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A Generalization of Duality Theorem in the Theory of Linear Programming

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We consider an $m\times n$ matrix (a_{ij}), a vector $\boldsymbol{b}$ with components $b_1, \cdots, b_m$ and a vector $\boldsymbol{c}$ with components $c_1, \cdots, c_n$. We denote by $\mathscr{M}$ the set of vectors $\boldsymbol{u}$ having non-negative components $u_1, \cdots, u_n$ and satisfying $\sum_{j=1}^{n} a_{ij}u_j \leqq b_i$ $(i=1, \cdots, m)$, and by $\mathscr{M}'$ the set of vectors $\boldsymbol{v}$ having non-negative components $v_1, \cdots, v_m$ and satisfying $\sum_{i=1}^{m} a_{ij}v_i \geqq c_j$ $(j=1, \cdots, n)$. We set

$$M=\sup_{\boldsymbol{u}\in\mathscr{M}} \boldsymbol{c}\cdot\boldsymbol{u}'=\sup_{\boldsymbol{u}\in\mathscr{M}} \sum_{j=1}^{n} c_j u_j \qquad \text{if } \mathscr{M}\neq\emptyset \text{ (the empty set)}$$

and

$$M'=\inf_{\boldsymbol{v}\in\mathscr{M}'} \boldsymbol{b}\cdot\boldsymbol{v}'=\inf_{\boldsymbol{v}\in\mathscr{M}'} \sum_{i=1}^{m} b_i v_i \qquad \text{if } \mathscr{M}'\neq\emptyset \; .$$

The well-known duality theorem in the theory of linear programming asserts that, if $\mathscr{M}\neq\emptyset$ and $M<\infty$, then $\mathscr{M}'\neq\emptyset$ and $M=M'$.

We shall generalize this theorem in the present paper. Let X and Y be compact Hausdorff spaces and $\Phi(x, y)$ a universally measurable[1] function on $X\times Y$ which is bounded below. Let $g(x)$ be a universally measurable function on X which is bounded below and $f(y)$ a universally measurable function on Y which is bounded above.

Under these general circumstances let $\mathscr{M}$ be the class of all non-negative Radon measures[2] μ on Y satisfying

$$\int_Y \Phi(x, y)d\mu(y)\leqq g(x) \qquad \text{on } X.$$

Such a measure is called *feasible*. In case $\mathscr{M}$ is not empty, we set

$$M=\sup_{\mu\in\mathscr{M}}\int f(y)d\mu(y) \; .$$

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1) A function in a compact space is universally measurable if it is measurable with respect to all Radon measures.

2) A measure means a non-negative Radon measure in this paper unless otherwise stated.

If there is a measure of $\mathscr{M}$ which attains the maximum, it is called *optimal.* Similarly, we define the class $\mathscr{M}'$ of (dually) feasible measures ν satisfying $\int \Phi(x, y) d\nu(x) \geqq f(y)$ on Y, and set

$$M' = \inf_{\nu \in \mathscr{M}'} \int g(x) d\nu(x)$$

if $\mathscr{M}' \neq \emptyset$. This is the dual problem. An element μ of $\mathscr{M}$ ($\mathscr{M}'$ resp.) is called *extreme* if there are no distinct $\mu_1, \mu_2 \in \mathscr{M}$ ($\mathscr{M}'$ resp.) such that $2\mu = \mu_1 + \mu_2$.

As in the discrete case we obtain

THEOREM 1. *Assume $\mathscr{M} \neq \emptyset$ and $\mathscr{M}' \neq \emptyset$. Then $M \leqq M'$. If $\mu \in \mathscr{M}$, $\nu \in \mathscr{M}'$ and $\int f d\mu = \int g d\nu$, then both μ and ν are optimal.*

PROOF. Suppose $\int \Phi d\mu \leqq g$ and $\int \Phi d\nu \geqq f$. Then

$$\int f d\mu \leqq \int\int \Phi d\nu d\mu = \int\int \Phi d\mu d\nu \leqq \int g d\nu$$

and $M \leqq M'$ is derived. If the equality $\int f d\mu = \int g d\nu$ holds, then $\int f d\mu = M = M' = \int g d\nu$. Thus both μ and ν are optimal.

