J. Sci. Hiroshima Univ. Ser. A-I
28 (1964), 101-112

Perspectivity of Points in Matroid Lattices

Fumitomo Maeda

(Received August 24, 1964)

1. Introduction

In the theory of modular lattices, the perspectivity plays an important role. We have the theorem of perspective mappings, the comparability theorem and the theorem of distributivity and perspectivity (cf. [6] Theorems (1.1), (1.2) and (1.3)). But for the non-modular lattices, the properties of perspectivity are unknown.

U. Sasaki and S. Fujiwara [12] proved that in the matroid lattices the perspectivity of points is transitive. In this paper, starting from this significant fact, I investigate some properties of perspectivity in matroid lattices, and obtain the theorem of distributivity and perspectivity ((4.9) below) and the comparability theorem ((5.4) below). But the theorem of perspective mappings is as yet unsolved, even if we use the symmetric perspectivity (cf. (2.12) below).

In this paper, I treat the matroid lattices from the standpoint of atomistic symmetric lattices.

2. Symmetric lattices and matroid lattices.

In this paper we deal with a given lattice L with 0.

Definition (2.1). Let $a, b \in L$. $(a, b)M$ means $(c \cup a) \cap b = c \cup (a \cap b)$ for every $c \leqq b$, and $a \perp b$ means $a \cap b = 0$ and $(a, b)M$. If $a \perp b$ implies $b \perp a$, then L is called a *symmetric lattice* (cf. [13] p. 495). If $(a, b)M$ implies $(b, a)M$, then L is called a *M-symmetric lattice* (cf. [14] p. 453). And if $a \cap b \neq 0$ implies $(a, b)M$, then we call L a *weakly modular lattice* (cf. [1] p. 68).

A lattice L is called *left complemented* if $a, b \in L$ implies the existence of b_1 such that

$$a \cup b = a \cup b_1, \qquad a \cap b_1 = 0, \qquad (b_1, a)M, \qquad b_1 \leqq b$$

(cf. [14] p. 453).

Lemma (2.2). *In a symmetric lattice L, the binary relation "$\perp$" satisfies* *the following axioms*:

$$(\perp 1) \quad a \perp a \quad \textit{implies} \quad a = 0;$$

$$(\perp 2) \quad a \perp b \quad \textit{implies} \quad b \perp a;$$

$$(\perp 3) \quad a \perp b, \quad a_1 \leqq a \quad \textit{imply} \quad a_1 \perp b;$$

$$(\perp 4) \quad a \perp b, \quad a \cup b \perp c \quad \textit{imply} \quad a \perp b \cup c.$$

PROOF. $(\perp 1)$ and $(\perp 2)$ are evident. $(\perp 3)$ and $(\perp 4)$ follow from [13] Theorem 1.1 and Lemma 1.3 respectively.

REMARK (2.3). Lemma (2.2) means that in a symmetric lattice L "$\perp$" is the semi-orthogonality in the sense of [7] and [8]. Hence we have the following definition.

DEFINITION (2.4). A subset S of a symmetric lattice L is called a *semi-orthogonal family*, in notation $(a; a \in S)\perp$, if for any pair of disjoint finite subsets F_1, F_2 of S, it holds that $\bigvee(a; a \in F_1) \perp \bigvee(a; a \in F_2)$.

REMARK (2.5). In a symmetric lattice L, if every finite subset of S is a semi-orthogonal family then so is S, and if $a_1 \cup \cdots \cup a_i \perp a_{i+1}$ for every $i=1, \cdots, n-1$, then $(a_i; 1 \leqq i \leqq n)\perp$ (cf. [8] p. 372).

In a symmetric lattice L, for a point p, $a \perp p$ is equivalent to $a \cap p = 0$. Hence by above, $(p_i; 1 \leqq i \leqq n)\perp$ if and only if $(p_1 \cup \cdots \cup p_i) \cap p_{i+1} = 0$ for every $i=1, \cdots, n-1$.

