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On the Symmetry of the Modular Relation in Atomic Lattices

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In some non-modular, atomic lattices, the covering property plays an important role instead of the modularity. For example, a geometric lattice, which is called a matroid lattice, is characterized as an upper continuous, relatively atomic lattice having the covering property (see [5] and [2; p. 264]). It is obvious that the symmetry of the modular relation implies the covering property, and it was proved by Sasaki [9] and Sacks [8] that in a matroid lattice the modular relation is symmetric. In other words, if a relatively atomic lattice is upper continuous, then "the covering property is equivalent to the symmetry of the modular relation". The main result of this paper (Theorem 2) is that the same statement holds if a relatively atomic lattice is orthocomplemented. We remark that the covering property has no meaning in non-atomic lattices. And, by the above consideration, it seems that in some non-atomic lattices the symmetry of the modular relation plays an important role like the covering property in atomic lattices.

§ **1. Definitions and preliminary lemmas.** In this section, we give some definitions and lemmas which are already known.

Definition 1. In a lattice L, we say that (a, b) is a *modular pair* and write $(a, b)M$ if $(c\cup a)\cap b=c\cup(a\cap b)$ for every $c\leqq b$. We say that (a, b) is a *dual modular pair* and write $(a, b)M^*$ if $(a\cup b)\cap c=a\cup(b\cap c)$ for every $c\geqq a$. A lattice L is called *M-symmetric* if in L the modular relation is symmetric, that is, $(a, b)M$ implies $(b, a)M$. L is called *symmetric* if $(a, b)M$, $a\cap b=0$ imply $(b, a)M$ (see [10]). Some authors ([4], [8]) call a M-symmetric lattice a semi-modular lattice.

Definition 2. Let L be a lattice with 0. In L, we say that b *covers* a and write $a\lessdot b$ if $a<b$ and there does not exist c with $a<c<b$. An element $p\in L$ is called a *point* (or an *atom*) if $0\lessdot p$. L is called *atomic* if every non-zero element contains a point. L is called *relatively atomic* if $a<b$ implies $a<a\cup p\leqq b$ for some point p. It is easy to show that L is relatively atomic if and only if every non-zero element of L is the join of some set of points.

Definition 3. We say that an atomic lattice L has the *covering property*, if the following statement holds in L:

(C) If p is a point and $p \nleqq a$ then $a \lessdot a \cup p$.

LEMMA 1. (i) *The covering property* (C) *is equivalent to each of the following statements.*

(C_1) *If p is a point then $(p, a)M$ for every a.*

(C_2) *If p is a point and $p \nleqq a \cup b$ then $a \cap b = (a \cup p) \cap b$.*

(ii) *If the lattice L is relatively atomic then moreover* (C) *is equivalent to each of the following statements.*

(C_3) *If p, q are points and $p \nleqq a$, $p \leqq a \cup q$ then $q \leqq a \cup p$ (exchange property).*

(C_4) *If $a \cap b \lessdot a$ then $b \lessdot a \cup b$.*

PROOF. The equivalence of (C_1) and (C) was proved by Theorem 1 of [2; p. 250]. The equivalence of (C_2), (C_3), (C_4) and (C) was proved by Theorem 2 of [5].

REMARK 1. Birkhoff's condition of semi-modularity is as follows ([1, Chap. VII]):

If $a \lessdot x$, $a \lessdot y$ and $x \neq y$, then $x \lessdot x \cup y$ and $y \lessdot x \cup y$. Here, it is easy to see that $a = x \cap y$. Hence (C_4) implies this semi-modularity.

LEMMA 2. *If an atomic lattice L is symmetric then it has the covering property.*

PROOF. If p is a point then $(a, p)M$ always holds. Since L is symmetric, $(p, a)M$ holds when $a \cap p = 0$. If $a \cap p \neq 0$, then $p \leqq a$, and hence $(p, a)M$ holds obviously. Therefore (C_1) holds.

REMARK 2. An upper continuous, relatively atomic lattice having the covering property is called a *matroid lattice* ([5]), and it was proved that a matroid lattice is M-symmetric ([9], Theorem 1; [8], Theorem 6) and hence symmetric. Therefore, if a relatively atomic lattice L is upper continuous then the following three statements are equivalent.

(α) L is M-symmetric.

(β) L is symmetric.

(γ) L has the covering property.

DEFINITION 4. Let L be an atomic lattice. An element of L is called a *line* if it is the join of two different points. An element of L is called *finite* if it is zero or is the join of a finite number of points.

LEMMA 3. *In a relatively atomic lattice with the covering property,*

(i) *if l is a line and $p \lessdot l$ then p is a point, and*
(ii) *all finite elements form an ideal.*

PROOF. These are consequences of Theorem 4.1 of [4] (it follows from Remark 1 that the assumption of the theorem is satisfied).

§ 2. Relatively atomic lattices with the covering property.

