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On the Connectedness Theorem on Schemes over Local Domains

By

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The Connectedness Theorem in algebraic geometry was first proved by Zariski, using the theory of holomorphic functions on an algebraic variety, and it was applied to show the Principle of Degeneration in [7]. Later on, Chow gave "a general Connectedness Theorem" which asserts essentially that the Connectedness Theorem on a projective scheme over a complete local domain holds true. Precisely let X be a projective scheme over $Y=\mathrm{Spec}(\mathfrak{O})$, where $\mathfrak{O}$ is a complete local domain. Then if X is connected, the fiber of X at the closed point of Y is also connected. On the other hand Grothendieck gave a generalization of this theorem to a proper prescheme over a locally noetherian prescheme Y with structure morphism f. He treated the case where the direct image $f_*(\mathcal{O}_X)$ is isomorphic to $\mathcal{O}_Y$ and applied it to the case where Y is the spectrum of a "*unibranche*" local domain (cf. (III, 4.3.) in [2]). In this paper we shall also give a generalization of the Connectedness Theorem on schemes over a complete local domain (Theorem 3). Although a complete local domain is "unibranche", our result is not merely a special case of his results but covers a little more, and moreover our method is direct and elementary compared with the elaborate one adopted in [2].

The first section is devoted to a summary of some basic results on proper schemes over a local domain. In §2 we shall show the equivalence of the following two properties (P_1) and (P_2) of a local domain $\mathfrak{O}$:

(P_1) *Let X be any integral scheme, proper and dominant over* $Y=\mathrm{Spec}(\mathfrak{O})$. *Then the fiber X_{y_0} of X at the closed point y_0 of Y is connected.*

(P_2) *Let X be any integral scheme of finite type and dominant over Y. Then X is proper over Y if the fiber X_{y_0} of Y at y_0 is non-empty and proper over* $\mathrm{Spec}(\mathfrak{O}/\mathfrak{m})$, *where $\mathfrak{m}$ is the maximal ideal of $\mathfrak{O}$.*

In other words (P_1) means that "the Connectedness Theorem" on a proper scheme over Y holds true, and (P_2) means that proper morphisms to Y are characterized by the fiber over the closed point y_0 of Y. Next we shall show that any complete local domain satisfies these two properties (P_1) and (P_2), using Chow's generalization of the Connectedness Theorem mentioned as above. From these results we shall obtain a generalization of the Connectedness Theorem to schemes over a local domain in §3. Lastly in §4 we shall generalize

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the Principle of Degeneration to a specialization of chains on an abstract algebraic variety which is not necessarily complete.

The formulation (P_2) arises from the following question raised by M. Miyanishi. *Is a* $\mathfrak{p}$*-simple* $\mathfrak{p}$*-variety* $(V, \bar{V})$ $\mathfrak{p}$*-complete, provided* $\bar{V}$ *is complete?* This is answered affirmatively in Theorem 4 and its Corollary. Miyanishi himself obtained another elementary proof for this question.

We shall borrow the notations and the terminologies from [2] as to the scheme theory, from [5] as to the theory of abstract algebraic varieties and from [6] as to the theory of models over discrete valuation rings.

§ 1. Preliminary results on proper schemes.

Let $\mathfrak{O}$ be a noetherian integral local domain with maximal ideal $\mathfrak{m}$, and put $Y=\mathrm{Spec}(\mathfrak{O})$. In the following we shall denote by X a Y-scheme of finite type. For convenience' sake we shall write down some basic results in the following lemmas.

LEMMA 1. *X is proper over Y if and only if each irreducible component of the reduced subscheme X_{red} of X is proper over Y (cf. (II, 5.4.5) and (II, 5.4.6) in [2]).*

LEMMA 2. *Let X be an integral scheme dominant over Y. Then X is proper over Y if and only if any valuation ring of the field of rational functions on X, which dominates a locality of Y, dominates a locality of X (cf. (II, 7.3.10) in [2]).*

LEMMA 3. *Let X be an integral scheme proper over Y. Then there exists an integral scheme projective over Y and a birational Y-morphism g of $\bar{X}$ onto X.*

This is a special case of the so-called Chow's lemma ((II, 5.6.1) in [2]). The proof of this case is given in the same way as the case of an algebraic variety over a field (cf. Lemma 7, p. 121 in [8]).