Reference. The above defined "semi-orthogonal family" means the symmetrically independent family in the sense of [13].

REMARK (2.6). When a symmetric complete lattice L is upper continuous, i.e. $a_\delta \uparrow a$ implies $a_\delta \cap b \uparrow a \cap b$, then since $a_\delta \uparrow a$ and $(a_\delta, b)M$ for every δ imply $(a, b)M$, the binary relation "$\perp$" satisfies furthermore the following axiom:

$$(\perp 5) \quad \text{If } a_\delta \uparrow a \text{ and } a_\delta \perp b \text{ for every } \delta \text{ then } a \perp b.$$

And if S is a semi-orthogonal family in L and S_1, S_2 are disjoint subsets (not necessarily finite) of S, then $\bigvee(a; a \in S_1) \perp \bigvee(a; a \in S_2)$. Cf. [8] p. 380.

REMARK (2.7). Since a left complemented lattice L is M-symmetric (cf. [14] p. 454), it is a relatively semi-orthocomplemented lattice in the sense of [7] and [8]. Hence we have the following theorem from [7] Theorem 3.

THEOREM (2.8). *Let L be a left complemented complete lattice and Z be its center.*

(I) *Z is a complete Boolean sublattice of L.*

(II) *When $a_\alpha \in Z$ for all $\alpha \in I$ or $b \in Z$,*

$$\bigvee(a_\alpha;\ \alpha \in I) \cap b = \bigvee(a_\alpha \cap b;\ \alpha \in L).$$

REMARK (2.9). The above two properties (I) and (II) mean that L is a Z-lattice in the sense of [4] Definition 1.1. From (I) we can define the *central cover* $e(a)$ of a as the least central element z such that $a \leqq z$, and we have the following lemma as in [3] p. 69 Hilfssatz 4·7 (IV).

LEMMA (2.10). *In a left complemented complete lattice L,*

$$e\big(\bigvee(a_\alpha;\ \alpha \in I)\big) = \bigvee\big(e(a_\alpha);\ \alpha \in I\big).$$

DEFINITION (2.11). Let a, b be elements in a lattice L. If there exists $x \in L$ such that

$$(1) \qquad a \cup x = b \cup x, \qquad a \cap x = b \cap x = 0,$$

then we say that a and b are *perspective* and write $a \sim_x b$ or simply $a \sim b$.

REMARK (2.12). In the symmetric lattice L, if we require that the perspectivity implies the equi-dimensionality, instead of (1) we must use the following condition:

$$(2) \qquad a \cup x = b \cup x, \qquad a \perp x, \quad b \perp x.$$

In this case, we may say that a and b are *symmetrically perspective* (or *s-perspective*) and write $a \overset{s}{\sim}_x b$ or simply $a \overset{s}{\sim} b$.

For the perspectivity of points p and q, these two definitions coincide, since $p \cap x = 0$ is equivalent to $p \perp x$.

LEMMA (2.13). *Let L be a lattice and Z be its center, when $a \sim b \leqq z$ and $z \in Z$, then $a \leqq z$,*

PROOF. Cf. [3] p. 31 Hilfssatz 3.8 (III).

LEMMA (2.14). *In a left complemented complete lattice L,*

$$a \sim b \qquad \textit{implies} \qquad e(a) = e(b).$$

PROOF. Since $a \sim b \leqq e(b)$, from (2.13) we have $a \leqq e(b)$. Hence $e(a) \leqq e(b)$. Similarly $e(b) \leqq e(a)$.

LEMMA (2.15). *In a symmetric lattice L, the following conditions hold:*

(η') *If p, q are points such that* $a \cap q = 0$ *and* $q \leqq a \cup p$, *then* $p \leqq a \cup q$.

(η'') *If p is a point and* $a \cap p = 0$, *then* $a \lessdot a \cup p$.

($a \lessdot a \cup p$ means that $a \cup p$ covers a.)

