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Semi-modular Lie Algebras

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Introduction

If L is a Lie algebra over a field Φ, we consider the lattice of all subalgebras of L. A Lie algebra will be called distributive, modular, upper semi-modular or lower semi-modular if its lattice of subalgebras has the corresponding property. In this paper we investigate the relation between the structure of the Lie algebra and the structure of its lattice of subalgebras. Analogous work has been done for groups by many investigators, and Suzuki [7] has written a comprehensive monograph describing the significant results in this area.

The Lie algebras considered in this paper will be finite dimensional. Also, the Lie algebras will, unless otherwise stated, be over a field of characteristic zero.

In this paper, if L is a Lie algebra $[L, L]$ will be denoted by L', and $[L', L']$ by L''. Also the subalgebra of L generated by $e_1, e_2, \cdots, e_k$ will be denoted by $\{e_1, e_2, \cdots, e_k\}$.

In this paper, we 1) characterize upper semi-modular Lie algebras over fields of characteristic zero, 2) characterize modular Lie algebras over fields of characteristic zero, 3) characterize lower semi-modular Lie algebras over algebraically closed fields of characteristic zero, 4) study other properties of distributive, modular, upper semi-modular, lower semi-modular Lie algebras.

1. Preliminaries and examples

Definition: A Lie algebra L over a field of any characteristic is called distributive, modular, upper semi-modular or lower semi-modular if its lattice of all subalgebras has the corresponding property.

If a Lie algebra is distributive, modular, upper semi-modular or lower semi-modular, then a subalgebra or a factor algebra has the corresponding property.

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DEFINITION: An $n+1$-dimensional ($n \geq 1$) Lie algebra, over a field of any characteristic, is called almost abelian if it has a basis $e_0, e_1, \cdots, e_n$ such that $[e_i, e_0] = e_i$ for $i \geq 1$, and $[e_i, e_j] = 0$ for $i, j, \geq 1$.

PROPOSITION 1.1 Let L be an $n+1$-dimensional, $n \geq 1$, almost abelian Lie algebra over a field of any characteristic. Then

1) $L'' = 0$, and thus L is solvable.
2) The nil radical N of L is abelian.
3) Every subspace of L is a subalgebra.

PROOF:

1) Clear.
2) The nil radical N of L is $\{e_1, e_2, \cdots, e_n\}$.
3) Let M be a subspace of L, and let $a = a_0e_0 + a_1e_1 + \cdots + a_ne_n$, $b = b_0e_0 + b_1e_1 + \cdots + b_ne_n$, be two elements of M. Then $[a, b] = b_0a - a_0b \in M$.

PROPOSITION 1.2 Let L be an $n+1$-dimensional almost abelian or n-dimensional abelian ($n \geq 1$) Lie algebra, over a field of any characteristic. Then L is modular (L is a projective geometry).

PROOF:

In either case, every subspace is a subalgebra.

PROPOSITION 1.3 Let L be the nilpotent three-dimensional Lie algebra, over a field of any characteristic, with basis e_1, e_2, e_3 defined by $[e_1, e_2] = e_3$; $[e_1, e_3] = 0$; $[e_2, e_3] = 0$. Then 1) L is lower semi-modular; 2) L is not upper semi-modular.

PROOF:

1) This follows from Theorem 3.1.
2) Define the following subalgebras of L:

$A = \{e_1, e_3\}$; $B = \{e_1\}$; $C = \{e_2\}$. Then both B and C cover $\{0\}$, but $B \cup C = L \supsetneqq A \supsetneqq B$.

PROPOSITION 1.4 Let L be the solvable Lie algebra, over a field of any characteristic, with basis e_1, e_2, e_3 defined by $[e_1, e_2] = 0$; $[e_1, e_3] = e_1$; $[e_2, e_3] = \alpha e_2$, $\alpha \neq 1$. Then L is not upper semi-modular.

PROOF:

Define the following subalgebras of L:

$A = \{e_1 + e_2, e_1\}$; $B = \{e_1 + e_2\}$; $C = \{e_3\}$. Then both B and C cover $\{0\}$, but since $\alpha \neq 1$, $B \cup C = L \supsetneqq A \supsetneqq B$.

THEOREM 1.1 (Kuranishi) Let L be a semi-simple Lie algebra. Then there exist elements $x, y \in L$ generating L.

PROOF: See [6].

In Theorem 1.1, we may take x in a Cartan subalgebra H of L and y in the orthogonal complement of H with respect to the Killing form.

The following proposition appears to be known, although the author has been unable to find any published proof of it.

PROPOSITION 1.5 (Analogue of Ito's Theorem in Group Theory [4]). If L is a Lie ring, such that $L=N_1+N_2$, and N_1 and N_2 are abelian subrings, then $L''=0$.