Let X be a Y-scheme and f the structure morphism of X into Y. Let y_0 be the closed point ef Y which corresponds to the maximal ideal $\mathfrak{m}$ of $\mathfrak{O}$.

LEMMA 4. *Let X be a proper scheme over Y and x be a point of X. Then there exists a point x' in the closed subset $f^{-1}(y_0)$ of X such that x' is a specialization of x.*

This is a special case of (II, 7.2.1) in [2], since a proper morphism is closed.

LEMMA 5. *Let X be an integral scheme of finite type and dominant over Y. Assume that there exist an integral scheme Y' and a surjective morphism g of Y' to Y, and that the fiber product $X'=X\times_Y Y'$ is proper over Y'. Then X is also proper over Y.*

PROOF. Let K and L be the field of rational functions on X and Y' respectively, and k the quotient field of $\mathfrak{O}$. Then K and L contain k. Since K is finitely generated over k, $K\otimes_k L$ is a noetherian ring and has no imbedded prime divisor of zero (cf. Lemma 5 in p. 368 of [3-II]). Let R be a valuation ring of K dominating a locality $\mathcal{O}_y$ at y on Y, and y' a point of Y' whose image in Y is y. Then the locality $\mathcal{O}_{y'}$ at y' dominates $\mathcal{O}_y$. It can be easily seen that $R\otimes_{\mathcal{O}_y}\mathcal{O}_{y'}$ is a subring of $K\otimes_k L=K\otimes_{\mathcal{O}_y}L$, since K and L are the quotient field of R and $\mathcal{O}_{y'}$ respectively. Let $\mathfrak{n}$ and $\mathfrak{m}'$ be the maximal ideals of R and $\mathcal{O}_{y'}$ respectively. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be the minimal prime divisors of zero in $K\otimes_k L$. If the ideal $(\mathfrak{n}, \mathfrak{m}', \mathfrak{p}_i\cap(R\otimes_{\mathcal{O}_y}\mathcal{O}_{y'}))$ in $R\otimes_{\mathcal{O}_y}\mathcal{O}_{y'}$ is equal to $R\otimes_{\mathcal{O}_y}\mathcal{O}_{y'}$ $(i=1, 2, \cdots, n)$, then there exist two elements m_i in $\mathfrak{m}'$ and n_i in $\mathfrak{n}$ such that $1+m_i+n_i$ is in $\mathfrak{p}_i$. Therefore there exist two elements m in $\mathfrak{m}'$ and n in $\mathfrak{n}$ such that $1+m+n=0$ in $R\otimes_{\mathcal{O}_y}\mathcal{O}_{y'}$. But this is a contradiction, since $R/\mathfrak{n}\otimes_{\mathcal{O}_y}\mathcal{O}_{y'}/\mathfrak{m}'=R/\mathfrak{n}\otimes_{\mathcal{O}_y/(\mathfrak{m}'\cap\mathcal{O}_y)}\mathcal{O}_{y'}/\mathfrak{m}'$ is not zero. Therefore the ideal $(\mathfrak{n}, \mathfrak{m}, \mathfrak{p}_i\cap(R\otimes_{\mathcal{O}_y}\mathcal{O}_{y'}))$ is not the unit ideal for some i. Put $\mathfrak{p}=\mathfrak{p}_i\cap(R\otimes_{\mathcal{O}_y}\mathcal{O}_{y'})$. Then $(R\otimes_{\mathcal{O}_y}\mathcal{O}_{y'})/\mathfrak{p}$ is an integral domain whose quotient field K' is equal to that of $(K\otimes_k L)/\mathfrak{p}_i$. It is easily seen that the canonical images of K and L in $K\otimes_k L/\mathfrak{p}_i$ is isomorphic to K and L respectively and that K' is the field of rational functions on an irreducible component X_1' of X' with the reduced structure. Since the ideal generated by $\mathfrak{n}$ and $\mathfrak{m}'$ in $(R\otimes_{\mathcal{O}_y}\mathcal{O}_{y'})/\mathfrak{p}$ is not the unit ideal, there exists a valuation ring R' of K' dominating both R and $\mathcal{O}_{y'}$. By assumption and Lemma 1, X_1' is proper over Y'. Therefore R' dominates a locality $\mathcal{O}_{x'}$ at a point x' on X_1'. On the other hand $\mathcal{O}_{x'}$ is a quotient ring of a ring which is generated by the image in $(K\otimes_k L)/\mathfrak{p}_i$ of an affine ring A in K corresponding to an affine open subset of X and the image of $\mathcal{O}_{y'}$. Then R' dominates the image of a locality $\mathcal{O}_x$ at a point x on X, and hence R dominates $\mathcal{O}_x$. This completes the proof by Lemma 2.