PROOF. For the proof of (η''), cf. [6] (2.3). (η') follovs directly from (η''), since $a < a \cup q \leqq a \cup p$.

REMARK (2.16). The matroid lattice can be defined in many ways (cf. [5] (1.8)). D. Sachs defined the matroid lattice as an atomistic, upper continuous lattice L which satisfies (η''), and proved that it is left complemented (cf. [9] pp. 330–331), hence as (2.7) it is symmetric. Therefore combining with (2.15), we may give the following definition (cf. [5] (1.4) and (1.5)).

DEFINITION (2.17). An atomistic, upper continuous, symmetric lattice is called a *matroid lattice*.

THEOREM (2.18). *A matroid lattice L is modular, if and only if L satisfies the following condition:*

(L) *If* $p \leqq q \cup a$ *where p, q are points, then there exists a point r such that* $p \leqq q \cup r$ *and* $r \leqq a$.

PROOF. (i) Necessity. Cf. [3] p. 76 Hilfssatz 2.8.

(ii) Sufficiency. I will show $(a, b)M$, it is evident that

$$(c \cup a) \cap b \geqq c \cup (a \cap b) \qquad \text{for} \quad c \leqq b.$$

Let p be any point such that $p \leqq (c \cup a) \cap b$. Since $p \leqq c \cup a$, by [5] (1.5), there is a finite set of points $q_i (i = 1, \cdots, n)$ such that $q_i \leqq c$ and

$$p \leqq q_1 \cup \cdots \cup q_n \cup a.$$

Hence by (L), there exists a point r_1 such that

$$p \leqq q_1 \cup r_1 \qquad \text{and} \qquad r_1 \leqq q_2 \cup \cdots \cup q_n \cup a.$$

Apply (L) again, then there exists a point r_2 such that

$$r_1 \leqq q_2 \cup r_2 \quad \text{and} \quad r_2 \leqq q_3 \cup \cdots \cup q_n \cup a.$$

And so on. Lastly we have a point r_n such that

$$r_{n-1} \leqq q_n \cup r_n \quad \text{and} \quad r_n \leqq a.$$

Then, putting $r = r_n$, we have

$$p \leqq q_1 \cup \cdots \cup q_n \cup r \quad \text{and} \quad r \leqq a.$$

Consequently by (L) we have a point q such that

$$p \leqq q \cup r \quad \text{and} \quad q \leqq q_1 \cup \cdots \cup q_n \leqq c, \quad r \leqq a.$$

When $p = q$, then $p \leqq c \leqq c \cup (a \cap b)$.

When $p \neq q$, by (γ') in (2.15), we have $r \leqq p \cup q$. Since $p \leqq b$ and $q \leqq c \leqq b$, we have $r \leqq b$. Hence $r \leqq a \cap b$. Therefore $p \leqq q \cup r \leqq c \cup (a \cap b)$.

Consequently $(c \cup a) \cap b \leqq c \cup (a \cap b)$, and $(a, b)M$ holds.

Reference. (2.18) is essentially due to [2] p. 194.

THEOREM (2.19). *A matroid lattice L is weakly modular, if and only if L satisfies the following condition:*

(SP) *If $p \leqq q \cup a$ and $r \leqq a$, where p, q, r are points, then there exists a point s such that $p \leqq q \cup r \cup s$ and $s \leqq a$.*

PROOF. Cf. [10] p. 232 and [11] p. 414.

3. Perspectivity of points.

LEMMA (3.1). *Let p, q be points in a symmetric lattice L. Then $p \sim_x q$ if and only if*

$$(1) \qquad q \leqq p \cup x \quad \textit{and} \quad q \cap x = 0.$$

PROOF. Necessity is evident from Definition (2.11).

Sufficiency. From (1), by (2.15) (γ') we have $p \leqq q \cup x$. Hence $p \cup x = q \cup x$. If $p \leqq x$, then $q \leqq x$ which contradicts $q \cap x = 0$. Hence $p \cap x = 0$. Consequently $p \sim_x q$.

