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Notes on the Theory of Differential Forms on Algebraic Varieties

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This note contains two rather separate topics. The first theorem is a version of Lefschetz' theorem in the language of differential forms. The second one is a characterization of abelian subvariety of an abelian variety. They are a continuation of our preceding papers [2] and [3][1]. As addenda we shall give corrections to the cited papers [2] and [3].

§ 1. Isomorphism of j_Y^*.

We shall prove in this § the following

Theorem 1.1. *Let X^n be a non-singular projective variety and let Y be a non-singular irreducible hypersurface section of X of order m. Let j_Y be the injection $Y \rightarrow X$ and j_Y^* be its adjoint map $H^0(X, \Omega_X) \rightarrow H^0(Y, \Omega_Y)$, where Ω_X, Ω_Y are the sheaves of germs of regular differential forms of degree 1 on X and Y respectively. Then if $n \geqslant 3$ and m is sufficiently large j_Y^* is an isomorphism of $H^0(X, \Omega_X)$ and $H^0(Y, \Omega_Y)$.*

We have proved already in [2] that j_Y^* is an injective map provided m is sufficiently large (Theorem 5 in [2]). Hence to prove Theorem 1 it suffices to prove the following:

Proposition 1.2. *Let X be as in* Theorem 1 *and let $\mathcal{O}$ be the structure sheaf of X and let m_0 be an integer such that*

(1) $$H^i(X, \mathcal{O}_X(-m)) = 0 \quad for \quad m \geqslant m_0 \quad and \quad i = 1, 2.$$

(2) $$H^1(X, \Omega_X(-m)) = 0 \quad for \quad m \geqslant m_0.$$

If Y is a generic hypersurface section of order $\geqslant m_0$, then j_Y^* is a surjective map.

Proof. Let us denote by $\mathscr{P}$ the sheaf of ideals defined by Y, i.e., the sheaf of germs of rational functions f such that $(f) > Y$. As before let Ω_X, Ω_Y be the sheaves of germs of regular differential forms on X and Y respectively.

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1) The numbers in the bracket refer to the bibliography at the end of the paper.

Then we have the following commutative diagram of cohomology groups (Cf. §5 of [2]).

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & H^0(X, \mathscr{P}\Omega_X) & & & \\
 & & & \downarrow & & & \\
 & & & H^0(X, \Omega_X) & \searrow^{j_Y^*} & & \\
 & & & \downarrow & & & \\
0 \longrightarrow & H^0(X, \mathscr{P}/\mathscr{P}^2) & \longrightarrow & H^0(X, \mathcal{O}_Y \otimes \Omega_X) & \longrightarrow & H^0(Y, \Omega_Y) & \longrightarrow H^1(X, \mathscr{P}/\mathscr{P}^2) \\
 & & & \downarrow & & & \\
 & & & H^1(X, \mathscr{P}\Omega_X) & & &
\end{array}
$$

Hence j_Y^* is certainly surjective if we have

$$\text{(i)} \quad H^1(X, \mathscr{P}/\mathscr{P}^2) = 0$$

$$\text{(ii)} \quad H^1(X, \mathscr{P}\Omega_X) = 0$$

Now assume that Y is linearly equivalent to a hypersurface section of order m. Then as is seen easily $\mathscr{P}/\mathscr{P}^2$ is isomorphic to $\mathcal{O}_X(-m)/\mathcal{O}_X(-2m)$ and $\mathscr{P}\Omega_X \cong \Omega_X(-m)$. The equivalence of conditions (2) and (ii) is visible. On the other hand we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(-2m) \longrightarrow \mathcal{O}_X(-m) \longrightarrow \mathcal{O}_X(-m)/\mathcal{O}_X(-2m) \longrightarrow 0$$

Hence if we have $H^1(X, \mathcal{O}_X(-m))=0$, $H^2(X, \mathcal{O}_X(-2m))=0$, then $H^1(X, \mathcal{O}_X(-m)/\mathcal{O}_X(-2m))=0$, i.e. the condition (i) follows from (1). q.e.d.

The existence of an integer m_0 satisfying the conditions of Proposition 1.2 follows from the general theory of algebraic coherent sheaves (Cf. [4]) and the assumption $n \geqslant 3$.

§ 2. A criterion of an abelian subvariety.

