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Proof of Ohtsuka's Theorem on the Value of Matrix Games(*)

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We shall give a simple proof to Ohtsuka's theorem in the preceding paper (**) in the finite dimensional case.

Formula (1) in (**) for the finite dimensional case can be put in the following form:

Set

$v(A)$ = *the value of the zero-sum game with a real matrix* A *as its pay-off matrix in which the maximizing player controls the rows and the minizing player controls the columns*;

$\alpha(A)=\min v(B)$ *over all principal minor matrices* B *of a square matrix* A.

Then, we have, if we denote by A' *the transpose of* A,

$$\alpha(A)=\alpha(A').$$

Proof. It suffices to see $\alpha(A)\geqq\alpha(A')$ for any A. We shall proceed by induction on n.

The case $n=1$ is trivial. Assume the truth of the theorem for A of order lower than n, and consider the case of A of order n. Noting that $\alpha(A)\leqq\alpha(B)$ for any principal minor matrix B of A, we divide the discussion into two cases:

Case (I). $\alpha(A)=\alpha(B)$ for some proper principal minor matrix B of A. In this case, $\alpha(B)\geqq\alpha(B')$ by the assumed inductive hypothesis. Hence $\alpha(A)=\alpha(B)\geqq\alpha(B')\geqq\alpha(A')$, so that $\alpha(A)\geqq\alpha(A')$.

Case (II). $\alpha(A)<\alpha(B)$ for any proper principal minor matrix B of A. Then $\alpha(A)=v(A)$. Let $x'=(x_1, \cdots, x_n)$ and $y'=(y_1, \cdots, y_n)$ be optimal strategies of the maximizing player and the minimizing player, respectively, in the game with the pay-off matrix A. Then, if $v=v(A)$, we have by definition

$$\sum_{j=1}^{n} a_{ij} y_j \leqq v \qquad (i=1, \cdots, n), \tag{1}$$

$$y_j \geqq 0, \quad \sum_{j=1}^{n} y_j = 1 \qquad (j=1, \cdots, n), \tag{2}$$

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(**) An application of the minimax theorem to the theory of capacity, this Journal.

(3) $$\sum_{i=1}^{n} a_{ij}x_i \geqq v \qquad (j=1, \cdots, n),$$

(4) $$x_i \geqq 0, \quad \sum_{i=1}^{n} x_i = 1 \qquad (i=1, \cdots, n).$$

It will be seen that $y_j > 0$ $(j=1, \cdots, n)$. If we assume the contrary, then $J=\{j \mid y_j > 0\}$ is a non-empty proper subset of $\{1, \cdots, n\}$. Let $B=(a_{ij})\,(i, j \in J)$, which is a proper principal minor matrix. Then,

$$\alpha(B) \leqq v(B) \leqq \max_{i \in J} \sum_{j \in J} a_{ij}\, y_j \leqq \max_{1 \leqq i \leqq n} \sum_{j \in J} a_{ij}\, y_j \leqq v = v(A) = \alpha(A)$$

by (1), which contradicts the basic assumption of case (II). Hence $y_j > 0$ $(j=1, \cdots, n)$, so that equality holds in all the relations of (3); that is

$$\sum_{i=1}^{n} a_{ij}\, x_i = v \qquad (j=1, \cdots, n),$$

whence

$$v(A') \leqq \max_{1 \leqq j \leqq n} \sum_{i=1}^{n} a_{ij}x_i = v = v(A),$$

which proves $v(A) \geqq v(A')$. Hence $\alpha(A) = v(A) \geqq v(A') \geqq \alpha(A')$.

Therefore in both cases (I), (II) we have $\alpha(A) \geqq \alpha(A')$, Q.E.D.

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