Reference. In a general lattice L, if (1) holds, then we say that p is subperspective to q. (Cf. [3] p. 71.)

LEMMA (3.2). *Let* $p, q_1, \cdots, q_n$ *be points in a symmetric lattice* L. *If*

$$p \leqq q_1 \cup \cdots \cup q_n \cup x \qquad \textit{and} \qquad p \cap x = 0,$$

then p *is perspective to one of* $q_1, \cdots, q_n$.

PROOF. When $p \cap (q_2 \cup \cdots \cup q_n \cup x) = 0$, by (3.1) we have $p \sim q_1$. When $p \leqq q_2 \cup \cdots \cup q_n \cup x$, then as above $p \sim q_2$ or $p \leqq q_3 \cup \cdots \cup q_n \cup x$. And so on. Hence p is perspective to some q_i.

LEMMA (3.3). *In a matroid lattice* L, *if* $a \sim b$ *and* p *is a point with* $p \leqq a$, *then there exists a point* q *such that* $p \sim q$ *and* $q \leqq b$.

PROOF. Since $a \sim b$, there exists x such that

$$a \cup x = b \cup x, \qquad a \cap x = b \cap x = 0.$$

Since $p \leqq a \cup x = b \cup x$, by [5] (1.5), there exist points q_i $(i=1, \cdots, n)$ such that $q_i \leqq b$ and

$$p \leqq q_i \cup \cdots \cup q_n \cup x.$$

Since $p \cap x \leqq a \cap x = 0$, from (3.2) p is perspective to one of $q_1, \cdots, q_n$.

THEOREM (3,4). *A matroid lattice* L *is modular, if and only if* L *satisfies the following condition*:

(L_1) *If* $p \sim_x q$ *where* p, q *are points, then there exists a point* r *such that* $p \sim_r q$ *and* $r \leqq x$.

PROOF. By (2.18) the modularity of L is equivalent to the following condition:

(L) If $p \leqq q \cup x$, then there exists a point r such that $p \leqq q \cup r$ and $r \leqq x$.

When $p \leqq x$, (L) always holds. Hence we may write (L) as follows:

(L') If $p \leqq q \cup x$, $p \cap x = 0$, then there exists a point r such that $p \leqq q \cup r$, $p \cap r = 0$ and $r \leqq x$.

By (3.1), (L') is equivalent to (L_1).

COROLLARY (3.5). *A matroid lattice L is modular, if and only if L satisfies the following condition:*

(L_2) *If $p\sim_x q$ where p, q are different points, then the line $p\cup q$ contains a third point r such that $r\leqq x$.*

PROOF. When $p=q$, (L_1) is evident. When $p\neq q$, by (2.15) (η'), $p\sim_r q$ is equivalent to $r\leqq p\cup q$ and $r\neq p$, $r\neq q$. Hence from (3.4), this corollary holds.

THEOREM (3.6). *A matroid lattice L is weakly modular, if and only if L satisfies the following condition:*

(SP_1) *If $p\sim_x q$ and $r\leqq x$ where p, q, r are points, then there exists either a point s such that $p\sim_s q$, $s\leqq x$, or a line l such that $p\sim_l q$, $r<l\leqq x$.*

PROOF. By (2.19) the weak modularity of L is equivalent to the following condition:

(SP) If $p\leqq q\cup x$ and $r\leqq x$, then there exists a point s such that $p\leqq q\cup r\cup s$ and $s\leqq x$.

When $p\leqq x$, (SP) always holds. Since $p\frown x=0$ implies $p\frown(r\cup s)=0$, we may write (SP) as follows:

(SP') If $p\leqq q\cup x$, $p\frown x=0$ and $r\leqq x$, then there exists a point s such that $p\leqq q\cup r\cup s$, $p\frown(r\cup s)=0$ and $r\cup s\leqq x$.