LEMMA 6. *Let X be an integral scheme of finite type over* $Y=\mathrm{Spec}(\mathfrak{O})$. *Assume that X is dominant over Y. Then there exists an integral scheme $\bar{X}$ of finite type over Y such that X is contained in $\bar{X}$ as an open subscheme.*

This is the main theorem in the paper [4].

LEMMA 7. *Let X be a proper scheme over* $Y=\mathrm{Spec}(\mathfrak{O})$ *such that the fiber X_{y_0} of X at the closed point y_0 of Y is connected. Then X is connected and X_{y_0} is proper over* $\mathrm{Spec}(\mathfrak{O}/\mathfrak{m})$.

PROOF. Let Z be any irreducible component of X. Then Z is proper over Y by Lemma 1 and hence $Z\cap f^{-1}(y_0)$ is non-empty by Lemma 4. Therefore it is easy to see that the connectedness of $f^{-1}(y_0)$ means that of X. On the other hand it is clear that X_{y_0} is proper over $\mathrm{Spec}(\mathfrak{O}/\mathfrak{m})$, since X_{y_0} is the fiber product of X and $\mathrm{Spec}(\mathfrak{O}/\mathfrak{m})$ over Y.

§ 2. Equivalence of two properties (P_1) and (P_2) of $Y=\mathrm{Spec}(\mathfrak{O})$.

In this section we shall show that two properties (P_1) and (P_2) of the spectrum $Y=\mathrm{Spec}(\mathfrak{O})$, which concern proper schemes over Y, are equivalent. Moreover if $\mathfrak{O}$ is a complete local domain, we see that these two properties are satisfied.

THEOREM 1. *$Y=\mathrm{Spec}(\mathfrak{O})$ satisfies (P_1) if and only if it satisfies (P_2).*

