} \quad \partial K. \end{cases}$$

Then, $f \in R_D(\Delta)$ if and only if $H_{f_K}^{(\Omega-K)*} \in HD(\Omega - K)$.

In this case,

$$v_{f,K} = \begin{cases} H_{f_K}^{(\Omega-K)*} & \text{on} \quad \Omega - K \\ 0 & \text{on} \quad K \end{cases}$$

is a BLD-function on Ω.

PROOF. If $f \in R_D(\Delta)$, then $H_f \in HD$. On the other hand, $H_{f_K}^{(\Omega-K)^*}=H_f+H_{g_1}^{\Omega-K}$, where $g_1=-H_f$ on ∂K. (See the proof of the above lemma.) Let φ be a BLD-function having compact support in Ω and $\varphi=-H_f$ on K. Then it is easy to see that $H_{g_1}^{\Omega-K}$ is the harmonic part of the Royden decomposition of φ on $\Omega-K$. Hence $H_{g_1}^{\Omega-K} \in HD(\Omega-K)$, so that $H_{f_K}^{(\Omega-K)^*} \in HD(\Omega-K)$. Furthermore, it can be seen that $v_{f,K}$ is a BLD-function (cf. Brelot [1]).

Conversely, if $H_{f_K}^{(\Omega-K)^*} \in HD(\Omega-K)$, then there exists a BLD-function ψ on Ω such that $\psi=H_{f_K}^{(\Omega-K)^*}$ outside a compact set. Then, we see that H_f is the harmonic part of the Royden decomposition of ψ, so that $H_f \in HD$.

§ 3. Normal derivatives

In this and the following sections, we always assume that Ω^* is a resolutive compactification.

3.1 *Definition.*

We fix $a_0 \in \Omega$ once for all and let $\mu \equiv \mu_{a_0}$.

DEFINITION. Let $u \in HD(\Omega-K)$ for some compact set K (may be empty) in Ω. We say that u has *a normal derivative* φ *on* Δ (relative to a_0), or φ is *a normal derivative of* u *on* Δ (relative to a_0), if $\varphi f \in L^1(\mu)$ and

$$\langle u, v_{f,K}\rangle_{\Omega-K} = -\int \varphi f \, d\mu$$

for any $f \in R_D(\Delta)$, where $v_{f,K}$ is the function defined in Lemma 5.

By virtue of the following lemma, we see that the definition above is independent of the choice of K:

LEMMA 6. *If K and K' are compact sets such that $K \subseteq K'$ and if $u \in HD(\Omega-K)$, then*

$$\langle u, v_{f,K}\rangle_{\Omega-K} = \langle u, v_{f,K'}\rangle_{\Omega-K'}$$

for any $f \in R_D(\Delta)$.

PROOF. We see that $H_{f_K}^{(\Omega-K)^*}$ is the harmonic part of the Royden decomposition of $v_{f,K'}$ on $\Omega-K$. Hence,

$$\langle u, v_{f,K'}\rangle_{\Omega-K'} = \langle u, v_{f,K'}\rangle_{\Omega-K} = \langle u, H_{f_K}^{(\Omega-K)^*}\rangle_{\Omega-K} = \langle u, v_{f,K}\rangle_{\Omega-K}.$$

The following properties are immediate consequences of the definition:

a) If $\varphi_i (i=1, 2)$ is a normal derivative of $u_i (i=1, 2)$, then $\alpha_1\varphi_1+\alpha_2\varphi_2$ is a normal derivative of $\alpha_1 u_1+\alpha_2 u_2$ for any real numbers α_1, α_2.

b) If $u\equiv$const., then it has a normal derivative zero.

c) If φ is a normal derivative of u and ψ is a function on Δ such that $\psi=\varphi$ μ-almost everywhere, then ψ is also a normal derivative of u.

THEOREM 2. *If Ω^* is a regular compactification, then the normal derivative of a function $u \in HD(\Omega-K)$ is, if it exists, uniquely determined μ-almost everywhere.*

PROOF. If φ_1 and φ_2 are normal derivatives of u, then $\int \varphi_1 f\, d\mu = \int \varphi_2 f\, d\mu$ for all $f \in C_D(\Delta)$. If Ω^* is regular, then this equation holds for any $f \in C(\Delta)$. Hence $\varphi_1=\varphi_2$ μ-almost everywhere.