THEOREM 1. *If L is a relatively atomic lattice with the covering property, then, in L, the following five statements are equivalent.*

(α) *If l is a line, then $(a, l)M$ holds for every a.*

(β) *If l is a line and $a \lessdot a \cup l$, then $a \cap l \lessdot l$.*

(γ) *If p, q are points and $p \leqq q \cup a$ $(a \neq 0)$, then there exists a point r such that $p \leqq q \cup r$, $r \leqq a$.*

(δ) *If p is a point, a or b is finite and $p \leqq a \cup b$ $(a \neq 0,\ b \neq 0)$, then there exist two points q, r such that $p \leqq q \cup r$, $q \leqq a$, $r \leqq b$.*

(ε) *If a is finite then $(a, b)M$, $(b, a)M$, $(a, b)M^*$ and $(b, a)M^*$ hold for every b.*

PROOF. $(\alpha) \Rightarrow (\beta)$. Let $a \lessdot a \cup l$ and $a \cap l \leqq c \leqq l$. It follows from (α) that $(c \cup a) \cap l = c \cup (a \cap l) = c$. On the other hand, since $a \leqq a \cup c \leqq a \cup l$ we have $a \cup c = a$ or $a \cup l$. Hence $c = a \cap l$ or a, which means $a \cap l \lessdot l$ ($a \cap l < l$ is obvious).

$(\beta) \Rightarrow (\gamma)$. Let $p \leqq q \cup a$. When $p = q$, any point $r \leqq a$ is the required, and when $q \leqq a$, $r = p$ is the required. Hence, assume that $p \neq q$ and $q \nleqq a$. Then, since $a \cup (p \cup q) = a \cup q \gtrdot a$ by the covering property, it follows from (β) that $a \cap (p \cup q) \lessdot p \cup q$. By Lemma 3 (i), $r = a \cap (p \cup q)$ is a point and $r \leqq a$. Since $q \nleqq a$ we have $q \neq r$, and hence $r \leqq p \cup q$ implies $p \leqq q \cup r$ by (C_3).

$(\gamma) \Rightarrow (\delta)$. Let p be a point and $p \leqq a \cup b$. And we may assume that a is finite, that is, there is a finite number of points $q_1, \cdots, q_n$ such that $a = q_1 \cup \cdots \cup q_n$. Since $p \leqq q_1 \cup \cdots \cup q_n \cup b$, it follows from ($\gamma$) that there exists a point r_1 such that $p \leqq q_1 \cup r_1$, $r_1 \leqq q_2 \cup \cdots \cup q_n \cup b$. And, from ($\gamma$) again, there exists a point r_2 such that $r_1 \leqq q_2 \cup r_2$, $r_2 \leqq q_3 \cup \cdots \cup q_n \cup b$. Continuing this process, lastly we have a point r_n such that $r_{n-1} \leqq q_n \cup r_n$, $r_n \leqq b$. Then, we have $p \leqq q_1 \cup \cdots \cup q_n \cup r_n$. Hence, it follows from (γ) that there exists a point q such that $p \leqq q \cup r_n$, $q \leqq a$.

$(\delta) \Rightarrow (\varepsilon)$. It suffices to prove that $(a, b)M$ and $(a, b)M^*$ hold if a or b is finite. It is evident that $(c \cup a) \cap b \geqq c \cup (a \cap b)$ for $c \leqq b$. Let p be a point such that $p \leqq (c \cup a) \cap b$, and we shall show that $p \leqq c \cup (a \cap b)$. Since $p \leqq c \cup a$ and c or a is finite by Lemma 3 (ii), it follows from (δ) that there exist two point q, r such that $p \leqq q \cup r$, $q \leqq c$, $r \leqq a$. When $p = q$, then $p \leqq c \leqq c \cup (a \cap b)$. When $p \neq q$, by (C_3) we have $r \leqq p \cup q \leqq b \cup c = b$. Hence $r \leqq a \cap b$ and $p \leqq q \cup r \leqq c \cup (a \cap b)$. Therefore $(a, b)M$ holds since L is relatively atomic. Next, it is evident that $(a \cup b) \cap c \geqq a \cup (b \cap c)$ for $c \geqq a$. If p is a point such that $p \leqq$

$(a\cup b)\cap c$ then we can show that $p\leqq a\cup(b\cap c)$ by the same way as above, provided that a or b is finite. Therefore $(a, b)M^*$ holds.

$(\varepsilon)\Rightarrow(\alpha)$ is trivial. This completes the proof.