When $s=r$, then $p\sim_s q$; and when $s\neq r$, put $l=r\cup s$, then $p\sim_l q$ and $r<l\leqq x$. Hence (SP') is equivalent to (SP_1).

THEOREM (3.7). *A matroid lattice L is weakly modular, if and only if L satisfies the following condition:*

(SP_2) *If $p\sim_x q$ and $r\leqq x$ where p, q, r are points with $p\neq q$, then either the line $p\cup q$ contains a third point s such that $s\leqq x$, or there exists a line l such that $p\cup q\parallel l$ and $r<l\leqq x$.*

PROOF. (i) Necessity. From (SP_1), there exists either (α) a point s such that $p\sim_s q$, $s\leqq x$, or (β) a line l such that $p\sim_l q$, $r<l\leqq x$.

In the case (α), $p\cup q$ contains a third point s with $s\leqq x$. In the case (β), since $p\leqq q\cup l$, two lines $p\cup q$ and l are contained in the plane $q\cup l$. When $p\cup q$ and l intersect at a point s, then since $p\frown l=0$, we have $p\neq s$. Similarly $q\neq s$. Hence $p\cup q$ contains a third point s such that $s<l\leqq x$. When $p\cup q$ and l do not intersect, then $p\cup q\parallel l$ (Cf. [5] (2.2)).

(ii) Sufficiency. We shall prove (SP). When $p=q$ or $p\leqq x$, (SP) is evident. Hence assume $p \leqq q \cup x$, $p \cap x = 0$ and $r \leqq x$, that is $p \sim_x q$ and $r \leqq x$, where $p \neq q$. Then by (SP_2), either (α) the line $p \cup q$ contains a third point s such that $s \leqq x$, or (β) there exists a line l such that $p \cup q \parallel l$ and $r < l \leqq x$.

In the case (α), $p \leqq q \cup s \leqq q \cup r \cup s$, $s \leqq x$. In the case (β), there exists a point s such that $l = r \cup s$. Then from $p \cup q \parallel r \cup s$ we have $p \leqq q \cup r \cup s$, and $s < l \leqq x$.

THEOREM (3.8). *A weakly modular matroid lattice L is modular if every line in L has no parallel line.*

PROOF. This is evident from (3.5) and (3.7).

Reference. (3.8) is proved in [1] p. 307 using the hyperplane, and in [5] (2.6) using the projective space.

4. Center and perspectivity.

DEFINITION (4.1). Let a, b be elements of any lattice L. $(a, b)D$ means $(c \cup a) \cap b = (c \cup b) \cup (a \cup b)$ for every $c \in L$, and $a \nabla b$ means $a \cap b = 0$ and $(a, b)D$.

If S is any subset of L, denote by S^{∇} the set of a such that $a \nabla b$ for all $b \in S$.

DEFINITION (4.2). Let $\{S_\alpha;\ \alpha \in I\}$ be a family of subsets with 0 of a complete lattice L. If

(1°) every $a \in L$ is expressible in the form
$$a = \bigvee (a_\alpha;\ \alpha \in I), \qquad a_\alpha \in S_\alpha \text{ for all } \alpha \in I,$$
(2°) $\alpha \neq \beta$ implies $S_\beta \leqq S_\alpha^{\nabla}$,

then we say that L is a *direct sum* of $S_\alpha (\alpha \in I)$, and write $L = \dot{\bigvee}(S_\alpha;\ \alpha \in I)$.

LEMMA (4.3). *Let L be an upper continuous lattice.*

(I) $a_\delta \uparrow a$, $a_\delta \nabla b$ *for all* δ *imply* $a \nabla b$.

(II) *When* $L = \dot{\bigvee}(S_\alpha;\ \alpha \in I)$ *any element* $a \in L$ *is expressible uniquely as*

$$a = \bigvee (a_\alpha;\ \alpha \in I), \qquad a_\alpha \in S_\alpha (\alpha \in I), \tag{1}$$

and there exist central elements $e_\alpha (\alpha \in I)$, *such that*

$$1 = \bigvee (e_\alpha;\ \alpha \in I) \qquad \textit{and} \qquad a_\alpha = e_\alpha \cap a \quad (\alpha \in I).$$

Hence $L = \dot{\bigvee}(L(0, e_\alpha);\ \alpha \in I)$.