Next we prove

LEMMA 1.[3] *Assume $\mathscr{M} \neq \emptyset$, $-\infty < M < \infty$ and either* (i) $f(y) > 0$ *on Y or* (ii) *there is $x_0 \in X$ such that $\Phi(x_0, y) > 0$ on Y and $g(x_0) < \infty$. Assume also that Φ is lower semicontinuous.*[4] *Let $D = \{\kappa\}$ be a directed set. Let $\{\Phi_\kappa; \kappa \in D\}$ be a net of lower semicontinuous functions increasing to Φ, and $\{X_\kappa; \kappa \in D\}$ be an increasing net of sets in X. If $g(x) = \infty$ on $X - \lim_D X_\kappa$ and $\int \Phi_\kappa(x, y) d\mu_\kappa(y) \leqq g(x) + const.$ (independent of κ) on X_κ for every $\kappa \in D$, then there is κ_0 such that $\{\mu_\kappa(Y); \kappa \geqq \kappa_0\}$ are bounded.*

PROOF. It is easy to obtain the conclusion under (ii). Hence assume (i). Suppose that $\mu_\kappa(Y) \to \infty$ along a subnet $\{\mu_\kappa; \kappa \in D'\}$ of $\{\mu_\kappa\}$. We set $\mu'_\kappa = \mu_\kappa / \mu_\kappa(Y)$, and choose a vaguely convergent subnet of $\{\mu'_\kappa; \kappa \in D'\}$. We shall denote it again by $\{\mu'_\kappa; \kappa \in D'\}$ and let μ'_0 be the limit. Let $g(x) < \infty$. For some $\kappa = \kappa_x$, $x \in X_{\kappa_x}$ and it holds that

3) This very useful lemma and some improvements in Theorem 3 were suggested orally by M. Yoshida, a colleague of the author. In the first manuscript, $\Phi \geqq 0$ was assumed in condition (i) of this theorem and $g \geqq 0$ was assumed at some place.

4) A lower (upper resp.) semicontinuous function is assumed to be bounded below (above resp.) in this paper.

$$\int \Phi(x, y) d\mu_0'(y) = \lim_D \int \Phi_\kappa(x, y) d\mu_0'(y) \leqq \lim_D \underline{\lim}_{D'} \int \Phi_\kappa(x, y) d\mu_{\kappa'}'(y)$$

$$\leqq \underline{\lim}_{D'} \int \Phi_{\kappa'}(x, y) d\mu_{\kappa'}'(y) \leqq \lim_{D'} \frac{g(x) + \text{const.}}{\mu_{\kappa'}(Y)} = 0 .$$

Take a $\mu \in \mathscr{M}$ such that $\int f d\mu$ is finite. Then

$$\int \Phi(x, y) d(\mu + N\mu_0') \leqq g(x) \qquad \text{everywhere on } X$$

for any $N > 0$, or $\mu + N\mu_0' \in \mathscr{M}$. On the other hand, since $f > 0$ on Y and $\mu_0'(Y) = 1$, we see that $\int f d\mu_0' > 0$ and

$$\int f d(\mu + N\mu_0') = \int f d\mu + N\int f d\mu_0' \to \infty \qquad \text{as } N \to \infty .$$

Thus $M = \infty$, which is contrary to our assumption.

LEMMA 2. *Assume* $\mathscr{M} \neq \emptyset$, $-\infty < M < \infty$ *and* $\sup_{X \times Y} \Phi < 0$. *Let* $h(x)$ *be a function on* X, *and* μ *be a measure in* Y *with finite* $\int f d\mu$. *If* $\int \Phi(x, y) d\mu(y) \leqq h(x)$ *on* X, *then we can find a measure* $\mu' \leqq \mu$ *such that* $\mu'(Y) \leqq -(\inf_X h)^- / \sup_{X \times Y} \Phi$, $\int f d\mu \leqq \int f d\mu'$ *and* $\int \Phi(x, y) d\mu'(y) \leqq h(x)$ *on* X, *where* $(\inf h)^- = \max(-\inf h, 0)$.