3.2 *Normal derivatives of Green functions.*

The harmonic measure μ_a with respect to $a \in \Omega$ is μ-absolutely continuous. Hence, there is a μ-measurable function χ_a on Δ such that $d\mu_a=\chi_a d\mu$. We may assume (by Harnack's principle) that χ_a are bounded functions.

PROPOSITION 1. *The Green function G_a of Ω has a normal derivative $q\chi_a$*[1] *on Δ.*

PROOF. Obviously, $f\chi_a \in L^1(\mu)$ for any $f \in R(\Delta)$. For a large $\alpha>0$, the set $K=\{b; G_a(b)\geq\alpha\}$ is compact and its boundary ∂K is a smooth surface in Ω. Then, $H_{f_K}^{(\Omega-K)^*}=H_f+H_{g^*}^{(\Omega-K)^*}$ on $\Omega-K$, where $g=-H_f$ on ∂K. Hence, for any $f \in R_D(\Delta)$,

$$<G_a, H_{f_K}^{(\Omega-K)^*}>_{\Omega-K}=<G_a, H_f>_{\Omega-K}+<G_a, H_g^{\Omega-K}>_{\Omega-K}.$$

Now,

$$<G_a, H_f>_{\Omega-K}=<\min(G_a, \alpha), H_f>=0$$

and

$$<G_a, H_g^{\Omega-K}>_{\Omega-K}=-\int_{\partial K} H_f \frac{\partial G_a}{\partial n}\, d\sigma=-qH_f(a).$$

Hence

1) q is a constant depending only on the dimension of Ω. See [4]. It is denoted by φ_τ in [2].

$$<G_a, H_{f_K}^{(\Omega-K)*}>_{\Omega-K} = -q\int f\chi_a d\mu.$$

Therefore, $q\chi_a$ is a normal derivative of G_a on Δ.

COROLLARY. *If a measure m on Ω has a compact support, then its Green potential $p^m(b)=\int G_a(b)\,dm(a)$ has a normal derivative $q\int\chi_a dm(a)$ on Δ.*

PROOF. If K is a compact set containing the support of m in its interior, then $p^m \in HD(\Omega-K)$ and

$$<p^m, v>_{\Omega-K} = \int <G_a, v>_{\Omega-K}\, dm(a)$$

for any $v \in HD(\Omega-K)$. Hence the corollary follows from the proposition and the definition of normal derivatives.

3.3 THEOREM 3. *If K is a compact set in Ω and if f is a resolutive function on ∂K with respect to $\Omega-K$, then $H_f^{\Omega-K}$ has a normal derivative on Δ.*

PROOF. (i) First, let f be non-negative constant $\equiv c$. Let

$$s_1 = \begin{cases} H_f^{\Omega-K} & \text{on } \Omega-K \\ c & \text{on } K \end{cases}$$

and let s be the regularization of s_1 (i.e., $s(b)=\varliminf_{a\to b} s_1(a)$). Then, s is a Green potential whose associated measure is supported on K. Hence, by the above corollary, s, hence $H_f^{\Omega-K}$, has a normal derivative on Δ.

(ii) Next, let f be a non-negative bounded function on ∂K. For a sufficiently large n,

$$K' = \{b \in \Omega-K;\, H_{f+n}^{\Omega-K}(b) \geq n-1\} \cup K$$

is compact in Ω. Then the regularization of

$$s_1 = \begin{cases} H_{f+n}^{\Omega-K} & \text{on } \Omega-K' \\ n-1 & \text{on } K' \end{cases}$$

is a Green potential whose associated measure is supported on K'. Hence, $H_{f+n}^{\Omega-K} = H_f^{\Omega-K} + H_n^{\Omega-K}$ has a normal derivative on Δ. Since $H_n^{\Omega-K}$ has a normal derivative on $\varDelta$ by (i), so does $H_f^{\Omega-K}$.