PROOF. (I) is [3] p. 23 Hilfssatz 2.5 (I).

First half of (II) is [3] p. 24 Hilfssatz 2.6. By [3] p. 21 Hilfssatz 2.3 and p, 23 Hilfssatz 2.5 (III), S_α is an ideal and complete sublattice of L, hence we have $S_\alpha=L(0, e_\alpha)$ where $e_\alpha=\bigvee(b; b \in S_\alpha)$. Since, by [3] p. 24 Satz 2.4, L is isomorphic to the product of $L(0, e_\alpha)\,(\alpha \in I)$, $e_\alpha(\alpha \in I)$ are central elements of L and $1=\bigvee(e_\alpha; \alpha \in I)$. By [3] p. 29 Hilfssatz 3.6, we have

$$a=\bigvee(e_\alpha; \alpha \in I) \cap a=\bigvee(e_\alpha \cap a; \alpha \in I).$$

Hence from the uniqueness of the expression (1), we have $a_\alpha=e_\alpha \cap a$.

LEMMA (4.4) *In a matroid lattice L, the following two propositions are equivalent.*

(α) $a \nabla b$.

(β) *There do not exist points p, q with $p \sim q$, $p \leqq a$, $q \leqq b$.*

PROOF. From (3.1), this lemma is a special case of [3] p. 72 Hilfssatz 1.3.

THEOREM (4.5). *In a matroid lattice L, the perspectivity of points is transitive, and L is a direct sum of irreducible sublatlice $S_\alpha(\alpha \in L)$ of L, that is $L=\dot{\bigvee}(S_\alpha; \alpha \in I)$. And any two points in the same S_α are perspective and two points which are contained in different S_α and S_β are not perspective.*

PROOF. Cf. [12] pp. 186–188 and (3.1).

REMARK (4.6). Applying (4.3) to the above direct sum decomposition of matroid lattice L, $L=\dot{\bigvee}(S_\alpha; \alpha \in I)$, there exist central elements $e_\alpha(\alpha \in I)$ such that $1=\bigvee(e_\alpha; \alpha \in I)$ and $S_\alpha=L(0, e_\alpha)\,(\alpha \in I)$, and any element $a \in L$ is expressible uniquely as

$$(1) \qquad a=\bigvee(a_\alpha; \alpha \in I), \qquad a_\alpha=e_\alpha \cap a \quad (\alpha \in I).$$

Since $S_\alpha=L(0, e_\alpha)$ is irreducible, e_α is a point in the center Z of L. If we denote the set of points $e_\alpha(\alpha \in I)$ by $\Omega(Z)$, then the center Z is isomorphic to the lattice of all subsets of $\Omega(Z)$.

LEMMA (4.7) *Let p, q be points in a matroid lattice L. Then $p \sim q$ if and only if $e(p)=e(q)$.*

PROOF. When $p \sim q$, by (2.14) we have $e(p)=e(q)$. Conversely $e(p)=e(q)$ means $e(p)=e(q)=e_\alpha$ for some α in (4.6). Hence by (4.5) we have $p \sim q$.

LEMMA (4.8). *In a matroid lattice L, if p is a point with* $p \leqq e(a)$, *then there exists a point q such that* $p \sim q$ *and* $q \leqq a$.

PROOF. Let $q_\gamma (\gamma \in J)$ be points such that $a = \bigvee(q_\gamma; \gamma \in J)$. Then by (2.10) we have $e(a) = \bigvee(e(q_\gamma); \gamma \in J)$. Since $p \leqq e(a)$, by (2.8) there exists γ such that $p \leqq e(q_\gamma)$. Hence $e(p) = e(q_\gamma)$ and by (4.7) we have $p \sim q_\gamma$.