PROOF:

Let $[a, b]$ and $[a', b'] \in L'$, where $a, a' \in N_1$, $b, b' \in N_2$. We have, by Jacobi's identity, $[[a', b'], [a, b]]=[a', [b', [a, b]]]+[[a', [a, b]], b']=[a', [[b', a], b]] +[[a, [a', b]], b']$. Now, since $[b', a]$ and $[a', b] \in L$, we can write $[b', a]= a''+b'''$ and $[a', b]=a'''+b''$, where $a'', a''' \in N_1$, and $b'', b''' \in N_2$. Then, $[[a', b'], [a, b]]=[a', [a'', b]]+[[a, b''], b']=[[b, a'], a'']+[[a, b'], b'']=-[b'', a'']-[a'', b'']=0$.

Hence, $L''=0$.

Let L be a simple three-dimensional Lie algebra. Then there exists a basis e_1, e_2, e_3 for L such that $[e_1, e_2]=e_3$; $[e_2, e_3]=\alpha e_1$; $[e_3, e_1]=\beta e_2$; $\alpha \neq 0$, $\beta \neq 0 \in \Phi$. The Killing form of L is $-2[\beta x_1 y_1+\alpha x_2 y_2+\alpha\beta x_3 y_3]$.

The question of when two simple three-dimensional Lie algebras are isomorphic is answered in [2], p. 133 as follows. Two simple three dimensional Lie algebras are isomorphic if and only if their repective Killing forms are equivalent up to a non-zero constant factor.

It is also known that L has no two-dimensional subalgebra if and only if L is non-split. Moreover, L is non-split if and only if there do not exist $a_1, a_2, a_3 \in \Phi$, not all zero such that $\beta a_1^2+\alpha a_2^2+\alpha\beta a_3^2=0$.

LEMMA 1.1 Let L be a simple three-dimensional Lie algebra. If $c=c_1e_1+c_2e_2+c_3e_3 \neq 0 \in L$, then there exists an element $x \in L$ such that c and x generate L.

PROOF:

Assume the contrary, and let $x=x_1e_1+x_2e_2+x_3e_3$. Then c, x and $[c, x]$ are linearly dependent for all $x \in L$. Hence,

$$\begin{vmatrix} \alpha(c_2x_3-c_3x_2) & c_1 & x_1 \\ \beta(c_3x_1-c_1x_3) & c_2 & x_2 \\ c_1x_2-c_2x_1 & c_3 & x_3 \end{vmatrix} \equiv 0$$

Since this holds for all x_i, we obtain the following relations.

$$\begin{array}{ll} c_2^2+\beta c_3^2=0 & 2c_1c_2=0 \\ c_1^2+\alpha c_3^2=0 & 2\beta c_1c_3=0 \\ c_1^2\beta+\alpha c_2^2=0 & 2\alpha c_2c_3=0 \end{array}$$

which imply that $c=0$, a contradiction.

PROPOSITION 1.6 Let L be a simple three dimensional Lie algebra. Then L is non-split if and only if L is upper semi-modular. Moreover, in this case L is also modular.

PROOF:

If L is non-split, then it does not contain a two-dimensional subalgebra and is thus upper semi-modular.

Conversely, suppose L is split. Then L contains a two-dimensional subalgebra $A=\{x, y\}$. Now by Lemma 1.1, there exists an element $z \in L$ such that x and z generate L. Since $A\cap\{z\}=0$, we have a contradiction to the upper semi-modularity of L.

DEFINITION A simple three-dimensional Lie algebra over a field of characteristic zero, satisfying any of the equivalent properties of Proposition 1.6 is called a special simple Lie algebra.

2. Upper Semi-modular Lie Algebras

We first establish the following theorem, which characterizes distributive Lie algebras over fields of any characteristic.

THEOREM 2.1 Let L be a Lie algebra over a field of any characteristic. Then L is distributive if and only if L is one-dimensional.

PROOF:

If L is one-dimensional, it is of course, distributive. Now suppose dim $L\geq 2$. If L is two-dimensional, then L is abelian or almost abelian, and the lattice is a projective geometry, which cannot be distributive.

If dim $L>2$, let e_1, e_2 be two linearly independent elements of L, and let

$e_3=[e_1, e_2]$. If e_3 is a linear combination of e_1 and e_2 then $\{e_1, e_2\}$ is a two-dimensional subalgebra of L, and thus L is not distributive. If e_3 is not a linear combination of e_1 and e_2, we define the the following subalgebras of L: $A=\{e_3\}$, $B=\{e_1\}$, $C=\{e_2\}$. Then $A\cap(B\cup C)\supset A$, and $(A\cap B)\cup(A\cap C)=0$. Thus, again L is not distributive.

Proposition 2.1 If L is an upper semi-modular Lie algebra, then either L is solvable or L/S, where S is the radical of L, is special simple.