COROLLARY. *If L is a matroid lattice* (see Remark 2), *then the following five statements are equivalent.*

(α) *If l is a line, then $(a, l)M$ for every a.*

(β) *If l is a line and $a\lessdot a\cup l$, then $a\cap l\lessdot l$.*

(γ) *If p, q are points and $p\leqq q\cup a$ $(a\neq 0)$, then there exists a point r such that $p\leqq q\cup r$, $r\leqq a$.*

(δ) *If p is a point and $p\leqq a\cup b$ $(a\neq 0,\ b\neq 0)$, then there exist two points q, r such that $p\leqq q\cup r$, $q\leqq a$, $r\leqq b$.*

(ε) *L is modular.*

PROOF. The implications $(\varepsilon)\Rightarrow(\alpha)\Rightarrow(\beta)\Rightarrow(\gamma)$ and $(\delta)\Rightarrow(\varepsilon)$ are obvious from the proof of the theorem. To prove $(\gamma)\Rightarrow(\delta)$, we only remark the following fact: If $p\leqq a\cup b$ then it follows from the upper continuity of L and Lemma 3 (ii) that there exists a finite element a_1 such that $p\leqq a_1\cup b$, $a_1\leqq a$.

This corollary is an extension of [6], Theorem 2.18.

LEMMA 4. *Let L be a relatively atomic lattice with the covering property. And assume that the statement* (γ) *of Theorem* 1 *holds.*

(i) $(a, b)M^*\Leftrightarrow$ *if p is a point and $p\leqq a\cup b$ then there exist two points q, r such that $p\leqq q\cup r$, $q\leqq a$, $r\leqq b$ $(a\neq 0,\ b\neq 0)$.*

(ii) $(a, b)M^*\Leftrightarrow(b, a)M^*$.

PROOF. (i) $\Rightarrow$. Let $p\leqq a\cup b$ and put $c=a\cup p$. When $b\cap c\nleqq a$, we have a point r such that $r\leqq b\cap c$, $r\nleqq a$. Since $r\leqq c=a\cup p$, it follows from (γ) that there exists a point q such that $r\leqq q\cup p$, $q\leqq a$. Since $r\nleqq a$, we have $r\neq q$ and hence $p\leqq q\cup r$ by (C_3). Next, when $b\cap c\leqq a$, it follows from $(a, b)M^*$ that $a=a\cup(b\cap c)=(a\cup b)\cap c\geqq p$. Hence, any point $r\leqq b$ and $q=p$ have the desired property.

$\Leftarrow$. This implication can be proved by the same way as $(\delta)\Rightarrow(\varepsilon)$ in the proof of Theorem 1.

(ii) Since the right side of the statement (i) is symmetric in a, b, we have the symmetry of the dual modular relation.

LEMMA 5. *If both L and the dual of L are relatively atomic lattices with the covering property, then L is M-symmetric.*

PROOF. It follows from Lemma 1 that (C_4) holds in L. Hence, in the dual of L, $a\cup b\gtrdot a$ implies $b\gtrdot a\cap b$. This implies the statement (β) of Theorem 1, and hence (γ) of Theorem 1 holds in the dual of L. Thus, it follows from Lemma 4 that, in the dual of L, the dual modular relation is symmetric. Since

$(a, b)M^*$ in the dual of L is equivalent to $(b, a)M$ in L, L is M-symmetric.

REMARK 3. If L satisfies the condition of Lemma 5, then, in L, all the statements $(\alpha)-(\varepsilon)$ of Theorem 1 hold. Because, (C_4) holds in the dual of L, which means that $a\cup b > a$ implies $b > a\cap b$ in L. This implies the statement (β) and other statements of Theorem 1.

§ 3. Orthocomplemented, relatively atomic lattices.

In an orthocomplemented lattice, the orthocomplement of an element a is denoted by $a^\perp$.

THEOREM 2. *If L is an orthocomplemented, relatively atomic lattice, then the following three statements are equivalent.*

(α) *L is M-symmetric.*

(β) *L is symmetric.*

(γ) *L has the covering property.*

PROOF. $(\alpha)\Rightarrow(\beta)$ is obvious, and $(\beta)\Rightarrow(\gamma)$ follows from Lemma 2. $(\gamma)\Rightarrow(\alpha)$. Since the orthocomplementation $a\to a^\perp$ is a dual-automorphism of L, if L has the covering property, then L satisfies the condition of Lemma 5, and hence L is M-symmetric. This completes the proof.

This theorem is a generalization of the main part of the corollary of Theorem 4.4 of [4].

EXAMPLES. (i) The lattice of all closed subspaces of a Hilbert space is orthocomplemented, relatively atomic and has the covering property. Hence, it is M-symmetric.

(ii) The system of propositions in quantum theory is formulated by Piron [7] as a relatively orthocomplemented (=orthomodular), atomic, complete lattice with the covering property. Since a relatively orthocomplemented, atomic lattice is relatively atomic, the above lattice is M-symmetric.

REMARK 4. It follows from Remark 3 that, in an orthocomplemented, relatively atomic, symmetric lattice, all the statements (α)—(ε) of Theorem 1 hold. Especially, if a or $a^\perp$ is finite then $(a, b)M$ and $(b, a)M$ hold for every b.

THEOREM 3. *Let L be an orthocomplemented, relatively atomic, complete, symmetric lattice. L is continuous if and only if it is modular.*

PROOF. The "if" part is a consequence of the well known theorem of Kaplansky [3]. The "only if" part is proved by Remark 4 and the corollary of Theorem 1.

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