THEOREM (4.9). (Distributivity and perspectivity). *Let a, b be elements in a matroid lattice L. Then the following three propositions are equivalent.*

(α) $a \nabla b$.
(β) *There do not exist nonzero elements* a_1, b_1 *with* $a_1 \sim b_1$, $a_1 \leqq a$, $b_1 \leqq b$.
(γ) $e(a) \frown e(b) = 0$

PROOF. (α)→(β). If there exist a_1, b_1 such that $0 < a_1 \leqq a$, $0 < b_1 \leqq b$ and $a_1 \sim b_1$, then from (3.3) for a point $p \leqq a_1$, there exist a point $q \leqq b_1$, such that $p \sim q$. By (4.4) this contradicts (α).

(β)→(γ). If there exists a point r such that $r \leqq e(a) \frown e(b)$, then by (4.8) there exist points p, q such that $r \sim p$, $p \leqq a$ and $r \sim q$, $q \leqq b$. By (4.5) $p \sim q$, which contradicts (β.)

(γ)→(α). If $a \nabla b$ does not hold, from (4.4) there exists points p, q such that $p \sim q$ and $p \leqq a$, $q \leqq b$. Then by (2.14) we have $e(p) = e(q)$. Hence $e(a) \frown e(b) \geqq e(p) \frown e(q) = e(p) > 0$, which contradicts ($\gamma$).

REMARK (4.10). Since (3.3) holds for $a \overset{s}{\sim} b$, we may write $a_1 \overset{s}{\sim} b_1$ instead of $a_1 \sim b_1$ in (4.9) (β).

Reference. (4.9) is a non-modular case of Theorem (1.3) in [6].

5. Point-weise perspectivity and comparability theorem.

DEFINITION (5.1). Let a be an element of a matroid lattice L and $\{p_i; i \in I\}$ be a semi-orthogonal family of points such that $a = \bigvee(p_i; i \in I)$, then $\{p_i; i \in I\}$ is called a *base* of a.

The existence of the base follows from Zorn's lemma.

DEFINITION (5.2). If two elements a, b of a matroid lattice L, have bases $\{p_i; i \in I\}$ and $\{q_i; i \in I\}$ respectively with the same cardinal number and $p_i \sim q_i$ for all $i \in I$, then we say that a and b are *point-weise perspective*, and write $a \overset{p}{\sim} b$. When $a = b = 0$, we write $a \overset{p}{\sim} b$.

LEMMA (5.3). *In a matroid lattice L, when* $a \overset{p}{\sim} b$, *we have* $e(a) = e(b)$.

PROOF. From (2.14), in (5.2) we have $e(p_i)=e(q_i)$ for all $i \in I$. Hence from (2.10) we have $e(a)=e(b)$.

THEOREM (5.4). (Comparability theorem). *Let a, b be any elements in a matroid lattice L. Then there exist a', a'', b', b'' such that*

$$(1^\circ) \qquad a=a' \cup a'', \qquad a' \perp a'',$$
$$b=b' \cup b'', \qquad b' \perp b'',$$
$$(2^\circ) \qquad a' \overset{p}{\sim} b' \qquad \text{and} \qquad e(a'') \cap e(b'')=0.$$

In this case $e(a')=e(b')=e(a) \cap e(b)$.

PROOF. Let T be the set of all pairs (p_α, q_α) of points such that $p_\alpha \leqq a$, $q_\alpha \leqq b$ and $p_\alpha \sim q_\alpha$. And let S be a subset of T, such that

(I) when (p_α, q_α) and (p_β, q_β) are different elements in S, then both $p_\alpha \neq p_\beta$ and $q_\alpha \neq q_\beta$ hold;

(II) $\{p_\alpha; (p_\alpha, q_\alpha) \in S\}$ and $\{q_\alpha; (p_\alpha, q_\alpha) \in S\}$ are semi-orthogonal families.