(iii) If f is any resolutive non-negative function, we consider a compact set K' containing K in its interior. The restriction f_1 of $H_f^{\Omega-K}$ on $\partial K'$ is a non-negative bounded function and we have $H_f^{\Omega-K}=H_{f_1}^{\Omega-K'}$ on $\Omega-K'$. Hence, by (ii), $H_f^{\Omega-K}$ has a normal derivative on $\varDelta$.

(iv) If f is arbitrary, then $H_f^{\Omega-K}=H_{\max(f,0)}^{\Omega-K}+H_{\min(f,0)}^{\Omega-K}$ and each term of the right hand side has a normal derivative on $\varDelta$. Hence, so does $H_f^{\Omega-K}$.

COROLLARY. *Suppose $u=H_f$ has a normal derivative on $\varDelta$ for an $f\in R_D(\varDelta)$. Let g be any resolutive function on ∂K for a compact set K and let*

$$f_1=\begin{cases} f & \text{on } \varDelta \\ g & \text{on } \partial K. \end{cases}$$

Then $H_{f_1}^{(\Omega-K)^*}$ has a normal derivative on $\varDelta$.

PROOF. This is immediate from the theorem, since $H_{f_1}^{(\Omega-K)^*}=H_f+H_{g_1}^{\Omega-K}$ with $g_1=g-H_f$ on K.

3.4 *Reproducing functions.*

The space $HD_0=\{u\in HD;\ u(a_0)=0\}$ is a Hilbert space with respect to the inner product $<u_1, u_2>$. The mapping $u\to u(a)$ is linear and bounded on HD_0 for each $a\in\Omega$. Hence there exists $u_a\in HD_0$ such that $<u_a, u>=u(a)$ for all $u\in HD_0$. Then $<u_a, u>=u(a)-u(a_0)$ for any $u\in HD$.

PROPOSITION 2. *The reproducing function u_a has a normal derivative $1-\chi_a$ on $\varDelta$.*

PROOF. If $f\in R_D(\varDelta)$, then

$$<u_a, H_f> = H_f(a)-H_f(a_0) = -\int f(1-\chi_a)d\mu.$$

COROLLARY. *$n_a=(1/q)G_a+u_a$ has a normal derivative $\equiv 1$ on $\varDelta$. $\nu_a=(1/q)(G_a-G_{a_0})+u_a$ has a normal derivative $\equiv 0$ on $\varDelta$.*

§ 4. The Neumann problem and the third boundary value problem

4.1 *The Neumann problem.*

If $u\in HD$ has a normal derivative φ on $\varDelta$, then it must satisfy $\int\varphi\, d\mu=0$.

We seek the converse problem, i.e., the Neumann problem. We have the following (cf. [4]):

THEOREM 4. *If* $\varphi \in L^2(\mu)$ *and* $\int \varphi \, d\mu = 0$, *then there exists a unique function* $u \in H_D(\Omega^*)$ *such that* $u(a_0)=0$ (or = any given real number) *and* φ *is a normal derivative of* u.

PROOF. Let $H_0 = H_D(\Omega^*) \cap HD_0$. By Theorem 1, H_0 is a Hilbert space as a closed subspace of HD_0. For any $v \in H_0$, there is $f \in R_D(\Delta)$ such that $v = H_f$. Then $\int \varphi f \, d\mu$ is finite. We see that $v \to -\int \varphi f \, d\mu$ is a well-defined mapping on H_0 (i.e., the value $\int \varphi f \, d\mu$ is independent of the choice of f for a fixed v) and is linear. By Lemma 3, we have

$$\left| \int \varphi f \, d\mu \right| \leq \sqrt{\int \varphi^2 \, d\mu} \cdot \sqrt{\int f^2 \, d\mu} \leq \sqrt{M} \sqrt{\int \varphi^2 \, d\mu} \, \|v\|.$$

Hence the mapping above is bounded on H_0. Therefore, there is $u \in H_0$ such that $\langle v, u \rangle = -\int \varphi f \, d\mu$. Now, for any $f \in R_D(\Delta)$, $H_f - H_f(a_0) \in H_0$. Hence $\langle u, H_f \rangle = \langle u, H_f - H_f(a_0) \rangle = -\int \varphi (f - H_f(a_0)) \, d\mu = -\int \varphi f \, d\mu$.