PROOF. We may assume $\mu \neq 0$. Suppose that there is a point $x_0 \in S_\mu$ with $f(x_0) > 0$. If a is a large number, $a\int \Phi(x, y) d\mu(y) \leqq g(x)$ on X because $\sup \Phi < 0$ and g is bounded below on X. The measure $a\mu + N\varepsilon_{x_0}$ belongs to $\mathscr{M}$ for any $N > 0$, where ε_{x_0} is the unit point measure at x_0. It holds that

$$M \geqq \int f d(a\mu + N\varepsilon_{x_0}) = a\int f d\mu + Nf(x_0) \to \infty \qquad \text{as } N \to \infty ,$$

contrary to our assumption. Therefore $f \leqq 0$ on S_μ. If $h \geqq 0$ on X, then $\mu' \equiv 0$ satisfies the required conditions. Otherwise, we set $m_h = \inf_X h$. If $\sup_X \int \Phi d\mu \geqq m_h$, then $m_h \leqq \sup \Phi \cdot \mu(Y)$ and μ itself satisfies the conditions. If $\sup_X \int \Phi d\mu < m_h$, we consider $\mu' = \mu m_h / \sup_X \int \Phi d\mu$. It holds that $\mu' \leqq \mu$, $\mu'(Y) \leqq m_h / \sup_{X \times Y} \Phi$ and $\int \Phi d\mu' \leqq m_h \leqq h$ on X. Since $f \leqq 0$ on S_μ, $\int f d\mu \leqq \int f d\mu'$. Thus μ' satisfies all the required conditions.

Let us prove

THEOREM 2. *Assume* $\mathscr{M}\neq\emptyset$, $M<\infty$ *and the lower semicontinuity of* Φ *and* $-f$. *Assume also one of the following conditions*: (i) $f(y)>0$ *on* Y, (ii) *there is* $x_0\in X$ *such that* $\Phi(x_0, y)>0$ *on* Y *and* $g(x_0)<\infty$, (iii) $f(y)<0$ *on* Y, (iv) $\sup_{X\times Y}\Phi(x, y)<0$. *Then there is an extreme optimal measure* $\mu_0\in\mathscr{M}$.

PROOF. If $M=-\infty$, every feasible measure is optimal. Therefore we assume $-\infty<M<\infty$. Choose $\mu_n\in\mathscr{M}$ such that $\int f d\mu_n\to M$ as $n\to\infty$. By Lemma 1, $\mu_n(Y)$ is bounded under (i) or (ii). It is easy to obtain the same conclusion under (iii). If we assume (iv), then we can find $\{\mu_n'\}$ in $\mathscr{M}$ by Lemma 2 such that $\mu_n'(Y)$ is bounded and $\int f d\mu_n'\to M$. Therefore we may assume from the beginning that $\mu_n(Y)$ is bounded in all cases. We can extract a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ which converges vaguely to a measure μ_0. It holds that

$$\int\Phi d\mu_0\leqq\varliminf_{k\to\infty}\int\Phi d\mu_{n_k}\leqq g \quad\text{and}\quad M=\lim_{k\to\infty}\int f d\mu_{n_k}\leqq\int f d\mu_0 .$$

Thus $\mu_0\in\mathscr{M}$ and it is optimal.