Proof:

Assume that L is upper semi-modular and set $L_1=L/S$. If rank $L_1\geq 2$, then dim $H\geq 2$, where H is a Cartan subalgebra of L_1. By Theorem 1.1, there exist $x\in H$, $y\notin H$ such that x and y generate L_1. We thus have the following sublattice of the lattice of all subalgebras of L (Fig. 1)

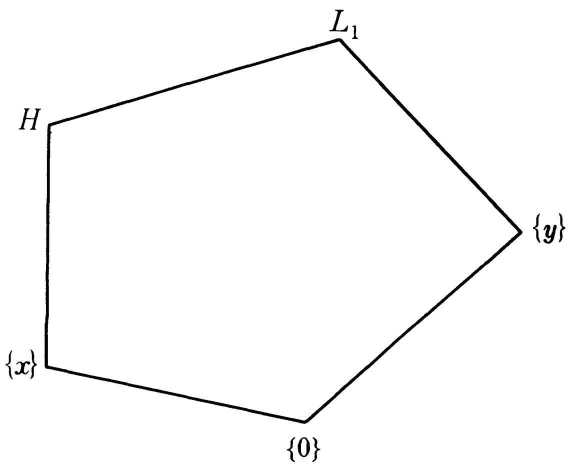

Fig. 1

This implies that L is not upper semi-modular. Hence, rank $L_1\leqq 1$. If rank $L_1=1$, and dim $L_1=3$, since L_1 is upper semi-modular, then L_1 is special simple. This completes the proof of Proposition 2.1.

Corollary 2.1. If L is an upper semi-modular Lie algebra, over an algebraically closed field, then L is solvable.

Theorem 2.2 If L is an upper semi-modular nilpotent Lie algebra, over a field of any characteristic, then L is abelian.

Proof:

If L is a non-abelian nilpotent Lie algebra then L contains a three-dimensional subalgebra L_1 with basis e_1, e_2, e_3 such that $[e_1, e_2]=e_3$; $[e_1, e_3]=0$; $[e_2, e_3]=0$. By Proposition 1.3, we conclude that L_1 is not upper semi-modular.

Proposition 2.2 If L is a solvable upper semi-modular Lie algebra then $L''=0$.

Proof:

Since L is solvable, $L=H+L'$, where H is a Cartan subalgebra of L and $L'=[L, L]$. Now H and L' are both upper semi-modular nilpotent subalgebras of L, and are thus abelian. By Proposition 1.5, we conclude that $L''=0$.

We now turn to the characterization of upper semi-modular Lie algebras.

Lemma 2.1 Let L be a Lie algebra of dimension $n+1$, $n\geq 1$, over a field of any characteristic. Let N be an abelian ideal of L of dimension n, and $x\in L$, $x\notin N$. If L is an upper semi-modular Lie algebra, then the linear transformation ad $x|_N$ is a scalar transformation.

Proof:

Let $f(z)$ be the minimal polynomial of ad $x|_N$. Factoring $f(z)$ into its irreducible factors, we have

$$f(z)=p_1(z)^{e_1}p_2(z)^{e_2}\cdots p_k(z)^{e_k},$$

where $p_i(z)$ are monic irreducible polynomials. If at least one of the $p_i(z)$ $(i=1, 2, \cdots, k)$, say $p_j(z)$, is of degree $r>1$, we write $p_j(z)=c_0+c_1z+c_2z^2+\cdots+c_{r-1}z^{r-1}+z^r$.

Now there exists a basis $f_1, f_2, \cdots, f_n$ of N such that ad $x|_N$ can be represented by a matrix in rational canonical form. Since one of the companion matrices will be the one corresponding to $p_j(z)$ we have, by possibly rearranging the f_i,

$$[f_1, x]=f_2;\ [f_2, x]=f_3;\ \cdots\cdots;\ [f_{r-1}, x]=f_r;$$

$$[f_r, x]=c_0f_1+c_1f_2+\cdots\cdots+c_{r-1}f_r.$$

We now assert that the subalgebra L_1 generated by $f_1, f_2, \cdots, f_r, x$ is not upper semi-modular. For consider the following subalgebras of L_1:

$A=\{f_1, f_2\}$; $B=\{f_1\}$; $c=\{x\}$. Then both B and C cover $\{0\}$, but $B\cup C=L_1\supsetneqq A\supsetneqq B$. Hence, all the irreducible factors of $f(z)$ are of degree one.

We now decompose N into its eigenspaces relative to ad $x|_N$, obtaining

$$N=N_\alpha+N_\beta+\cdots\cdots,$$

where $\alpha, \beta, \cdots$ are the eigenvalues of ad $x|_N$. Then there exist $e_\alpha\in N_\alpha$, $e_\beta\in N_\beta$ such that $[e_\alpha, x]=\alpha e_\alpha$; $[e_\beta, x]=\beta e_\beta$; $[e_\alpha, e_\beta]=0$. Thus $\{x, e_\alpha, e_\beta\}$ is a three-dimensional subalgebra L_2 of L. If $\alpha=0$ and $\beta\neq 0$, we show that L_2 is not upper semi-modular. For define the following subalgebras of L_2:

$$A=\{e_\beta+x,\ e_\alpha\};\quad B=\{e_\beta+x\};\quad C=\{e_\alpha+e_\beta\}.$$

Then $B\cup C=L_2\supsetneqq A\supsetneqq B$. Hence, either all the eigenvalues are zero or they are all $\neq 0$.

