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On a Trace Theorem for the Space $H^{\mu}(R^N)$

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Consider the space $H_m(R^N)$, R^N being an N-dimensional Enclidean space, composed of temperate distributions u defined in R^N such that the Fourier transform $\hat{u}(\xi)$ is a locally integrable function satisfying

$$\int_{\Xi^N} |\hat{u}(\xi)|^2(1+|\xi|^2)^m d\xi < \infty .$$

Let m be a positive number $>\frac{1}{2}$ and l the largest integer such that $l<m-\frac{1}{2}$. It is known that the trace mapping

$$u \in H_m(R^N) \to \left(u(x',0), \cdots, \frac{\partial^l}{\partial x_N^l} u(x',0)\right) \in \prod_{j=0}^{l} H_{m-j-\frac{1}{2}}(R^{N-1})$$

is an epimorphism, where x' stands for $(x_1, x_2, \cdots, x_{N-1})$.

$H_m(R^N)$ is a particular instance of the spaces $H^{\mu}(R^N)$, μ being a temperate weight function defined in Ξ^N. The discussion on the spaces $H^{\mu}(R^N)$ is given in full detail in L. Hörmander [1] and in L.R. Volevič and B.P. Paneyah [5]. As a result of J. L. Lions' theorems on the Hilbert spaces [2], the trace theorem as mentioned above remains valid for $H^{\mu}(R^N)$ when $\mu(\xi)$ is equivalent to

$$\mu_1(\xi')+|\xi_N|^a \mu_2(\xi')$$

where $\mu_1(\xi')$, $\mu_2(\xi')$ are temperate weight functions in Ξ^{N-1}.

Recently M. Pagni has shown the theorem for a special $H^{\mu}(R^N)$, to which Lions' theorem is not applicable [3].

Our main aim of this paper is to investigate the trace theorem of the above type for general $H^{\mu}(R^N)$. We have obtained the necessary and sufficient conditions for the validity of the theorem (cf. Theorem 1 below). It is to be noticed that a sufficient condition to the effect that $\mu(\xi', 2\xi_N) \geqq C\mu(\xi', \xi_N)$, C being a constant, seems convenient to guarantee the theorem in most cases as enumerated above.

1. Notations and Terminologies. Let R^N be an N-dimensional Euclidean space and let Ξ^N be its dual space. For $x=(x_1, \cdots, x_N) \in R^N$ and $\xi=(\xi_1, \cdots, \xi_N) \in \Xi^N$, the scalar product $<x, \xi>$ and the length of the vector

x are defined by $<x, \xi> = \sum_{j=1}^{N} x_j \xi_j$, $|x| = (\sum_{j=1}^{N} |x_j|^2)^{\frac{1}{2}}$, and similarly for $|\xi|$. We shall use the multi-indices notation. If α is an N-tuple $(\alpha_1, \cdots, \alpha_N)$ of non-negative integers, the sum $\sum_{j=1}^{N} \alpha_j$ will be denoted by $|\alpha|$ and the product $\alpha_1! \cdots \alpha_N!$ by $\alpha!$. With $D = (D_1, \cdots, D_N)$, $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$, we set $D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$ and similarly $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_N^{\alpha_N}$. For a polynomial $P(\xi) = \sum a_\alpha \xi^\alpha$ in ξ, we put $P(D) = \sum a_\alpha D^\alpha$, $\bar{P}(\xi) = \sum \overline{a_\alpha} \xi^\alpha$ and $\tilde{P}(\xi) = \{\sum_{|\alpha| = 0} |P^{(\alpha)}(\xi)|^2\}^{\frac{1}{2}}$, where $\overline{a_\alpha}$ is the complex conjugate of a_α and $P^{(\alpha)}$ means $i^{|\alpha|} D^\alpha P$.

Let us denote by $\mathcal{D}(R^N)$, or $\mathcal{D}$, the space of all C^∞-functions in R^N with compact supports with usual topology of L. Schwartz [4] and by $\mathcal{D}'$ its strong dual, whose elements are called distributions. Also by $\mathcal{S}(R^N)$, or $\mathcal{S}$, we denote the space of all rapidly decreasing C^∞-functions ϕ in R^N with the semi-norms $\sup_x |x^\alpha D^\beta \phi|$ and by $\mathcal{S}'$ its strong dual, whose elements are called temperate distributions. For $\phi \in \mathcal{D}$, $u \in \mathcal{D}'$ (or $\phi \in \mathcal{S}$, $u \in \mathcal{S}'$), $<u, \phi>$ means the scalar product between them. For any $\phi \in \mathcal{S}$, its Fourier transform $\mathcal{F}\phi$, or $\hat{\phi}$ is defined by the formula

$$(\mathcal{F}\phi)(\xi) = \hat{\phi}(\xi) = \int_{R^N} \phi(x) e^{-i<x,\xi>} dx .$$