We choose a sequence $\{\mu^{(n)}\}$ in $\mathscr{M}_0$ such that $\mu^{(n)}(K)$ tends to $m_0=\inf\{\mu(K); \mu\in\mathscr{M}_0\}$. A subsequence converges vaguely to some measure which lies in $\mathscr{M}_0$. Consequently, $\mathscr{M}_0^*=\{\mu\in\mathscr{M}_0;\ \mu(K)=m_0\}$ is convex and vaguely compact. As a convex compact set in the locally convex linear Hausdorff space of measures of general sign with vague topology, $\mathscr{M}_0^*$ has an extreme point; see [1] for instance. If $\mu\in\mathscr{M}_0^*$ is not extreme in $\mathscr{M}$, there exist distinct $\mu_1, \mu_2\in\mathscr{M}$ such that $2\mu=\mu_1+\mu_2$. Since $M=\int f d\mu=2^{-1}\left(\int f d\mu_1+\int f d\mu_2\right)\leqq M$, both μ_1 and μ_2 lie in $\mathscr{M}_0^*$. Consequently an extreme point of $\mathscr{M}_0^*$ is extreme as a point of $\mathscr{M}$.

REMARK 1. M. Yoshida [2] gave an example in which all Φ, g, f are continuous non-negative and $M<\infty$ but in which no optimal measure exists.

REMARK 2. The existence of $x_0\in X$ such that $\inf_Y\Phi(x_0, y)<0$ is not sufficient to obtain the above conclusion. Let $X=\{1, 2\}$ and $Y=\{1, 2, \cdots, \omega\}$, where n is supposed to tend to the point ω. We set $\Phi(1, y)=-1$ on Y, $\Phi(2, n)=-1/n$ and $\Phi(2, \omega)=0$. Let $g(x)\equiv-1$, $f(n)=-1/n^2$ and $f(\omega)=0$. Then the point measure μ_n at $x=n$ with total mass n satisfies $\int\Phi(x, y)d\mu_n(y)\leqq-1$ on X and $\int f d\mu_n=-1/n$. Thus $M=0$ but there is no optimal measure in $\mathscr{M}$.

Next we ask if the duality theorem holds, i.e. if the assumptions $\mathscr{M}\neq\emptyset$ and $-\infty<M<\infty$ imply $\mathscr{M}'\neq\emptyset$ and $M'=M$. This is not true in general.

Actually there is an example in which Φ, g, f are continuous non-negative and an optimal measure exists in $\mathcal{M}$, but in which $\mathcal{M}'=\emptyset$; see [2]. Another example in [2] shows that even if Φ, g, f are continuous non-negative and if $\mu\in\mathcal{M}$ and $\nu\in\mathcal{M}'$ are optimal, M may be strictly smaller than M'.

Our main result is the following duality theorem:

THEOREM 3. *Assume that* $\mathcal{M}\neq\emptyset$ *and* $-\infty<M<\infty$, *and that* Φ, g *and* $-f$ *are lower semicontinuous. Assume at least one of the following conditions*: (i) $f(y)>0$ *on* Y, (ii) *there is* $x_0\in X$ *such that* $\Phi(x_0, y)>0$ *on* Y *and* $g(x_0)<\infty$, (iii) $f(y)<0$ *on* Y, (iv) $\sup_{X\times Y}\Phi<0$. *Then* $\mathcal{M}'\neq\emptyset$ *and* $M=M'$.

PROOF. First we consider the case where Φ is continuous and g is bounded. We denote by (h, a) the couple of a bounded function h on X and a finite number a. Let E be the linear space consisting of all such couples (h, a). We regard it as a metric space by introducing the distance