If all the eigenvalues are $\neq 0$, we assert that they are all equal. Thus, if $\alpha\neq\beta$ then the three-dimensional subalgebra L_2 just defined is isomorphic to the Lie algebra defined in Proposition 1.4, which is not upper semi-modular. Thus, in all cases ad $x|_N$ has only one eigenvalue α, which may be zero.

We next show that ad $x|_N$ is a semi-simple linear transformation. Now $f(z)=(z-\alpha)^k$, and we must show that $k=1$. Assume $k>1$. Now there exists a basis $f_1, f_2, \cdots, f_n$ of N such that ad $x|_N$ can be represented by a matrix in Jordan normal form, and if $k>1$ then at least one of the matrices along the diagonal is of order k. We thus have,

$$[f_1, x]=\alpha f_1;\quad [f_2, x]=f_1+\alpha f_2.$$

Hence, we have a three-dimensional subalgebra L_4 with basis f_1, f_2, x and we now show that L_4 is not upper semi-modular by defining the following subalgebras of L_4:

$A=\{f_1, f_2\};\quad B=\{f_2\};\quad C=\{x\}$. Then

$$B\cup C=L_4\supsetneqq A\supsetneqq B.$$

Hence, ad $x|_N$ is a semi-simple transformation, which implies that it is a scalar transformation. This completes the proof of Lemma 2.1.

THEOREM 2.3 Let L be a solvable Lie algebra. Then L is upper semi-modular if and only if L is abelian or almost abelian.

PROOF:

Let L be upper semi-modular. If L is not abelian, let N be the nil radical of L. By Theorem 2.2, we conclude that N is abelian. Let $x\in L$, $x\notin N$, and consider ad $x|_N$. Since N is an ideal, $\{x, N\}$ is a subalgebra satisfying the hypotheses of Lemma 2.1. Hence, ad $x|_N$ is a scalar transformation, and thus $[z, x]=\alpha z$ for $z\in N$, where α is the unique eigenvalue of ad $x|_N$. If $\alpha=0$, we consider the subspace $\{x, N\}=M$ of L. Since L is solvable, $M\supset N\supset L'$. Hence, M is a nilpotent ideal of L, contradicting the maximality of N. Thus, $\alpha\neq 0$.

We now show that dim L-dim $N=1$.

Let $C(N)$ be the centralizer of N. Since ad $z|_N\neq 0$ for $z\notin N$, it follows that $C(N)=N$. Now let $y\in L$, $y\notin N$. Then there exists $\beta\neq 0\in\Phi$, such that $[z, y]=\beta z$ for $z\in N$. Consider the element $\beta x-\alpha y$. We have $[z, \beta x-\alpha y]=0$

for all $z \in N$. Thus, $\beta x-\alpha y \in C(N)$, and hence $\dim L/C(N)=\dim L/N=1$.

We have thus obtained the structure of L. It has a basis $e_0, e_1, \cdots, e_n$ such that

$$[e_i, e_0]=\alpha e_i, \quad \alpha\neq 0; \quad [e_i, e_j]=0 \quad \text{for} \quad i, j\geq 1.$$

If we let $e_0'=\frac{e_0}{\alpha}$ and $e_i'=e_i$ for $i\geq 1$, we find that L is almost abelian.

The converse follows from Proposition 1.2.

This completes the proof of Theorem 2.3.

Corollary 2.2 Let L be a Lie algebra over an algebraically closed field. Then L is upper semi-modular if and only if L is abelian or almost abelian.

We can now strengthen Proposition 2.1.

Theorem 2.4 A Lie algebra L is upper semi-modular if and only if L is abelian, almost abelian, or special simple.

Proof:

Let L be upper semi-modular. Then Proposition 2.1 implies that L is solvable or L/S, where S is the radical of L, is special simple.

If L is solvable, then Theorem 2.3 implies that L is abelian or almost abelian. Thus, we wish to show that if L is non-solvable then $S=0$. If L is non-solvable, then by Levi's Theorem $L=S\oplus L_1$, where L_1 is a semi-simple subalgebra of L. Then $L_1\simeq L/S$, and thus L_1 is special simple. L_1 then has a basis e_1, e_2, e_3 such that $[e_1, e_2]=e_3$; $[e_2, e_3]=\alpha e_1$; $[e_3, e_1]=\beta e_2$, $\alpha\neq 0$, $\beta\neq 0 \in \Phi$.