If $u \in \mathcal{S}'$, the Fourier transform $\hat{u}$ is defined by

$$<\hat{u}, \phi> = <u, \hat{\phi}>, \quad {}^\forall\phi \in \mathcal{S} .$$

A positive-valued continuous function $\mu(\xi)$ defined in Ξ^N is called a temperate weight function [1] if there exist positive constants C and k such that

$$\mu(\xi + \eta) \leqq C(1 + |\xi|^k)\mu(\eta), \quad {}^\forall\xi, \eta \in \Xi^N .$$

For temperate weight functions $\mu_1(\xi)$ and $\mu_2(\xi)$, $\mu_1(\xi) + \mu_2(\xi)$, $\mu_1(\xi)\mu_2(\xi)$ and $\mu_1(\xi)^{-1}$ are also temperate weight functions. If there exist positive constants C_1, C_2 such that

$$C_1 \leqq \frac{\mu_1(\xi)}{\mu_2(\xi)} \leqq C_2 ,$$

then we shall call that $\mu_1(\xi)$ and $\mu_2(\xi)$ are equivalent and write $\mu_1(\xi) \sim \mu_2(\xi)$. By $H^\mu(R^N)$, or H^μ, we shall understand the space of $u \in \mathcal{S}'(R^N)$ such that $\hat{u}$ is a function satisfying

$$||u||_\mu^2 = \left(\frac{1}{2\pi}\right)^N \int_{\Xi^N} |\hat{u}(\xi)|^2 \mu^2(\xi) d\xi < \infty ,$$

that is, $\hat{u} \in L^2_{\mu^2}(\Xi^N)$, the space of square integrable functions with respect to

$\mu^2 d\xi$. $H^{\mu}(R^N)$ is a Hilbert space with the inner product

$$(u\,|\,v)=\left(\frac{1}{2\pi}\right)^N\int_{\Xi^N}\hat{u}(\xi)\overline{\hat{v}(\xi)}\mu^2(\xi)d\xi\,.$$

Its strong dual space is $H^{\frac{1}{\mu}}(R^N)$ where for any $u\in H^{\mu}(R^N)$ and $w\in H^{\frac{1}{\mu}}(R^N)$, we have

$$<w,\bar{u}>=\left(\frac{1}{2\pi}\right)^N\int_{\Xi^N}\hat{w}(\xi)\overline{\hat{u}(\xi)}d\xi\,.$$

Let $N=n+m$. It will be convenient to employ the notations:

$$x=(x',t),\qquad x'=(x_1,\cdots,x_n),\qquad t=(t_1,\cdots,t_m),$$

$$\xi=(\xi',\tau),\qquad \xi'=(\xi_1,\cdots,\xi_n),\qquad \tau=(\tau_1,\cdots,\tau_m),$$

$$D^{\alpha}=D_{x'}^{\alpha'}D_t^{\alpha''},\qquad D_{x'}^{\alpha'}=D_1^{\alpha_1}\cdots D_n^{\alpha_n},\qquad D_t^{\alpha''}=D_{n+1}^{\alpha_{n+1}}\cdots D_N^{\alpha_{n+m}}\,.$$

The scalar product then takes the form $<x,\xi>=<x',\xi'>+<t,\tau>$.

By $R_{x'}^n$, or R^n, we shall denote the subspace of all the points $(x',0)$ and by R_t^m, or R^m, the subspace of all the points $(0,t)$ in R^N. The partial Fourier transforms are defined as follows: Let $\phi\in\mathcal{S}$, then

$$(\mathcal{F}_{x'}\phi)(\xi',t)=\hat{\phi}_{x'}(\xi',t)=\int_{R^n}\phi(x',t)e^{-i<x',\xi'>}dx'\,,$$

$$(\mathcal{F}_t\phi)(x',\tau)=\hat{\phi}_t(x',\tau)=\int_{R^m}\phi(x',t)e^{-i<t,\tau>}dt\,.$$

For $u\in\mathcal{S}'$, we define $\hat{u}_{x'}$, $\hat{u}_t$ by the relations

$$<\hat{u}_{x'},\phi>=<u,\hat{\phi}_{x'}>,\quad <\hat{u}_t,\phi>=<u,\hat{\phi}_t>,\quad \phi\in\mathcal{S}\,.$$

For a temperate weight function $\mu(\xi)$ in R^{n+m}, the integral $\int_{\Xi^m}\mu(\xi',\tau)d\tau$ diverges for every point $\xi'\in\Xi^n$, or converges for every point $\xi'\in\Xi^n$ and it is a temperate weight function in Ξ^n ([5], p. 10).