PROOF. Assume that Y satisfies (P_1). Let X be an integral scheme of finite type and dominant over Y such that the fibre X_{y_0} is proper over $\mathrm{Spec}(\mathfrak{O}/\mathfrak{m})$. If X is not proper over Y, then there exists a proper scheme $\bar{X}$ over Y such that X is contained in $\bar{X}$ as an open subscheme by Lemma 6. Let $\bar{f}$ be the structure morphism of $\bar{X}$ onto Y and f the restriction to X of $\bar{f}$. Then $\bar{f}^{-1}(y_0)$ is not equal to $f^{-1}(y_0)$. In fact let x be a point of $\bar{X}-X$. Then there exists a point x' in $\bar{f}^{-1}(y_0)$ such that x is a generalization of x' by Lemma 4. If x' is a point of X, then x must be in X. Therefore x' is not in $f^{-1}(y_0)=X\cap\bar{f}^{-1}(y_0)$. Now we show that $f^{-1}(y_0)$ is a closed subset of $\bar{f}^{-1}(y_0)$. Let x be a point of the closure of $f^{-1}(y_0)$ in $\bar{X}$. Then x is a specialization of a point x_1 in $f^{-1}(y_0)$. Therefore the locality O_{x_1} at x_1 of $\bar{X}$ is the quotient ring of the locality O_x at x of $\bar{X}$ with respect to a prime ideal $\mathfrak{p}$. It is clear that O_{x_1} is the locality at x_1 of X. Let Z be an irreducible component of X_{y_0} containing x_1. We may assume that x_1 is the generic point of Z. Denote by K the field of rational functions on X. Let R be a valuation ring of K dominating O_{x_1} and K' the residue field of R. Then K' contains the local domain $O_x/\mathfrak{p}$. Let $\bar{S}$ be a valuation ring of K' dominating $O_x/\mathfrak{p}$ and S the complete inverse image of $\bar{S}$ in R. Then it is easily seen that S is a valuation ring of K dominating O_x and that $\mathfrak{O}/\mathfrak{m}$ is dominated by $\bar{S}$. On the other hand there exists a point x' in Z such that the locality $O_{x',Z}$ at x' of the reduced scheme Z is dominated by $\bar{S}$ from the assumptions and Lemmas 1 and 2, since K' contains the field of rational functions on Z. Therefore we see that the locality $O_{x'}$ at x' of X is dominated by S. This means that x is equal to x', since $\bar{X}$ is an integral scheme. Hence x is a point of Z. This means that $f^{-1}(y_0)$ is a closed subset of $\bar{f}^{-1}(y_0)$. Now we see that $f^{-1}(y_0)$ is an open subset of $\bar{f}^{-1}(y_0)$ and hence that $\bar{f}^{-1}(y_0)$ is not connected. In fact $f^{-1}(y_0)$ is equal to $\bar{f}^{-1}(y_0)\cap X$ and X is an open subset of $\bar{X}$. However $\bar{f}^{-1}(y_0)$ is homeomorphic to $\bar{X}_{y_0}$ which is connected by (P_1). Therefore X must be proper over Y.

Conversely assume (P_2). Let X be an integral scheme proper and dominant over Y. If X_{y_0} is not connected, it is a disjoint sum of two closed subsets F_1 and F_2 of X_{y_0} which are not empty. Identifyng F_1 with the corresponding subset in X, F_1 is also a closed subset of X. Then the open subscheme $X-F_1$ is proper over Y by (P_2), since $(X-F_1)_{y_0}$ is a proper scheme over $\mathrm{Spec}(\mathfrak{O}/\mathfrak{m})$ having F_2 as the base space. Let R be a valuation ring of K dominating a locality O_x of X at a point x in F_1. Since X and $X-F_1$ are birational, R

dominates a locality $O_{x'}$ at a point x' of $X-F_1$ by Lemma 2. Then x must be x', since X is an integral scheme. This is a contradiction. Therefore X_{y_0} must be connected. This means that Y satisfies (P_1).

THEOREM 2. *If $\mathfrak{O}$ is a complete local domain, $Y=\mathrm{Spec}(\mathfrak{O})$ satisfies the properties (P_1) and (P_2).*

PROOF. By Theorem 1 it is sufficient to show that $Y=\mathrm{Spec}(\mathfrak{O})$ satisfies (P_1). Let X be an integral scheme proper and dominant over Y. Then, by Lemma 3, there exist a projective scheme $\bar{X}$ over Y and a birational morphism g of $\bar{X}$ onto X over Y. Let $\bar{f}$ be the structure morphism of $\bar{X}$ onto Y. Then we have $\bar{f}=f\circ g$, and hence $g(\bar{f}^{-1}(y_0))=f^{-1}(y_0)$. Since g is continuous, $f^{-1}(y_0)$ is connected if $\bar{f}^{-1}(y_0)$ is so. However $\bar{f}^{-1}(y_0)$ is connected by Theorem 3 in [1]. Since $f^{-1}(y_0)$ is homeomorphic to X_{y_0}, this completes the proof.