Since S is solvable and upper semi-modular, it is abelian or almost abelian. If S is abelian, let $x \in L_1$, $x \notin S$. Note that $\{x, S\}$ is a subalgebra of L and consider ad $x|_S$. From Lemma 2.1, it follows that ad $x|_S$ is a scalar transformation. For $x \in L_1$, $x \in S$, define $\varphi(x)=\text{ad } x|_S$. Then φ is a representation of L_1, and since $\varphi(x)$ is a scalar and L_1 is semi-simple it follows that $\varphi=0$, which implies that $S=$center of L.

If $S\neq 0$, let $e_4\neq 0 \in S$, and consider the subalgebra $L_2=\{e_1, e_2, e_3, e_4\}$ of L.

Define the following subalgebras of L_2: $A=\{e_1+e_4, e_4\}$; $B=\{e_1+e_4\}$; $C=\{e_2\}$. Then $B\cup C=L_2\supsetneqq A\supsetneqq B$, which contradicts the upper semi-modularity of L. Hence, $S=0$.

Now, suppose S is almost abelian. Let N be the nil radical of S and consider $N\oplus L_1$. It then follows that $N\oplus L_1$ is a subalgebra of L and N is its radical. Moreover, by Proposition 1.2, N is abelian, and we can thus apply the preceding proof to conclude that $N=0$, which is a contradiction.

The converse follows from Proposition 1.2.

This completes the proof of Theorem 2.4.

We have thus completely characterized upper semi-modular Lie algebras over fields of characteristic zero.

REMARK: Since abelian, almost abelian and special simple Lie algebras are modular, it follows that an upper semi-modular Lie algebra over a field of characteristic zero is modular.

3. Lower Semi-modular Lie Algebras

THEOREM 3.1 If L is a nilpotent Lie algebra, over a field of any characteristic, then L is lower semi-modular.

PROOF:

Let A and C be subalgebras of L with $C \supsetneqq A$. Then, since the normalizer of A in C is $\neq A$, there exists a subalgebra A_1 of L such that $C \supset A_1 \supset A$, and $\dim A_1 - \dim A = 1$. Thus, if A, B and C are subalgebras of L such that C covers both A and B, then both A and B cover $A \cap B$.

LEMMA 3.1 Let L be a split simple three-dimensional Lie algebra over a field Φ not containing $\sqrt{-1}$. Then 1) there exists $y \in L$ such that L covers the subalgebra $\{y\}$; 2) L is not lower semi-modular.

PROOF:

1) As usual, L has a basis e_1, e_2, e_3 such that $[e_1, e_2]=e_3$; $[e_1, e_3]=2e_1$; $[e_2, e_3]=-2e_2$. Consider the element e_1+e_2. We assert that it is not contained in any two-dimensional subalgebra of L. Suppose there exists $a=a_1e_1+a_2e_2+a_3e_3 \in L$ such that $\{e_1+e_2, a\}$ is a two-dimensional subalgebra. Then there exist $\lambda, \mu \in \Phi$ such that

$$[e_1+e_2, a]=\lambda(e_1+e_2)+\mu a.$$

Hence, the following system of equations has a solution in Φ for λ, μ.

$$\lambda+\mu a_1=2a_3$$

$$\lambda+\mu a_2=-2a_3$$

$$\mu a_3=(a_2-a_1).$$

If $a_2-a_1=0$, then $\mu a_3=0$, and we need only consider $a_3=0$. But in this case $a=a_1(e_1+e_2)$.

Thus, $a_2-a_1\neq 0$, and so $a_3\neq 0$. Solving these equations for λ and μ, we find that that $-4a_3^2=(a_1-a_2)^2$, which is impossible if $\sqrt{-1} \notin \Phi$.

2) Consider the following subalgebras of L: $A=\{e_1, e_3\}$, $B=\{e_1+e_2\}$.

Then L covers both A and B, and $A\cap B=0$. However, we also have $A\supset\{e_1\}\supset 0 =A\cap B$, which implies that L is not lower semi-modular.

THEOREM 3.2 If L is a split semi-simple Lie algebra over a field Φ not containing $\sqrt{-1}$, then L is not lower semi-modular.

PROOF:

If L is a split semi-simple Lie algebra, then L contains a split simple three-dimensional Lie algebra.

PROPOSITION 3.1 Let L be a simple three-dimensional Lie algebra. 1) If L is non-split, then L is lower semi-modular; 2) If Φ is algebraically closed, then L is lower semi-modular.

PROOF:

1) This follows from Proposition 1.6.

2) If $x\in L$, then either x is a regular element or ad x is a nilpotent transformation. If ad x is nilpotent it follows that x can be embedded in a two-dimensional subalgebra of L. If x is regular, then $H=\{x\}$ is a Cartan subalgebra of L and $L=H+\Phi e_\alpha+\Phi e_{-\alpha}$, which implies that x is contained in $H+\Phi e_\alpha$, a two-dimensional subalgebra.

It then follows that L is lower semi-modular.

LEMMA 3.2 Let Φ be an algebraically closed field, and L_1 a simple three-dimensional Lie algebra over Φ. If $L=L_1\oplus L_1$, then L is not lower semi-modular.