For any function $u(x)\in\mathcal{D}(R^{n+m})$, the trace $u(x',0)$ on R^n clearly belongs to $\mathcal{D}(R^n)$. $\mathcal{D}(R^{n+m})$ is dense in $H^{\mu}(R^{n+m})$. If the mapping $u\to u(x',0)$ can be continuously extended from $H^{\mu}(R^{n+m})$ into $\mathcal{D}'(R^n)$, then the extended mapping is called a trace mapping on R^n. The trace $u(x',0)$ on R^n exists for every $u\in H^{\mu}(R^{n+m})$ if and only if $\dfrac{1}{\mu(0,\tau)}\in L^2$ ([5], p. 36), and we can write

$$\widehat{u(x',0)}(\xi')=\left(\frac{1}{2\pi}\right)^m\int_{\Xi^m}\hat{u}(\xi',\tau)d\tau\,.$$

2. Preliminary Discussions. Let $P(D)$ be a differential operator, where $P(\xi)=P(\xi',\tau)$ is a non-trivial polynomial in the vector (ξ',τ), i.e. $P(\xi)$

$\neq 0$. For any $u(x) \in \mathcal{D}(R^{n+m})$, $P(D)u(x', 0)$ belongs to $\mathcal{D}(R^n)$. If the mapping $u \to P(D)u(x', 0)$ can be continuously extended from $H^{\mu}(R^{n+m})$ into $\mathcal{D}'(R^n)$, then we shall say that the trace $P(D)u(x', 0)$ on R^n exists for every $u \in H^{\mu}(R^{n+m})$. We start with making an improvement of a result of L.R. Volevič and B.P. Paneyah ([5], p. 39).

PROPOSITION 1. *Let $\mu(\xi)$ be a temperate weight function in R^{n+m}. In order that the trace $P(D)u(x', 0)$ on R^n may exist for every $u \in H^{\mu}(R^{n+m})$, it is necessary and sufficient that either of the following conditions (1), (2) is satisfied:*

(1) $\displaystyle \frac{1}{\mu_{\tilde{P}}^2(\xi')} = \int_{\Xi^m} \frac{\tilde{P}^2(\xi', \tau)}{\mu^2(\xi', \tau)} d\tau < \infty$ *for some* $\xi' \in \Xi^n$;

(2) $\displaystyle \int_{\Xi^m} \frac{|P(\xi', \tau)|^2}{\mu^2(\xi', \tau)} d\tau < \infty$ *for every* $\xi' \in \Xi^n$;

and then $P(D)u(x', 0) \in H^{\mu_{\tilde{P}}}(R^n)$.

In addition, $P(D)u(x', 0)$ *belongs to* $H^{\nu}(R^n)$ *for every* $u \in H^{\mu}(R^{n+m})$ *if and only if either of* (1)′, (2)′ *holds:*

(1)′ $\nu(\xi') \leqq C_1 \mu_{\tilde{P}}(\xi')$ *with a constant* C_1;

(2)′ $\displaystyle \nu^2(\xi') \int_{\Xi^m} \frac{|P(\xi', \tau)|^2}{\mu^2(\xi', \tau)} d\tau \leqq C_2$ *with a constant* C_2.

PROOF: For any $\eta \in \Xi^{n+m}$, the mapping $u \to e^{i<x,\eta>}u$ of $H^{\mu}(R^{n+m})$ into $H^{\mu}(R^{n+m})$ is continuous. If the trace $P(D)u(x', 0)$ is defined for every $u \in H^{\mu}(R^{n+m})$, then $P(D)e^{i<x,\eta>}u(x) = e^{i<x,\eta>}P(D+\eta)u(x)$ has the trace $e^{i<x',\eta'>}P(D+\eta)u(x', 0)$ on R^n. Therefore the mapping

$$u \to P(D+\eta)u(x', 0)$$

of $H^{\mu}(R^{n+m})$ into $\mathcal{D}'(R^n)$ is continuous. That is,

$$\bar{P}(D+\eta)(\phi \otimes \delta) \in (H^{\mu})' = H^{\frac{1}{\mu}}, \quad \forall \phi \in \mathcal{D}(R^n),$$

where δ is the Dirac measure in R^m. This means that

$$\bar{P}(\xi+\eta)\hat{\phi}(\xi') \in L^2_{\frac{1}{\mu^2}} \quad \text{for every } \eta \in \Xi^{n+m}.$$