§ 3. A generalization of "the Connectedness Theorem" to schemes.

Now we give our main results in the following

THEOREM 3. *Let $\mathfrak{O}$ be a complete local domain, and X a scheme of finite type over $Y=\mathrm{Spec}(\mathfrak{O})$. Then the following conditions are equivalent;*

(i) *X is connected and the fiber X_{y_0} of X at the closed point y_0 of Y is non-empty and proper over* $\mathrm{Spec}(\mathfrak{O}/\mathfrak{m})$.

(ii) *X is proper over Y and the fiber X_{y_0} at y_0 is connected.*

PROOF. By Lemma 7 it is sufficient to see that (ii) is a consequence of (i). By Lemma 1 we may assume that X is reduced. Let $X_1, \cdots, X_n$ be the irreducible components of X. If X_1 intersects with $f^{-1}(y_0)$, the fiber X_{1,y_0} at y_0 of X_1 is proper over $\mathrm{Spec}(\mathfrak{O}/\mathfrak{m})$. In fact X_{1,y_0} is a closed subscheme of the fiber X_{y_0} at y_0 of X. Since the closure of the image of X_1 in Y by f is an irreducible closed subset of Y and X_1 is an integral scheme, there exists a prime ideal $\mathfrak{p}$ of $\mathfrak{O}$ such that X_1 is a scheme dominant over $Y'=\mathrm{Spec}(\mathfrak{O}/\mathfrak{p})$. Then it is casily seen that the fiber of X_1 at y_0 as a Y'-scheme is the same one as a Y-scheme. Since $\mathfrak{O}$ is complete, $\mathfrak{O}/\mathfrak{p}$ is also a complete local domain. Therefore X_1 is proper over $Y'=\mathrm{Spec}(\mathfrak{O}/\mathfrak{p})$ and hence over $Y=\mathrm{Spec}(\mathfrak{O})$, and X_{1,y_0} is connected by Theorem 2. This means that any component X_i which has non-empty intersection with $f^{-1}(y_0)$ is proper over Y and that $X_i\cap f^{-1}(y_0)$ is connected for such X_i. Assume that X_i is proper over Y and that $X_i\cap X_j$ is not empty. Let x be a point of $X_i\cap X_j$. Then, by Lemma 4, there exists a point x' in $f^{-1}(y_0)\cap X_i$ which is a specialization of x. Since X_j is closed in X and x is in X_j, x' is also in X_j. This means that X_j is also proper over Y and that $f^{-1}(y_0)\cap X_i\cap X_j$ is not empty. By assumptions $f^{-1}(y_0)$ is not empty and X is connected. Therefore all the components of X are proper over Y and hence X

is proper over Y by Lemma 1. It is also easily seen that $f^{-1}(y_0)$ is connected and hence that X_{y_0} is connected.

COROLLARY. *Let $\mathfrak{O}$ be a complete local domain. Then the fiber X_{y_0} of a proper scheme X over $Y=\mathrm{Spec}(\mathfrak{O})$ is connected if X is connected.*

REMARK: Let $\mathfrak{O}$ be a local domain satisfying (P_1) and (P_2). Then the above theorem and its corollary also hold true if any irreducible component of X is dominant over $Y=\mathrm{Spec}(\mathfrak{O})$. In fact the completeness of $\mathfrak{O}$ is necessary in the above proof only for the condition that the integral domain $\mathfrak{O}/\mathfrak{p}$ satisfies (P_1) and (P_2), where $\mathfrak{p}$ corresponds to an image in Y of an irreducible component of X.