PROOF:

We set $L=L_1\oplus L_2$, where $L_2=L_1$, and consider the diagonal subalgebra D of L, $D=\{(e_1, e_1), (e_2, e_2), (e_3, e_3)\}$. We assert that D is a maximal subalgebra of L.

Let M be a four-dimensional subalgebra of L such that $M\supsetneqq D$, and let R=radical of M. It then follows that $M=D\oplus R$, and thus, dim $R=1$. Let $d\in D$, and define $\varphi(d)=\text{ad } d|_R$. Then φ is a representation of D which is one-dimensional, and hence, $\varphi=0$. Thus, $R\subset$center of M. Let $R=\Phi(r_1, r_2)$. Now, if $[(x, x), (r_1, r_2)]=0$ for all $(x, x)\in D$ then $[x, r_i]=0$ for all $x\in L_i$ $(i=1, 2)$, which implies that $R=0$. Thus, D is not contained in any four-dimensional subalgebra of L.

Now let M be a five-dimensional subalgebra of L such that $M\supsetneqq D$. Then dim $(M\cap L_1)=2$. Moreover, since Φ is algebraically closed and L is semi-simple, all maximal solvable subalgebras of L are conjugate to $H+\Phi e_\alpha+\Phi e_\beta+\cdots\cdots$, where H is a Cartan subalgebra of L and $\alpha, \beta, \cdots$ are positive roots with respect to some ordering. Thus, all two-dimensional subalgebras of L_1 are conjugate to the subalgebra $\{e_1, e_3\}$.

Hence, we may take M as

$$M=\{(e_1, e_1), (e_2, e_2), (e_3, e_3), (e_1, 0), (e_3, 0)\}.$$

Since $(e_2, 0) \notin M$, it follows that M is not a subalgebra. Hence, D is not contained in any five-dimensional subalgebra of L.

We now show that L is not lower semi-modular. Consider the following subalgebras of L: $A=\{(e_1, 0), (e_2, 0), (e_3, 0), (0, e_1), (0, e_3)\}$; $B=D$; $C=\{(e_1, 0), (0, e_1), (e_3, 0), (0, e_3)\}$. Then $A\cap B=\{(e_1, e_1), (e_3, e_3)\}$. Also, L covers both A and B. However, $A\supsetneqq C\supsetneqq A\cap B$, which implies that L is not lower semi-modular. This completes the proof of Lemma 3.2.

THEOREM 3.3 Let Φ be an algebraically closed field. If L is a semi-simple Lie algebra over Φ which is not simple, then L is not lower semi-modular.

PROOF:

If L is semi-simple and not simple, then $L=L_1\oplus L_2\oplus\cdots\oplus L_r$, where the L_i are ideals of L which are simple Lie algebras. Since Φ is algebraically closed, $L_1\supset M_1$, $L_2\supset M_2$, where M_1 and M_2 are simple three-dimensional subalgebras of L_1 and L_2, respectively. If we let $M=M_1\oplus M_2$, it follows that M is a subalgebra of L which is not lower semi-modular.

PROPOSITION 3.2 The classical Lie algebra A_2, over an algebraically closed field, is not lower semi-modular.

PROOF:

We consider the usual matrix basis f_{ij} of Φ_3, where f_{ij} is a 3×3 matrix with a 1 in position (i, j) and 0's elsewhere. Then a basis of A_2 is given by

$$e_1=f_{11}-f_{33} \qquad e_2=f_{22}-f_{33} \qquad e_3=f_{12}$$

$$e_4=f_{13} \qquad e_5=f_{21} \qquad e_6=f_{23}$$

$$e_7=f_{31} \qquad e_8=f_{32}$$

Then $M=\{e_2, e_5, e_6, e_7, e_8\}$ is a subalgebra of A_2. Now consider the following subalgebras of M: $A=\{e_2, e_5, e_6, e_7\}$; $B=\{e_2, e_6, e_8\}$; $C=\{e_2, e_5, e_6\}$. B is a simple three-dimensional Lie algebra, and using essentially the same proof as that of Lemma 3.2 we can show that B can't be embedded in any four-dimensional subalgebra of M. Thus, M covers B, and M covers A. We then have $A\supsetneqq C\supsetneqq A\cap B$, which implies that A_2 is not lower semi-modular.

LEMMA 3.3 Let L be a split semi-simple Lie algebra. If there exist two roots α and β which are orthogonal and such that $\pm\alpha\pm\beta$ is not a root then $L\supset A_1\oplus A_1$ as a subalgebra.

Proof:

If we let, as usual, $h_\alpha=[e_\alpha, e_{-\alpha}]$ and $h_\beta=[e_\beta, e_{-\beta}]$, then $L_1=\{h_\alpha, e_\alpha, e_{-\alpha}\}$ and $L_2=\{h_\beta, e_\beta, e_{-\beta}\}$ are two subalgebras of L, both of which are isomorphic to A_1. Since α and β are orthogonal, $[h_\alpha, e_{\pm\beta}]=0$. Hence, $L_1\oplus L_2$ or $A_1\oplus A_1$ is a subalgebra of L.