Consequently we have for every $\eta \in \Xi^{n+m}$

$$\hat{\phi}(\xi')\bar{P}(\xi+\eta) = \sum_{|\alpha| \geqq 0} \frac{\eta^{\alpha}}{\alpha!} \hat{\phi}(\xi')\bar{P}^{(\alpha)}(\xi) \in L^2_{\frac{1}{\mu^2}}(\Xi^{n+m}).$$

$\{\eta^{\alpha}\}$ being linearly independent, we can conclude that $\hat{\phi}(\xi')\bar{P}^{(\alpha)}(\xi) \in L^2_{\frac{1}{\mu^2}}(\Xi^{n+m})$, which implies

$$\int_{\Xi^n} |\hat{\phi}(\xi')|^2 d\xi' \int_{\Xi^m} \frac{\tilde{P}^2(\xi', \tau)}{\mu^2(\xi', \tau)} d\tau < \infty.$$

As a result,

$$\int_{\Xi^m} \frac{\tilde{P}^2(\xi',\tau)}{\mu^2(\xi',\tau)}\, d\tau < \infty \qquad \text{a.e.} \quad \text{in } \Xi^n .$$

Since $\tilde{P}(\xi)$ and $\mu(\xi)$ are temperate weight functions, it follows that the integral is finite at every point of Ξ^n ([5], p. 10).

Clearly the condition (1) implies (2).

Now suppose (2) holds. For any $u \in \mathcal{D}(R^{n+m})$, we have

$$\widehat{(P(D)u(x',0))}(\xi') = \left(\frac{1}{2\pi}\right)^m \int_{\Xi^m} P(\xi)\hat{u}(\xi)d\tau .$$

Then we have for any $\phi \in \mathcal{D}(R^n)$

$$|<P(D)u(x',0),\bar{\phi}>| = \left(\frac{1}{2\pi}\right)^{n+m} \left|\int_{\Xi^{n+m}} P(\xi)\hat{u}(\xi)\overline{\hat{\phi}(\xi')}d\xi\right|$$

$$\leqq \left(\frac{1}{2\pi}\right)^{n+m} \left(\int_{\Xi^n} |\hat{\phi}(\xi')|^2 \left(\int_{\Xi^m} \frac{|P(\xi)|^2}{\mu^2(\xi)} d\tau\right) d\xi'\right)^{\frac{1}{2}} \left(\int_{\Xi^{n+m}} |\hat{u}(\xi)|^2 \mu^2(\xi) d\xi\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{n+m}{2}} \left(\int_{\Xi^n} |\hat{\phi}(\xi')|^2 \left(\int_{\Xi^m} \frac{|P(\xi)|^2}{\mu^2(\xi)} d\tau\right) d\xi'\right)^{\frac{1}{2}} \|u\|_{\mu} .$$

$\mathcal{D}(R^{n+m})$ being dense in $H^{\mu}(R^{n+m})$, in order to prove the existence of the trace under consideration, it is sufficient to show that $\int_{\Xi^m} \frac{|P(\xi)|^2}{\mu^2(\xi)} d\tau$ is a slowly increasing function in ξ'. Taking into account the formula $P(\xi',\tau) = \sum_{|\alpha|\geqq 0} \frac{\xi'^{\alpha'}}{\alpha'!} P^{(\alpha')}(0,\tau)$, we see that $\int_{\Xi^m} \frac{|P^{(\alpha')}(0,\tau)|^2}{\mu^2(0,\tau)} d\tau < \infty$. Since there exist positive constants C, k such that $\mu(0,\tau) \leqq C(1+|\xi'|^k)\mu(\xi',\tau)$, it follows that $\int_{\Xi^m} \frac{|P^{(\alpha')}(0,\tau)|^2}{\mu^2(\xi',\tau)} d\tau < \infty$. We can therefore conclude that $\int_{\Xi^m} \frac{|P(\xi)|^2}{\mu^2(\xi)} d\tau$ is a slowly increasing function in ξ'.