$$d((h_1, a_1), (h_2, a_2))=\max(\sup_{x\in X}|h_1(x)-h_2(x)|, |a_1-a_2|) .$$

Let F be the set of all couples $(h, a)\in E$ such that there is a measure μ which satisfies $\int\Phi d\mu\leqq h$ and $\int f d\mu\geqq a$. It is certainly a convex cone in E. We shall show that the closure F^a of F does not contain any point of the form $(g, M+\delta)$ with $\delta>0$. Suppose that there is a sequence $\{(h_n, a_n)\}$ in F tending to $(g, M+\delta)$. Then there exists a sequence $\{\mu_n\}$ such that $\int\Phi d\mu_n\leqq h_n$ and $\int f d\mu_n\geqq a_n$ for each n and such that $h_n\to g$ uniformly and $a_n\to M+\delta$. By Lemma 1, $\mu_n(Y)$ is bounded under (i) or (ii). It is easy to have the same conclusion under (iii). Under (iv) we apply Lemma 2 and find $\{\mu_n'\}$ such that $\mu_n'\leqq\mu_n$, $\mu_n'(Y)\leqq-(\inf h_n)^-/\sup\Phi$ and $\int\Phi d\mu_n'\leqq h_n$ on X. Certainly $\{\mu_n'(Y)\}$ are bounded. The inequality $\int f d\mu_n'\geqq a_n$ remains true. Consequently, we may assume from the beginning that $\mu_n(Y)$ is bounded under (iv) too. We extract a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ which converges vaguely to a measure μ_0. It follows that

$$\lim_{k\to\infty}\int\Phi d\mu_{n_k}=\int\Phi d\mu_0 \quad \text{and} \quad \overline{\lim_{k\to\infty}}\int f d\mu_{n_k}\leqq\int f d\mu_0 ,$$

and hence $\int\Phi d\mu_0\leqq g$ and $\int f d\mu_0\geqq M+\delta$. This is impossible because $M=\sup_{\mu\in\mathcal{M}}\int f d\mu$.

By Theorem 2 there is μ satisfying $\int\Phi d\mu\leqq g$ and $\int f d\mu=M$. Hence $g^*=$

$(g, M)\in F$. Let us write e^* for $(0, 1)$. On account of the fact that F^a is a closed convex set and $g^*+\delta e^*$ is outside F^a, there exists a hyperplane in E separating F^a and $g^*+\delta e^*$; see [1]. Namely, there exists a linear functional φ on E such that $\varphi\geqq\alpha$ on F^a and $\varphi(g^*+\delta e^*)<\alpha$. Since F^a is a cone, $\varphi\geqq 0$ on F^a and $\varphi(0)=0$. Hence we may take $\alpha=0$. In particular, $\varphi(g^*)\geqq 0$. Hence $0>\varphi(g^*)+\delta\varphi(e^*)\geqq\delta\varphi(e^*)$. We may assume $\varphi(e^*)=-1$ without loss of generality. The inequality $\varphi\geqq 0$ on F^a implies that $\varphi((h, 0))\geqq 0$ for any $h\geqq 0$. If $\varphi((h, 0))$ is regarded as a linear functional on the family of continuous fnnctions in X, it is a positive functional. By Riesz's theorem, there is a measure ν such that $\varphi((h, 0))=\int h d\nu$ for continuous h. The couple $(\Phi(\cdot, y), f(y))$ belongs to F if $f(y)>-\infty$, and hence $\varphi((\Phi(\cdot, y), f(y)))\geqq 0$ holds. It follows that

$$0\leqq\varphi((\Phi(\cdot, y), f(y)))$$
$$=\varphi((\Phi(\cdot, y), 0))+f(y)\varphi(e^*)=\int\Phi(x, y)d\nu(x)-f(y),$$

whence

$$\int\Phi(x, y)d\nu(x)\geqq f(y) \qquad \text{if } f(y)>-\infty .$$

This is evidently true if $f(y)=-\infty$. Hence $\nu\in\mathscr{M}'$.

Let h be any continuous function satisfying $h\leqq g$. We have

$$\delta>\varphi(g^*)=\varphi((g, M))=\varphi((g, 0))+M\varphi(e^*)$$
$$\geqq\varphi((h, 0))+M\varphi(e^*)=\int h d\nu+M\varphi(e^*)=\int h d\nu-M,$$

or $\int h d\nu<M+\delta$. On account of the arbitrariness of h, we have $M'\leqq\int g d\nu\leqq M+\delta$, whence $M'\leqq M$. This yields $M=M'$ in virtue of Theorem 1.