Proposition 3.3 The classical Lie algebra B_2, over an algebraically closed field, is not lower semi-modular.

Proof:

The root system of B_2 is given by the vectors $\alpha=(1, 0)$, $\beta=(1, 1)$, $\gamma=(0, 1)$, $\delta=(-1, 1)$ and their negatives (see van der Waerden [8], p. 450). Since β and δ satisfy the conditions of Lemma 3.3, $B_2\supset A_1\oplus A_1$.

Theorem 3.4 Let L be a semi-simple Lie algebra over an algebraically closed field. L is lower semi-modular if and only if L is of rank one.

Proof:

Let L be lower semi-modular. We need only consider the simple Lie algebras. Since A_2 and B_2 are not lower semi-modular, then A_n and B_n, for $n>2$, are not lower semi-modular. Now since $A_2\subset G_2$ (van der Waerden [8], p. 461, or Dynkin [3], table 11, p. 149) and $G_2\subset F_4\subset E_6\subset E_7\subset E_8$ (Dynkin [3], p. 192), we conclude that the exceptional Lie algebras are not lower semi-modular. Moreover, we have $A_3\subset D_4$ and $B_2\subset C_3$ ([8], p. 461).

Thus, C_n and D_n are not lower semi-modular. Hence, L is of rank one.

The converse has been proved in Proposition 3.1.

Proposition 3.4 Let L be a Lie algebra over an algebraically closed field. If L is lower semi-modular then either L is solvable or L/S, where S is the radical of L, is of rank one.

Proof:

If L is non-solvable, then by Levi's Theorem $L=S\oplus L_1$, where S is the radical of L and L_1 is a semi-simple subalgebra of L. If rank $L_1>1$, then L is not lower semi-modular.

We next consider the question of whether the converse of Proposition 3.4 is true. First, we have the following

Theorem 3.5 If L is a solvable Lie algebra, over an algebraically closed field, then L is lower semi-modular.

Proof:

We proceed by induction on dim L. If dim $L=1$, then L is obviously lower semi-modular. Now let dim $L=n$, and assume that the result holds for Lie algebras of dim $\leq n-1$. By Lie's Theorem, L contains a one-dimensional ideal R. Let A, X, Y be subalgebras of L such that A covers both X and Y. We then show that both X and Y cover $X\cap Y$. We may assume, without loss of generality that $A+R=L$. We also note that by induction hypothesis, L/R is lower semi-modular.

Now if A covers X, it then follows that either $A+R$ covers $X+R$ or $A+R=X+R$. Thus, we now consider the following four cases:

1) $A+R=X+R$, $A+R=Y+R$.
2) $A+R=X+R$, $A+R$ covers $Y+R$.
3) $A+R$ covers $X+R$, $A+R$ covers $Y+R$.
4) $A+R=Y+R$, $A+R$ covers $X+R$.

We first consider case 1). It follows that $A=X\oplus R$, and $A=Y\oplus R$. Since R is one-dimensional, both X and Y cover $X\cap Y$.

Now in case 2) we have $A+R=A$. Since A covers Y, $Y+R=Y$, which implies that $R\subset Y$. We also have $A=X\oplus R$ and $Y=(X\cap Y)\oplus R$, and thus Y covers $X\cap Y$. Moreover, since A/R covers Y/R, we conclude that X covers $X\cap Y$.

In case 3), we first note that if any one of the following equalities, $A\cap R=R$, $X\cap R=R$, $Y\cap R=R$, hold then the other two also hold. Thus, suppose all three equalities hold. Then A/R covers both X/R and Y/R, and thus, by induction hypothesis both X/R and Y/R cover $(X\cap Y)/R$. Hence, X and Y both cover $X\cap Y$. If $X\cap R=Y\cap R=A\cap R=0$, then since X and Y are subalgebras of A, we conclude, by induction hypothesis that both X and Y cover $X\cap Y$.

Case 4) is handled in the same manner as case 2).

This completes the proof of Theorem 3.5.

The following example shows that if L/S is a semi-simple Lie algebra of rank one, where S is the radical of L, then L need not be lower semi-modular.

Example Let L_1 be the simple Lie algebra of rank one over an algebraically closed field Φ. Let V be any L_1-module such that the module multiplication is non-trivial. It then follows that dim $V>1$.

Now since L_1 is semi-simple V is completely reducible, i.e., V is a direct sum of subspaces V_i irreducible with respect to L_1. It then follows that there exists a V_i, say V_1 such that L_1 acting on V_1 is non-trivial. Hence, dim $V_1>1$. We now define $L_1^*=L_1\oplus V_1$, the split extension of L_1 by V_1, and show that L_1^* is not lower semi-modular.