Let G be a group variety and let a, b be two points on G. Let T_a, T_b be tangent spaces to G at points a and b and let U_a and U_b be subspaces of T_a and T_b respectively. By the translation τ sending the point a to the point b, the tangent space T_a is mapped onto T_b and U_a is mapped onto a subspace $\tau(U_a)$ of T_b. If $\tau(U_a)=U_b$ we say that U_a and U_b are *parallel*. The main result in this paragraph is the following:

THEOREM 2.1. *Let A be an abelian variety and let X be a non-singular* *subvariety of A such that the tangent spaces to X at various points are parallel to each other. Then there exists an abelian subvariety B of A such that X is a translation of B*[2].

To prove the Theorem 2.1 we need several auxiliary results. Following the conventions used in [2] we shall denote by k the universal domain of our geometry. Let G be a group variety and let x be a point of G (rational over k) and let $(\mathcal{O}_x, \mathcal{M}_x)$[3] be the local ring of x on G. We shall denote by Ω_x the module of k-differentials of $\mathcal{O}_x$ and let $\Omega_G = \bigcup_{x \in G} \Omega_x$ be the sheaf of germs of regular differential forms of degree 1 on G. Then for any given element w_x of Ω_x there exists a unique left invariant differential form ω on G such that $1 \otimes \omega(x) = 1 \otimes w_x$ in $\mathcal{O}_x/\mathcal{M}_x \otimes \Omega_x$ (Th. 1 of [2]) which will be called the left invariant differential form associated with w_x. Let X be a non-singular subvariety of G and let ω be a left invariant differential form on G. Then we have $j_x^*(\omega)=0$ if and only if ω is orthogonal to the tangent space T_x of X at any point $x \in X$. The following proposition is a generalization of the Proposition 3 in [3].

Proposition 2.2. *Let G^n be a group variety and let X^r be a non-singular subvariety of G such that for any point x on X, the tangent space T_x to X at x is parallel to the one and the same tangent space T_0. Let $\omega_1, \cdots, \omega_r$ be r-independent left invariant differential forms on G such that $j^*(\omega_i) \neq 0$. Then the r-fold differential form $j^*(\omega_1) \wedge \cdots \wedge j^*(\omega_r)$ on X has no zero at all on X, where j^* is the adjoint map associated with injection map $X \to G$.*

Proof. Let $\Omega_1, \cdots, \Omega_n$ be left invariant differential forms on G. We shall show for any choice of indices $i_1, \cdots, i_r$ $(1 \leq i \leq n)$ the r-fold differential $\bigwedge_{\alpha=1}^{r} j^*(\Omega_{i_\alpha})$ can be written as $a\left(\bigwedge_{i=1}^{r} j^*(\omega_i)\right)$ with $a \in k$. In fact, as a basis of the left invariant differential forms, we can take, $\omega_1, \cdots, \omega_r$ and $\tau_1, \cdots, \tau_{n-r}$ such as $\tau_1, \cdots, \tau_{n-r}$ are contained in the orthogonal complement of T_0. Then $\Omega_i = \sum_{j=1}^{r} a_{ij}\omega_j + \sum_{s=1}^{n-r} b_{is}\tau_s$. Since $j^*(\tau_s)=0$ we see immediately the assertion with $a = \det |a_{ij}|$. Next we shall show that for any point x on X, there exist r-differential forms $\Omega'_i (1 \leq i \leq r)$ such that $\bigwedge_{i=1}^{r} j^*(\Omega'_i)$ is not 0 at x. Take for instance a system of local parameters $t_1, \cdots, t_r, t_{r+1}, \cdots, t_n$ such that the subvariety X is defined locally at x by the ideal $(t_{r+1}, \cdots, t_n)$, and let Ω'_i be left invariant differential forms associated with $1 \otimes dt_i$ at $x (1 \leq i \leq r)$. Then clearly we have $\bigwedge_{i=1}^{r} j^*(\Omega'_i)$ is not zero at x, a fortiori $\bigwedge_{i=1}^{r} j^*(\Omega'_i) \neq 0$ on X. The assertion

2) The case where $\dim X = 1$ is proved in [3], and this result was presented as a conjecture there. According to the Review (MR 28 #93), J. P. Serre obtained the affirmative answer soon after the publication of [3].

3) This means that O_x is a local ring with the maximal ideal $\mathcal{M}_x$.

now follows easily from these considerations.