If $u_1 \in H_D(\Omega^*)$ is another solution, then $\langle u - u_1, H_f \rangle = 0$ for every $f \in R_D(f)$. Hence, in particular, $\langle u - u_1, u - u_1 \rangle = 0$, which implies $u = u_1$.

4.2 *Uniqueness of the Neumann problem in HD.*

In general, we don't have uniqueness of the solution in *HD* (up to an additive constant) for the Neumann problem.

THEOREM 5. *There exists no non-constant HD-function with a normal derivative zero if and only if* Ω^* *is a D-normal compactification.*

PROOF. If $u \in HD$ has a normal derivative zero, then $\langle u, H_f \rangle = 0$ for all $f \in R_D(\Delta)$. Hence, if $H_D(\Omega^*) = HD$, then u is a constant. If $H_D(\Omega^*)$ is not equal to *HD*, then there exists a non-constant *HD*-function u_0 such that $\langle u_0, H_f \rangle = 0$ for all $f \in R_D(\Delta)$.

By this theorem, we have the uniqueness for the Neumann problem in *HD* if and only if Ω^* is a *D*-normal compactification.

4.3 THEOREM 6. *If* Ω^* *is a D-normal compactification, then the family*

$\{u \in HD;\, u$ has a normal derivative $\varphi \in L^2(\mu)\}$ *is dense in HD.*

PROOF. Let ND be the family. Suppose ND is not dense in HD. Then there is $v \in HD$ such that $v(a_0)=0$, $v \neq 0$ and $<v, u> = 0$ for all $u \in ND$. Since Ω^* is D-normal, $v=H_f$ for some $f \in R_D(\Delta)$. Lemma 3 implies $f \in L^2(\mu)$ and since $v \neq 0$, $\int f^2 d\mu \neq 0$. On the other hand, $\int f\, d\mu = H_f(a_0)=0$. Hence, by Theorem 4, there is $u \in ND$ such that f is a normal derivative of u. Then

$$0 = <u, v> = <u, H_f> = -\int f^2\, d\mu \neq 0,$$

which is a contradiction. Hence, ND is dense in HD.

4.4 *The operator B.*

LEMMA 7. *If* $f \in R_D(\Delta)$, $u=H_f$ *has a normal derivative* $\varphi \in L^2(\mu)$ *and* $u(a_0)=0$, *then*

$$\|u\|^2 \leq M \int \varphi^2\, d\mu,$$

$$\int f^2\, d\mu \leq M^2 \int \varphi^2\, d\mu.$$

($M=M_{a_0}$ in Lemma 3.)

PROOF. By Lemma 3, $\int f^2 d\mu \leq M\|u\|^2$. Hence, $\|u\|^2 = <u, H_f> = -\int \varphi f d\mu$ $\leq \sqrt{\int \varphi^2\, d\mu} \cdot \sqrt{\int f^2\, d\mu} \leq \sqrt{M}\, \|u\| \cdot \sqrt{\int \varphi^2\, d\mu}$.

Hence we have the first inequality. Applying the above inequality again, we have the second inequality.

Now, let Ω^* be a D-normal compactification. Let

$$L = \{\varphi \in L^2(\mu);\ \int \varphi\, d\mu = 0\}.$$

We identify functions which are equal μ-almost everywhere. Then L becomes a Hilbert space with respect to the inner product $(f, g) = \int fg\, d\mu$. For any $\varphi \in L$, there is a unique $u_\varphi \in HD$ (Theorems 4 and 5) such that φ is a normal derivative of u_φ and $u_\varphi(a_0)=0$. Since we have assumed that Ω^* is D-normal,

$u_\varphi = H_f$ for some $f \in R_D(\Delta)$. Then $f \in L$, so that $\varphi \to f$ is a mapping from L into itself. It is clear that this mapping is linear. The above lemma implies that this mapping is bounded.