Let $A=L_1$, a subalgebra of L_1^*. Since V_1 is irreducible, L_1^* covers A. Now let $L_2=\{e_1, e_3\}$, a maximal subalgebra of L_1, which is solvable. We also define $B=L_2\oplus V_1$, the split extension of L_2 by V_1. Note that L_1^* covers B and that $A\cap B=L_2$. Also, L_2 is a maximal subalgebra of B if and only if V_1 is irreducible with respect to L_2. Since L_2 is solvable, we conclude, by Lie's Theorem, that if V_1 is irreducible with respect to L_2, then dim $V_1=1$. Since dim $V_1>1$, it follows that L_1^* is not lower semi-modular.

We can, however, prove the following theorem, which characterizes lower semi-modular Lie algebras over algebraically closed fields of characteristic zero.

THEOREM 3.6 Let L be a Lie algebra over an algebraically closed field. Then L is lower semi-modular if and only if L is solvable, or L is a direct sum of its radical S and a simple ideal L_1 isomophic to A_1.

PROOF:

Let $L=S\oplus L_1$. Since S is solvable, then by Lie's Theorem, S contains a one-dimensional ideal A, which turns out to be an ideal of L. We now proceed iductively as in the proof of Theorem 3.5, and conclude that L is lower semi-modular.

We now turn to the converse and proceed by induction on dim L. Thus, suppose that the result is true if dim $L\leq n-1$. If L is not solvable, then by Levi's Theorem and Proposition 3.4, $L=S\oplus L_1$, as a vector space direct sum. We show that $[S, L_1]=0$.

Let A be a minimal ideal of L which is contained in S. It then follows that A is abelian. Also, A is then completely reducible, i.e., $A=\sum_{i=1}^{n}\oplus A_i$, where A_i is an irreducible subspace of A with respect to L_1. Now $L_1\oplus A_i$, the split extension of L_1 by A_i is a lower semi-modular subalgebra of L. By the argument used in the discussion of the preceding example, A_i is one-dimensional. Thus $[A_i, L_1]=0$. It also follows that A is the minimal ideal of S, which means that A is an irreducible S-module, and hence, A is one-dimensional. Then by induction hypothesis, $L/A=S/A\oplus(L_1+A)/A$. Thus, $[S, L_1]\subseteq A$. We also have $[[S, L_1], L_1]\subseteq[A, L_1]=0$. Hence, $0=[[S, L_1], L_1]=[[L_1, L_1], S]+[[S, L_1], L_1]=[L_1, S]+[[S, L_1], L_1]=[L_1, S]$.

This completes the proof of Theorem 3.6.

Theorem 3.5 does not hold over a non-algebraically closed field. An example is provided by the following

PROPOSITION 3.5 Let L be a solvable three-dimensional Lie algebra with basis e_1, e_2, e_3 over the field of real numbers defined by

$$[e_1, e_2]=0; \quad [e_1, e_3]=\alpha e_1+\beta e_2; \quad [e_2, e_3]=\gamma e_1+\delta e_2$$

where $A=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$ is non-singular. Then L is lower semi-modular if and only if the eigenvalues of A are real.

PROOF:

Let the eigenvalues of A be non-real. We now show that e_3 is not embeddable in any two-dimensional subalgebra of L. Thus, suppose $\{e_3, a\}$ is a two-dimensional subalgebra of L, where $a=a_1e_1+a_2e_2$. Then there exist real numbers λ, μ such that $[e_3, a]=\lambda e_3+\mu a$. Thus,

$$\lambda=0$$

$$\mu a_1=-a_1\alpha-a_2\gamma$$

$$\mu a_2=-a_1\beta-a_2\delta$$

Hence, $-\mu v={}^tAv$, where $v=\begin{bmatrix}a_1\\a_2\end{bmatrix}$ and ${}^tA$ is the transpose of A. Thus, μ is an eigenvalue of A, which contradicts the hypothesis that A has only non-real eigenvalues. Thus, L covers the subalgebras $Y=\{e_3\}$ and $X=\{e_1, e_2\}$. However, $X\supsetneqq\{e_1\}\supsetneqq\{0\}=X\cap Y$, which implies that L is not lower semi-modular.

Conversely, let A have only real eigenvalues. We prove that each element of L is embeddable in a two-dimensional subalgebra of L. Since $\{e_1, e_2\}$ is a subalgebra of L, it suffices to show that the element $x=x_1e_1+x_2e_2+e_3$ is contained in some two-dimensional subalgebra of L. The subalgebra $\{x, a\}$ is obtained by solving ${}^tAv=-\mu v$ for a given eigenvalue μ of ${}^tA$ and setting $a=a_1e_1+a_2e_2$, where $v=\begin{bmatrix}a_1\\a_2\end{bmatrix}$. This completes the proof of Proposition 3.5.

We also note that all other solvable three-dimensional real Lie algebras are lower semi-modular. Thus, we have obtained a complete classification of real three-dimensional lower semi-modular Lie algebras.

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