Denote by Φ the set of S which satisfies (I) and (II). Then by (2.5), a subset S of T belongs to Φ if and only if all finite subsets of S belong to Φ. Therefore by Zorn's lemma, there exists a maximal set S^* in Φ.

Put $a'=\bigcup(p_\alpha; (p_\alpha, q_\alpha) \in S^*)$ and $b'=\bigcup(q_\alpha; (p_\alpha, q_\alpha) \in S^*)$. Then by (5.20) we have $a' \overset{p}{\sim} b'$.

By (2.16), since L is left complemented and symmetric, there exist a'' and b'' such that

$$a=a' \cup a'', \qquad a' \perp a'',$$
$$b=b' \cup b'', \qquad b' \perp b''.$$

If $e(a'') \cap e(b'') \neq 0$, by (4.4) and (4.9), there exist points p_γ, q_γ such that $p_\gamma \leqq a''$, $q_\gamma \leqq b''$ and $p_\gamma \sim q_\gamma$. By (2.5), this contradicts the maximability of S^*. Hence $e(a'') \cap e(b'')=0$.

By (5.3) and (2.10), we have

$$e(a')=e(b'), \qquad e(a)=e(a') \cup e(a''), \qquad e(b)=e(b') \cup e(b'').$$

Hence $e(a) \cup e(b)=\{e(a') \cup e(a'')\} \cap \{e(a') \cup e(b'')\}=e(a') \cup \{e(a'') \cap e(b'')\}=e(a')$.

REMARK (5.5). Since matroid lattices are relatively semi-orthocomplemented upper continuous lattices (cf. (2.7)), we can apply the dimension theory [8] to matroid lattices, using the point-weise perspectivity "$\overset{p}{\sim}$".

Bibliography

1. M. L. Dubreil-Jacotin, L. Lesieur and R. Croisot, *Leçons sur la théorie des treillis des structures algébriques ordonnées et des treillis geométriques*. Paris, 1953.
2. B. Jónsson, *Lattice-theoretic approach to projective and affine geometry*, Proc. Internat. Sympos. Axiomatic Method (1957-1958), 188-203. Amsterdam, 1959.
3. F. Maeda, *Kontinuierliche Geometrien*. Berlin, 1958.
4. ———, *Decomposition of general lattices into direct summands of types* I, II *and* III, J. Sci. Hiroshima Univ. Ser. A, **23** (1959), 151-170.
5. ———, *Modular centers of affine matroid lattice*, J. Sci. Hiroshima Univ. Ser. A-I, **27** (1963), 73-84.
6. ———, *Parallel mappings and comparability theorem in affine matroid lattices*, J. Sci. Hiroshima Univ. Ser. A-I, **27** (1963), 85-94.
7. S. Maeda, *On relatively semi-orthocomplemented lattices*, J. Sci. Hiroshima Univ. Ser. A, **24** (1960), 155-161.
8. ———, *Dimension theory on relatively semi-orthocomplemented complete lattices*, J. Sci. Hiroshima Univ. Ser. A-I, **25** (1961), 369-404.
9. D. Sachs, *Partition and modulated lattices*, Pacific J. Math., **11** (1961), 325-345.
10. U. Sasaki, *Lattice theoretic characterization of an affine geometry of arbitrary dimensions*, J. Sci. Hiroshima Univ. Ser. A, **16** (1953), 223-238.
11. ———, *Semi-modularity in relatively atomic, upper continuous lattices*, J. Sci. Hiroshima Univ. Ser. A, **16** (1953), 409-416.
12. U. Sasaki and S. Fujiwara, *The decomposition of matroid lattices*, J. Sci. Hiroshima Univ. Ser. A, **15** (1952), 183-188.
13. L. R. Wilcox, *Modularity in the theory of lattices*, Annals of Math., (2) **40** (1939), 490-505.
14. ———, *A note on complementation in lattices*, Bull. Amer. Math. Soc., **48** (1942), 453-458.