Secondly we consider the case where Φ is continuous but g may not be bounded. It is not necessary to discuss the case subject to (iii). Under (iv), $\mathscr{M}$ is equal to $\left\{\mu; \int\Phi(x, y)d\mu(y)\leqq -g^-(x)\right\}$, where $g^-=\max(-g, 0)$. As we saw above, $\mathscr{M}'$ is not empty and $M=\inf\left\{-\int g^- d\nu;\ \nu\in\mathscr{M}'\right\}$. Naturally $M'\geqq\inf\left\{-\int g^- d\nu;\ \nu\in\mathscr{M}'\right\}$. Let us show the inverse inequality. Take any $\nu\in\mathscr{M}'$ and denote by ν^- its restriction to the set $\{x;\ g(x)\leqq 0\}$. As a measure in X, ν^- belongs to $\mathscr{M}'$ because $\Phi<0$, and it holds that $-\int g^- d\nu=\int g d\nu^-\geqq M'$. Thus we obtain $M=M'$. Next, assuming (i) or (ii), we set

$$X_n=\{x\in X;\ g(x)\leqq n\} \quad \text{and} \quad X_\infty=\{x\in X;\ g(x)=\infty\}\ .$$

We see that X_n is a closed set and $X_n \nearrow X-X_\infty$ as $n\to\infty$. By Lemma 1 there exists n_0 with the property that the inequality $\int\Phi(x, y)d\mu(y)\leqq g(x)$ on any X_n $(n\geqq n_0)$ implies the boundedness of $\mu(Y)$. Hence there is a finite number c_0 such that $\int\Phi(x, y)d\mu(y)<c_0$ on X whenever $\int\Phi(x, y)d\mu(y)\leqq g(x)$ on some X_n $(n\geqq n_0)$. For $n_1\geqq\max(c_0, n_0)$ we have $\int\Phi(x, y)d\mu(y)\leqq g(x)$ on X whenever $\int\Phi(x, y)d\mu(y)\leqq g(x)$ is true on X_{n_1}. Since $g(x)$ is bounded on X_{n_1},

$$\sup\left\{\int f d\mu;\ \int\Phi(x, y)d\mu(y)\leqq g(x) \text{ on } X_{n_1}\right\}$$

$$=\inf\left\{\int g d\nu;\ \int_{X_{n_1}}\Phi(x, y)d\nu(x)\geqq f(y) \text{ on } Y\right\}\ .$$

We shall denote both sides by M_1 and M_1' respectively. We observe readily that $M=M_1$. We take ν which satisfies $\int_{X_{n_1}}\Phi(x, y)d\nu(x)\geqq f(y)$ on Y, and regard it as a measure in X. Then $\nu\in\mathscr{M}'$ and $M'\leqq M_1'$. Since $M_1=M\leqq M'$ by Theorem 1 and $M_1=M_1'$, $M=M'$ is concluded.

Finally we consider the general case where Φ may not be continuous. We consider the directed set D of continuous functions not greater than Φ, and use the notations $\mathscr{M}_\Psi$, $\mathscr{M}_\Psi'$, M_Ψ and M_Ψ' when $\Psi\in D$ is taken as kernel. If Ψ, $\Psi'\in D$ and $\Psi\leqq\Psi'$, then $\mathscr{M}_\Psi\supset\mathscr{M}_{\Psi'}\supset\mathscr{M}$. Hence $M_\Psi\searrow$ along D and $\lim_D M_\Psi\geqq M$. If $M=\infty$, $\lim M_\Psi=M$. Suppose next $M<\infty$ and $\lim M_\Psi>M$. We choose δ, $0<\delta<\lim M_\Psi-M$, and $\mu_\Psi\in\mathscr{M}_\Psi$ for each $\Psi\in D$ such that $\int f d\mu_\Psi >M+\delta$. First we assume (i) or (ii). By Lemma 1 there exists $\Psi_0\in D$ such that $\{\mu_\Psi(Y);\ \Psi\in D, \Psi\geqq\Psi_0\}$ are bounded under any one of (i) and (ii). Since $M+\delta<\sup f\cdot\mu_\Psi(Y)$ for every $\Psi\in D$, $\{\mu_\Psi(Y);\ \Psi\in D\}$ are bounded under (iii). We do not need pay attention to (iv). We choose a subnet $\{\mu_\Psi;\ \Psi\in D'\}$ vaguely convergent to μ_0, and have