THEOREM 4. *Let $(\mathfrak{O}, \mathfrak{m})$ and $(\mathfrak{O}', \mathfrak{m}')$ be two local domains such that $\mathfrak{O}$ is a subring of $\mathfrak{O}'$ and such that the canonical morphism g of $Y'=\mathrm{Spec}(\mathfrak{O}')$ to $Y=\mathrm{Spec}(\mathfrak{O})$ is surjective. Let X be a scheme of finite type over Y such that each irreducible component of X is dominant over Y. Assume that the fiber X_{y_0} of X at the closed point y_0 of Y is non-empty and proper over $\mathrm{Spec}(\mathfrak{O}/\mathfrak{m})$ and that the fiber product $X'=X\times_Y Y'$ is connected. Then if $\mathfrak{O}'$ satisfies (P_1) and (P_2), X is proper over Y and X_{y_0} is connected.*

PROOF. We may assume that X is reduced by Lemma 1. Let y'_0 be the closed point of Y' corresponding to the maximal ideal $\mathfrak{m}'$ of $\mathfrak{O}'$. Since g is surjective, $\mathfrak{O}'$ dominates $\mathfrak{O}$. Therefore we have the following commutative diagram

$$\begin{array}{ccccc}
X'_{y'_0}=X_{y_0}\times_Y\mathfrak{O}'/\mathfrak{m}' & \longrightarrow & X'=X\times_Y Y' & \xrightarrow{f'} & Y' \\
\downarrow & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\
X_{y_0}=X\times_Y\mathfrak{O}/\mathfrak{m} & \longrightarrow & X & \xrightarrow{f} & Y
\end{array}$$

since $X'_{y'_0}=(X\times_Y Y')\times_{Y'}\mathfrak{O}'/\mathfrak{m}'$ is equal to $X_{y_0}\times_Y\mathfrak{O}'/\mathfrak{m}'$. Therefore the fiber $X'_{y'_0}$ of X' at y'_0 is proper over $\mathrm{Spec}(\mathfrak{O}'/\mathfrak{m}')$ and hence our assumptions mean by the remark after Corollary of Theorem 3 that X' is proper over Y' and that $Y'_{y'_0}$ is connected, since it can be easily seen that any irreducible component of X' is dominant over Y'. Applying Lemma 5 to each irreducible compenent of X we can easily see that X is proper over Y. On the other hand let Z be the inverse image of $f^{-1}(y_0)$ under g'. To show the connectedness of $f^{-1}(y_0)$, it is sufficient to see that of Z. Since Z is a closed subset of a proper scheme X' over Y', any irreducible component of Z has non-empty intersection with $f'^{-1}(y'_0)$ by Lemma 4. This means that Z is connected because $f'^{-1}(y'_0)$ is connected as shown in the above. Therefore X_{y_0} is connected.

COROLLARY. *Let $\mathfrak{O}$ be a local domain such that the completion $\bar{\mathfrak{O}}$ of $\mathfrak{O}$ is* *also a local domain. Let X be an integral scheme of finite type and dominant over $Y=\mathrm{Spec}(\mathfrak{O})$ such that the field of rational functions of X is a primary extension of the quotient field of $\mathfrak{O}$. Then if the fiber X_{y_0} of X at the closed point y_0 of Y is proper over $\mathrm{Spec}(\mathfrak{O}/\mathfrak{m})$, X is proper over Y and X_{y_0} is connected.*

PROOF. Put $\bar{Y}=\mathrm{Spec}(\bar{\mathfrak{O}})$. Then the canonical morphism of $\bar{Y}$ into Y is surjective, since $\bar{\mathfrak{O}}$ is faithfully flat over $\mathfrak{O}$. Let $\bar{X}$ be the fiber product $X\times_Y\bar{Y}$. Then $\bar{X}$ is an irreducible scheme by the assumption. In particular $\bar{X}$ is connected. Therefore the corollary is a direct consequence of Theorem 4.

§ 4. Applications to specializations of chaisns on algebraic varieties.

In this section we shall generalize the Principle of Degeneration to a specialization of chains on an abstract algebraic variety which is not necessarily complete. For this purpose let us recall the definition of a model over a discrete valuation ring of rank 1 which plays an essential rôle in the theory of specialization of chains on algebraic varieties.