COROLLARY 2.3. *Under the same assumptions as in* Proposition 2.2. *and assume moreover that G is an abelian variety, then the canonical divisor of X is the zero divisor.*

COROLLARY 2.4. *Under the same assumptions and notations, $j^*(\omega_1),\cdots, j^*(\omega_r)$ form a basis of $D_k(K)$ over K where K is the function field of X over k.*

PROPOSITION 2,5. *Let G and X be as is* Prop. 2.2. *and assume moreover that G is an abelian variety. Let ω be a differential form of the first kind on X, then ω has no zero on X.*

PROOF. In fact assume that ω has zero at the point x on X. Since $j^*(\omega_i)$ $(1\leq i\leq r)$ form a basis of the vector space $D_k(K)$ over K (where K is the function field of X over k) it is possible to find $r-1$ forms, say $j^*(\omega_i)$, $i=1,\cdots, r-1$, such that $\omega, j^*(\omega_1),\cdots, j^*(\omega_{r-1})$ form a K-basis of $D_k(K)$. Then $\omega\wedge j^*(\omega_1)\wedge \cdots\wedge j^*(\omega_{r-1})$ is not 0 and we see easily that the r-fold differential $\Omega= \omega\wedge j^*(\omega_1)\wedge\cdots\wedge j^*(\omega_{r-1})$ has 0 at the point x. Hence the divisor of the differential form Ω must contain a positive divisor. This is a contradiction to Corollary 2.3, and thereby the Proposition is proved.

PROPOSITION 2.6. *Let A be an abelian variety and let X be a non-singular subvariety of X and let j be the injection $X\longrightarrow A$. Then the adjoint map j^*: $H^0(A, \Omega_A)\longrightarrow H^0(X, \Omega_X)$ is surjective, and dim $H^0(X, \Omega_X)=\dim X$.*

PROOF. Let $\omega\in H^0(X, \Omega_X)$ and let x be an arbitrary point of X. Then $1\otimes\omega$ is not 0 in $\mathcal{O}'/\mathcal{M}'\otimes D(\mathcal{O}')$, where $(\mathcal{O}', \mathcal{M}')$ is the local ring of x on X. We shall denote by $(\mathcal{O}, \mathcal{M})$ the local ring of the point x on A and let $\mathcal{P}$ be the defining ideal of X in $\mathcal{O}$. Then $\mathcal{O}'=\mathcal{O}/\mathcal{P}$ and $\mathcal{M}'=\mathcal{M}/\mathcal{P}$. Since $(\mathcal{O}/\mathcal{P})\otimes D(\mathcal{O}) \longrightarrow D(\mathcal{O}')$ is surjective, $\mathcal{O}/\mathcal{M}\otimes D(\mathcal{O})\longrightarrow(\mathcal{O}'/\mathcal{M}')\otimes D(\mathcal{O}')$ is also surjective. Take an element w of $D(\mathcal{O})$ such that $1\otimes w$ is mapped onto $1\otimes\omega$. If we denote by Ω the left invariant differential associated with $1\otimes w$ we see easily that $j_X^*(\Omega)-\omega$ has 0 at x. $j_X^*(\Omega)-\omega$ is also a differential form of the first kind, hence $i_X^*(\Omega)-\omega=0$ on X by Prop. 2.5. proving the assertion.

PROOF of Theorem 2.1. Assume that X contains the neutral element. If we denote by q the dimension of the Albanese variety of X we know that dim $H^0(X, \Omega_X)\geqslant q$ ([1]). In our case we have $r=\dim H^0(X, \Omega_X)$ by Proposition 2.6. and hence $r\geqslant q$. Let B be the abelian subvariety of A generated by X, then there is a surjective homomorphism of the Albanese variety of X onto B, hence $q\geqq\dim B\geqq r$. Combining these inequalities we have $q=r$, i.e., X is itself the Albanese variety of X. q.e.d.

Bibliography

[1] Igusa, J-I. A fundamental inequality in the theory of the Picard Varieties, Proc. N.A.S., U.S.A. 41 (1955), 317-320.

[2] Nakai, Y. On the theory of differentials on algebraic varieties, J. Sci. Hiroshima Univ. Vol. 27 (1963), 7-34.

[3] Nakai, Y. Note on invariant differentials on abelian varieties, J. Math. Kyoto Univ. Vol. 3 (1963), 127-135.

[4] Serre, J. P. Faisceaux algébriques cohérénts, Ann. Math. 61 (1955), 197-278.

Addenda

1. Corrections to the paper [2].
   The curves which are denoted by Γ in Prop. 20, Th. 8 and Cor. 1 and by C in Theorem 9 should be non-singular.
2. Corrections to the paper [3].
   - p. 127, Abolish the sentence beginning at line 15 by the word "As a" and ending in the line 17 and footnote 2).
   - p. 130, line 2. Insert "if X is non-singular" after $i_X^*(\omega)=0$.
   - p. 131, line 11. Insert "non-singular" after "let X be a".
   - p. 131, line 8. Abolish "outside a bunch of subvarieties".
   - p. 133, line 12. Insert "if $\Gamma\oplus\Gamma$ is non-singular".