Proposition 3. *The mapping $B: \varphi \to f$ is a symmetric, negative definite bounded operator of L into itself.*

Proof. We have seen that B is a bounded operator. For any $\varphi_1, \varphi_2 \in L$,

$$(B\varphi_1, \varphi_2) = \int (B\varphi_1)\varphi_2 d\mu = - \langle u_{\varphi_1}, u_{\varphi_2} \rangle.$$

Hence, B is symmetric and $(B\varphi, \varphi) = -\|u_\varphi\|^2 \leq 0$, so that B is negative definite.

4.5 *The third boundary value problem.*

Theorem 7. *Let Ω^* be a D-normal compactification of Ω. Let α be a positive real number. Given $\varphi \in L^2(\mu)$, there exists $f \in R_D(\Delta)$, uniquely determined μ-almost everywhere, such that $\alpha f + \varphi$ is a normal derivative of H_f.*

Proof. Let $\varphi_0 = \varphi - \int \varphi \, d\mu$. Then $\varphi_0 \in L$, so that $-(1/\alpha)B\varphi_0 \in L$. By the above proposition, $B-(1/\alpha)I$ is an invertible operator on L. Hence, there exists $f_0 \in L$ such that $\left(B - \frac{1}{\alpha}\right) f_0 = -\frac{1}{\alpha} B\varphi_0$. There exists a resolutive function which is equal to f_0 μ-almost everywhere. Hence, we may assume that f_0 is resolutive. By definition, $B\varphi_0$ (resp. Bf_0) is resolutive and φ_0 (resp. f_0) is a normal derivative of $H_{B\varphi_0}$ (resp. H_{Bf_0}). Now, let $f = f_0 - \frac{1}{\alpha}\int \varphi d\mu$. Then f is resolutive, $f \in L^2(\mu)$ and

$$\begin{aligned} H_f &= H_{f_0} - \frac{1}{\alpha}\int \varphi \, d\mu \\ &= H_{f_0 - B\varphi_0} + H_{B\varphi_0} - \frac{1}{\alpha}\int \varphi \, d\mu \\ &= \alpha H_{Bf_0} + H_{B\varphi_0} - \frac{1}{\alpha}\int \varphi \, d\mu \in HD. \end{aligned}$$

Furthermore, $\alpha f_0 + \varphi_0 = (\alpha f + \int \varphi \, d\mu) + (\varphi - \int \varphi \, d\mu) = \alpha f + \varphi$ is a normal derivative of H_f.

If f_1 is another function such that $\alpha f_1 + \varphi$ is a normal derivative of H_{f_1}, then $f_1+\frac{1}{\alpha}\int\varphi\,d\mu\in L$ and $B(\alpha f_1+\varphi)=f_1$ (in L). Hence

$$\left(B-\frac{1}{\alpha}\right)\left(f_1+\frac{1}{\alpha}\int\varphi\,d\mu\right)=-\frac{1}{\alpha}B\varphi_0.$$

Since $B-(1/\alpha)I$ is one-to-one on L, we have

$$f_0=f_1+\frac{1}{\alpha}\int\varphi\,d\mu \quad\text{or}\quad f=f_1 \text{ (in } L).$$

REMARK. We may be able to extend this theorem to the case α is a μ-measurable positive function on Δ. Cf. Doob [4].

§ 5. Functions with normal derivative zero

5.1 *Function f^K.*

Let f be a BLD-function on Ω and K be a non-polar compact set in Ω. Then there exists a uniquely determined BLD-function f^K which minimizes $\|g\|$ among BLD-functions g such that $f=g$ q.p. on K (Cf. [3] and [5]). f^K is harmonic outside K. Let $D^K=\{f$; BLD-function on Ω such that $f=0$ q.p. on $K\}$. Then $f=f^K$ if and only if f is orthogonal to D^K.