Let $\mathfrak{O}$ be a discrete valuation ring of rank 1 and k the quotient field of $\mathfrak{O}$. Let K be a finitely generated field over k. Then a model M of K over $\mathfrak{O}$ is, by definition, nothing else than an integral scheme of finite type and dominant over $\mathrm{Spec}(\mathfrak{O})$ such that the field of rational functions on M is k-isomorphic to K (cf. [6]). We denote by M_k the open subset of the points in M whose localities dominate k. In other words M_k is the fiber of M over the generic point of $\mathrm{Spec}(\mathfrak{O})$. Then the fiber of M over the closed point of $\mathrm{Spec}(\mathfrak{O})$ is $M-M_k$. Let F be a closed subset of M and $M(F)$ the closed subscheme of M with the reduced structure whose base space is F. We shall say that F is complete over $\mathfrak{O}$ if $M(F)$ is proper over $\mathrm{Spec}(\mathfrak{O})$ in the scheme theoretic sense.

Next we shall recall relations between models and algebraic varieties. Let V be an abstract algebraic variety defined over a field k_0, and k an extension of k_0 such that there is a discrete valuation ring $\mathfrak{O}$ of k containing k_0 in the unit group of $\mathfrak{O}$. Let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{O}$. Let (x) be a generic point of V over k and put $K=k(x)$. Then there exists a $\mathfrak{p}$-simple model M of K over $\mathfrak{O}$ such that the fiber M_k over k corresponds to V over k in the sense of §9, Chap. 2 in [3-I] and such that the fiber $M-M_k$ over $\mathfrak{O}/\mathfrak{p}$ corresponds to V over $\mathfrak{O}/\mathfrak{p}$ (cf. §5 in [6]). Moreover V is complete if and only if M is complete over $\mathfrak{O}$. We denote by $M(V,\mathfrak{O})$ this model M corresponding to V. Then a rational chain on V over k (resp. over $\mathfrak{O}/\mathfrak{p}$) corresponds naturally to a generalized cycle on the fiber M_k (resp. the fiber $M-M_k$).

Let Z and Z' be two positive chains on V. Then we say that Z' is a specialization of Z over k_0, if there exists a discrete valuation ring $\mathfrak{O}$ of rank 1 containing k_0 which satisfies the following conditions.

(i) Z is rational over the quotient field k of $\mathfrak{O}$.

(ii) Z' is rational over the residue field $\kappa=\mathfrak{O}/\mathfrak{p}$.
(iii) Put $M=M(V, \mathfrak{O})$, and let Z_M be the generalized cycle on M_k corresponding to k-rational chain Z on V. Then the generalized cycle $\rho(Z_M)$ on $M-M_k$ obtained from Z_M by the reduction modulo $\mathfrak{p}$ corresponds to the κ-rational chain Z' on V (cf. §3 in [6]).

Then Theorem 3 means the following

THEOREM 3'. *Let V be an abstract variety defined over k_0. Let Z and Z' be two positive chains on V such that Z' is a specialization of Z over k_0. Then if each component of Z' is complete and if the support of Z is connected in the absolute Zariski topology on V, the support of Z' is also connected and each component of Z is complete.*

In particular if V is complete, any component of Z' is complete. Therefore the connectedness of the support of Z means always that of Z'.

PROOF. By assumptions there exists a valuation ring $\mathfrak{O}$ of rank 1 containing k_0 such that Z and Z' satisfy the conditions (i), (ii) and (iii). Then we may assume that $\mathfrak{O}$ is complete since the operation ρ is compatible with extension of ground rings (cf. Prop. 5 in [6]). Let F be the closure of the support of the generalized cycle Z_M on $M=M(V, \mathfrak{O})$ corresponding to the chain Z on V. Then it is seen that the closed subset $(M-M_k)\cap F$ is the support of the generalized cycle $\rho(Z_M)$ from the definition of ρ. Therefore we must show that $(M-M_k)\cap F$ is connected. Now we consider the scheme $M(F)$ over $\mathrm{Spec}(\mathfrak{O})$. Then the fiber $M(F)_{y_0}$ of $M(F)$ at the closed point y_0 of $\mathrm{Spec}(\mathfrak{O})$ is proper over $\mathrm{Spec}(\mathfrak{O}/\mathfrak{p})$, since each component of Z' is complete. The connectedness of $M(F)$ follows from that of the support of Z. Therefore $M(F)$ is proper over $\mathrm{Spec}(\mathfrak{O})$ and $(M-M_k)\cap F$ is connected by Theorem 3. This means that each component of Z is complete and that the support of Z' is connected.