$$\int\Phi d\mu_0\leqq\varliminf_{D'}\int\Psi d\mu_\Psi\leqq g\ .$$

Thus $\mu_0\in\mathscr{M}$. On the other hand

$$M\geqq\int f d\mu_0\geqq\varlimsup_{D'}\int f d\mu_\Psi\geqq M+\delta\ .$$

This is impossible. Thus $\lim_D M_\Psi=M$. We know that $\mathscr{M}_\Psi'\neq\emptyset$ and $M_\Psi=M_\Psi'$

for each $\Psi \in D$. Since $\mathscr{M}'_\Psi \subset \mathscr{M}'$, $\mathscr{M}' \neq \emptyset$ and $M'_\Psi \geqq M'$. Hence by Theorem 1

$$M = \lim_D M_\Psi = \lim_D M'_\Psi \geqq M' \geqq M$$

and $M = M'$ is derived. Our theorem is completely proved.

Finally we prove

THEOREM 4. *We have the same result as in Theorem 3 if the lower semicontinuity of g is replaced by the condition that g is upper semicontinuous and bounded above*[5]*, while the other assumptions are kept.*

PROOF. Let $D = \{h\}$ be the directed set of continuous functions satitfying $h \geqq g$. We use the notations $\mathscr{M}_h$, $\mathscr{M}'_h$, M_h, M'_h in an obvious manner. Evidently $\mathscr{M}'_h = \mathscr{M}'$ for every $h \in D$. Since $\mathscr{M}_h \supset \mathscr{M}_{h'} \supset \mathscr{M}$ if $h, h' \in D$ and $h \geqq h'$, M_h decreases along D and $\lim_D M_h \geqq M$. The same relation is true for M'_h and M'. Assume that $\lim_D M_h > M$ and choose $\delta > 0$ such that $\lim_D M_h > M + \delta$. We select $\mu_h \in \mathscr{M}_h$ for each $h \in D$ such that $\int f d\mu_h \geqq M + \delta$. If (i) or (ii) is assumed, then by Lemma 1 there exists $h_0 \in D$ with the property that $\{\mu_h(Y);\ h \in D,\ h \leqq h_0\}$ are bounded. The same is true under (iii). Let us assume (iv). By Lemma 2 we may assume $\mu_h(Y) \leqq -(\inf_X g)^- / \sup_{X \times Y} \Phi$ for every $h \in D$. Thus we may assume that $\{\mu_h(Y);\ h \in D\}$ are bounded in all cases.

We choose a subnet $\{\mu_h;\ h \in D'\}$ which converges vaguely to some measure μ_0. We have

$$g = \lim_{D'} h \geqq \varliminf_{D'} \int \Phi d\mu_h \geqq \int \Phi d\mu_0 \ .$$

Namely, $\mu_0 \in \mathscr{M}$. Consequently

$$M + \delta \leqq \varlimsup_{D'} \int f d\mu_h \leqq \int f d\mu_0 \leqq M \ ,$$

which is impossible. Accordingly $\lim_D M_h = M$. On the other hand, we see $\mathscr{M}' = \mathscr{M}'_h \neq \emptyset$ on account of Theorem 3 and obtain

$$M = \lim_D M_h = \lim_D M'_h \geqq M' \geqq M$$

by Theorems 1 and 3. We have now $M = M'$.

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5) Since $\mathscr{M} \neq \emptyset$, there is μ satisfying $\int \Phi d\mu \leqq g$. This shows that g is bounded. Thus the assumption on g made at the beginning holds.

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