THEOREM 8. *Let Ω^* be a resolutive compactification of Ω and let $u\in HD(\Omega-K_0)$ for some compact set K_0.*

(i) *If $u=u^K$ for some non-polar compact set K containing K_0 in its interior, then u has a normal derivative zero on Δ.*

(ii) *If Ω^* is D-normal and u has a normal derivative zero on Ω, then $u=u^K$ for any non-polar compact set K containing K_0 in its interior.*

PROOF. (i) For any $f\in R_D(\Delta)$, $v_{f,K}\in D^K$ (Lemma 5). Hence, $u=u^K$ implies $\langle u, v_{f,K}\rangle_{\Omega-K}=0$, so that zero is a normal derivative of u.

(ii) Suppose u has a normal derivative zero on Δ and let K be any non-polar compact set containing K_0 in its interior. Let $g\in D^K$ and let $g=h+g_0$ be the Royden decomposition of g. Ω^* being D-normal, there is $f\in R_D(\Delta)$ such that $h=H_f$. Then $H_{f_K}^{(\Omega-K)^*}$ is the harmonic part of g on $\Omega-K$. Hence $\langle g, u\rangle_{\Omega-K}=\langle v_{f,K}, u\rangle_{\Omega-K}=0$. Hence, u is orthogonal to D^K, so that $u=u^K$.

5.2 *A mixed boundary value problem.*

THEOREM 9. *Let K be a compact set in Ω with non-empty interior. Suppose a BLD-function f on Ω and $\varphi\in L^2(\mu)$ are given. Then, there exists $u\in HD(\Omega-$*

K) such that φ is a normal derivative of u on Δ and

$$u^* = \begin{cases} u & \text{on} \quad \Omega - K \\ f & \text{on} \quad K \end{cases}$$

is a BLD-function on Ω.[2] *u is unique if Ω^* is D-normal.*

PROOF. Let a_1 be an interior point of K. Then the Green function G_{a_1} is bounded on $\Omega - K$. Let $\alpha = \sup_{b \in \Omega - K} G_{a_1}(b)$ and let $g_0 = \min(\alpha, G_{a_1})$. Then, g_0 is a BLD-function on Ω and $g_0|_{\Omega-K} \in HD(\Omega - K)$.

Let $c = \int \varphi d\mu$ and $\psi = \varphi - c\chi_{a_1}$. Then, $\psi \in L^2(\mu)$ and $\int \psi d\mu = 0$. By Theorem 4, there is $u_\psi \in HD$ such that ψ is a normal derivative of u_ψ. Now, let $v = \left(f - u_\psi - \frac{c}{q} g_0\right)^K$. Then $v|_{\Omega-K} \in HD(\Omega - K)$ and v has a normal derivative zero on Δ by Theorem 8, (i). Hence $u_1 = u_\psi + \frac{c}{q} g_0 + v$ is a BLD-function such that $u_1|_{\Omega-K} \in HD(\Omega - K)$ and $\psi + \frac{c}{q}(q\chi_{a_1}) = \varphi$ is a normal derivative of u_1. Furthermore, $u_1 = f$ q.p. on K. Hence if we take $u = u_1|_{\Omega-K}$, then it is the required function.

If $\tilde{u}$ is another solution, then $u^* - \tilde{u}^* \in D^K$ and it has a normal derivative zero on Δ. Hence, if Ω^* is D-normal, then $u^* = \tilde{u}^*$ by Theorem 8, (ii), or $u = \tilde{u}$.

5.3 *An application.*

We have seen that the Kuramochi compactification Ω_N^* is D-normal. On the other hand, we have seen that $\nu_a = \frac{1}{q}(G_a - G_{a_0}) + u_a$ has a normal derivative zero on Δ_N for each $a \in \Omega$. By Theorem 8, (ii) and by the definition of the Kuramochi boundary (cf. [3] and [5]), ν_a can be continuously extended over Δ_N. We denote the extended function on Δ_N by ν_a^*.

LEMMA 8. *$\{\nu_a^*; a \in \Omega\}$ separates points of Δ_N.*

PROOF. Let $\xi_1, \xi_2 \in \Delta_N$ and $\xi_1 \neq \xi_2$. Then there exists a continuous function f on Ω_N^* such that

1) f is a twice continuously differentiable BLD-function on $\check{\Omega}$,
2) $f = f^K$ for some non-polar compact set K in Ω,

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2) If f is continuous on ∂K, then the condition that u^* is a BLD-function is equivalent to the condition that $\lim_{a \to b, a \in \Omega - K} u(a) = f(b)$ for any regular boundary point b of $\Omega - K$.