Now we shall show a generalization of a result on connectedness of the carrier of an algebraic family, which is shown in [7] for an algebraic family of cycles in a projective space.

Let V and W be two abstract varieties defined over k_0, and $\mathfrak{F}$ an algebraic family of positive cycles on V parametrized by W. This means here that there exists a positive cycle Z on $V\times W$ rational over k_0, that Z and $V\times P$ intersect properly at every component of $Z\cap(V\times P)$ for any point P in W, and that $\mathfrak{F}$ consists of all the cycles $Z(P)$ for P in W where $Z(P)\times P=Z\cdot(V\times P)$.

If P is a generic point of W over k_0, then we call $Z(P)$ a generic member of $\mathfrak{F}$ over k_0. Let F be a k_0-closed subset of W, and denote by $\mathfrak{F}(F)$ all the cycles $Z(P)$ of $\mathfrak{F}$ such that P is in F. We call $\mathfrak{F}(F)$ the algebraic subfamily of $\mathfrak{F}$ parametrized by F. Moreover let $C(\mathfrak{F}(F))$ be the set of points Q in V such that Q is in the support of $Z(P)$ for a point P of F, and call it the carrier of $\mathfrak{F}(F)$. Then we have the following

THEOREM 5. *Let V and W be two abstract varieties defined over k_0 and $\mathfrak{F}$ an algebraic family of positive cycles on V parametrized by W. Let F be a k_0-connected closed subset of W and assume that the carrier $C(\mathfrak{F}(F))$ of the algebraic subfamily $\mathfrak{F}(F)$ is k_0-closed subset of V whose irreducible components are all complete. Then if the support of a generic member of $\mathfrak{F}$ is absolutely connected, $C(\mathfrak{F}(F))$ is k_0-connected.*

PEOOF. The proof is essentially the same as the one that was given by Zariski in [7]. But we repeat the proof for convenience' sake. If $C(\mathfrak{F}(F))$ is decomposed to two k_0-closed subset C_1 and C_2, let F_i be the geometric projection of W of $|Z|\cap(C_i\times F)$ $(i=1, 2)$. Then F_1 and F_2 are k_0-closed subsets and F is the union of F_1 and F_2. We show that F_1 and F_2 have no points in common. In fact let Q be a point of F. Then we can easily see that $Z(Q)$ is a specialization of a generic member of $\mathfrak{F}$ over k_0, since specialization and intersection of positive cycles are compatible (cf. Theorem 2 in [6]). Moreover each irreducible component of $Z(Q)$ is complete, since $Z(P)$ is contained in $C(\mathfrak{F}(F))$. Therefore $Z(Q)$ is absoluty connected by Theorem 3′, and hence the support of $Z(Q)$ is contained in one of the two closed subset C_1 and C_2. If $Z(Q)$ is in C_1, Q is in F_1 and not in F_2. This shows that F_1 and F_2 have no points in common. On the other hand F is k_0-connected by hypothesis. Therefore one of F_1 and F_2 is empty. This means that one of C_1 and C_2 is empty, and hence that $C(\mathfrak{F}(F))$ is k_0-connected.

REMARK: If V and W are complete, we can easily see that $C(\mathfrak{F}(F))$ is k_0-closed subset of V, and that all the irreducible components of $C(\mathfrak{F}(F))$ are complete. Therefore, in this case, the absolute connectedness of the support of a generic member in $\mathfrak{F}$ and k_0-connectedness of F means k_0-connectedness of the carrier $C(\mathfrak{F}(F))$.

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