3) $f(\xi_1) \neq f(\xi_2)$.

Since we may assume ∂K does not contain points of infinity, we can choose f in such a way that $f \equiv$const. on a neighborhood of each point of infinity in the interior of K and also on a neighborhood of a_0. Then $\Delta f \neq 0$ only on a compact set A of finite volume. From property 2) of f, it follows that $\int_A \Delta f(a) dv(a) = 0$. We can see that the function

$$h(b) = f(b) + \int_A \nu_a(b) \Delta f(a) dv(a)$$

is harmonic on Ω and has a normal derivative zero on Δ_N. (Cf. [3], p. 170) Hence, $h \equiv$const. $=c$ by Theorem 5. Thus,

$$f(b) = -\int_A \nu_a(b) \Delta f(a) dv(a) + c.$$

As $b \to \xi \in \Delta_N$, $\nu_a(b) \to \nu_a^*(\xi)$ uniformly for $a \in A$. Hence,

$$f(\xi) = -\int_A \nu_a^*(\xi) \Delta f(a) dv(a) + c.$$

Since $f(\xi_1) \neq f(\xi_2)$, there exists $a \in A$ such that $\nu_a^*(\xi_1) \neq \nu_a^*(\xi_2)$.

COROLLARY. *The $\{\nu_a\, ;\, a \in \Omega\}$-compactification coincides with the Kuramochi compactification.*

THEOREM 10. *Let $U = \{u_a;\, a \in \Omega\}$, where u_a is the reproducing function defined in 3.4. Then $U \subseteq H_C(\Omega_N^*)$ and $H_C(\Omega_N^*)$ is the smallest subspace of HB with the properties that it contains constants and U, it is closed under $\vee$ and $\wedge$ operations*[3] *and closed under uniform convergence.*

PROOF. Since ν_a is bounded outside a compact set, u_a is a bounded function on Ω for each $a \in \Omega$. Thus, it is obvious that

$$u_a - \frac{1}{q} G_{a_0} \in \underline{\mathfrak{S}}_{\nu_a^*} \qquad \text{and} \qquad u_a + \frac{1}{q} G_a \in \overline{\mathfrak{S}}_{\nu_a^*}.$$

Hence,

$$u_a - \frac{1}{q} G_{a_0} \leq H_{\nu_a^*} \leq u_a + \frac{1}{q} G_a,$$

3) If $u, v \in HP$, $u \vee v$ is the least harmonic majorant of max (u, v). Similar for $u \wedge v$.

so that $u_a = H_{\nu_a^*} \in H_C(\Omega_N^*)$.

Let $\mathfrak{H}$ be the smallest subspace of HB with the properties given in the theorem. Since $H_C(\Omega_N^*)$ satisfies the properties, we have $\mathfrak{H} \subseteq H_C(\Omega_N^*)$.

Let $\tilde{C} = \{f \in C(\Delta_N);\, H_f \in \mathfrak{H}\}$. Then the properties of $\mathfrak{H}$ imply that $\tilde{C}$ contains constants, $\{\nu_a^*;\, a \in \Omega\} \subseteq \tilde{C}$, $\tilde{C}$ is closed under max. and min. operations and under uniform convergence. Since $\{\nu_a^*;\, a \in \Omega\}$ separates points of Δ_N (by the previous lemma), we conclude that $\tilde{C} = C(\Delta_N)$ by the Stone-Weierstrass theorem. Hence, $\mathfrak{H} = H_C(\Omega_N^*)$.

Corollary. *The U-compactification Ω_U^* is regular and D-normal.*

Proof. Since $U \subset HD$, Ω_U^* is regular. It follows from the theorem that $H_C(\Omega_U^*) = H_C(\Omega_N^*)$. Hence $H_D(\Omega_U^*) = H_D(\Omega_N^*) = HD$, i.e., Ω_U^* is D-normal.

Remark. It can be seen that Ω_U^* is also metrizable.

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