If the trace $P(D)u(x',0)$ exists for every $u \in H^{\mu}(R^{n+m})$, then we have for any $u \in \mathcal{D}(R^{n+m})$

$$\|P(D)u(x',0)\|^2_{\mu_{\tilde{P}}} = \left(\frac{1}{2\pi}\right)^{n+2m} \int_{\Xi^n} \mu^2_{\tilde{P}}(\xi') \left|\int_{\Xi^m} P(\xi)\hat{u}(\xi)d\tau\right|^2 d\xi'$$

$$= \left(\frac{1}{2\pi}\right)^{n+2m} \int_{\Xi^n} \mu^2_{\tilde{P}}(\xi') \left(\int_{\Xi^m} \frac{|P(\xi)|^2}{\mu^2(\xi)} d\tau\right) \left(\int_{\Xi^m} \mu^2(\xi)|\hat{u}(\xi)|^2 d\tau\right) d\xi'$$

$$\leqq \left(\frac{1}{2\pi}\right)^{n+2m} \int_{\Xi^n} \mu^2_{\tilde{P}}(\xi') \left(\int_{\Xi^m} \frac{\tilde{P}^2(\xi)}{\mu^2(\xi)} d\tau\right) \left(\int_{\Xi^m} \mu^2(\xi)|\hat{u}(\xi)|^2 d\tau\right) d\xi'$$

$$= \left(\frac{1}{2\pi}\right)^m \|u\|^2_{\mu} .$$

Therefore, $\mathcal{D}(R^{n+m})$ being dense in $H^{\mu}(R^{n+m})$, the trace $P(D)u(x',0) \in H^{\mu_{\tilde{P}}}(R^n)$.

Thus the proof of the first part of Proposition 1 is complete. Along the same line as above, if $P(D)u(x',0)$ belongs to $H^{\nu}(R^n)$ for every $u \in H^{\mu}(R^{n+m})$, then

$$\bar{P}(D+\eta)(\phi\otimes\delta) \in (H^{\mu})' = H^{\frac{1}{\mu}}, \qquad \forall\phi \in (H^{\nu})' = H^{\frac{1}{\nu}}$$

for any $\eta \in \Xi^{n+m}$. This implies that $\hat{\phi}(\xi')\tilde{P}(\xi) \in L^2_{\frac{1}{\mu^2}}(\Xi^{n+m})$. That is,

$$\int_{\Xi^n} \frac{|\hat{\phi}(\xi')|^2}{\nu^2(\xi')}\left(\nu^2(\xi')\int_{\Xi^m} \frac{\tilde{P}^2(\xi)}{\mu^2(\xi)}\, d\tau\right)d\xi' < \infty .$$

Then for some constant $C > 0$

$$\nu^2(\xi')\int_{\Xi^m} \frac{\tilde{P}^2(\xi)}{\mu^2(\xi)}\, d\tau \leqq C^2 \qquad \text{a.e.}$$

Since $\mu_{\tilde{P}}(\xi')$ and $\nu(\xi')$ are temperate weight functions, we have for every $\xi' \in \Xi^n$

$$\nu(\xi') \leqq C\mu_{\tilde{P}}(\xi') .$$

Thus (1)′ follows.

Clearly (1)′ implies (2)′.

Suppose (2)′ holds. After calculation, as in the proof of the first part, we have for some constant C_1

$$||P(D)u(x',0)||_{\nu} \leqq C_1||u||_{\mu}, \qquad \forall u \in \mathcal{D}(R^{n+m}) .$$

$\mathcal{D}(R^{n+m})$ is dense in $H^{\mu}(R^{n+m})$. Therefore, for every $u \in H^{\mu}(R^{n+m})$, the trace $P(D)u(x',0)$ exists and belongs to $H^{\nu}(R^n)$.

Thus the proof is complete.

REMARK 1. Let Q be a non-trivial polynomial weaker than P, that is, $Q(\xi) \leqq C\tilde{P}(\xi)$, $\xi \in \Xi^{n+m}$ with a constant C. Then $\tilde{Q}(\xi) \leqq C\tilde{P}(\xi)$ with a constant C ([1], p. 73). Proposition 1 shows that if the trace $P(D)u(x',0)$ exists for every $u \in H^{\mu}(R^{n+m})$, then $Q(D)u(x',0)$ exists, too.

PROPOSITION 2. *Suppose* $\frac{1}{\mu_{\tilde{P}}^2} = \int_{\Xi^m} \frac{\tilde{P}^2(\xi',\tau)}{\mu^2(\xi',\tau)}\, d\tau < \infty$. *The trace mapping* $\mathcal{T}$: $u \rightarrow P(D)u(x',0)$ *of* $H^{\mu}(R^{n+m})$ *into* $H^{\mu_{\tilde{P}}}(R^n)$ *is an epimorphism if and only if each of the following conditions is satisfied*:

(1) *the range of the transposed mapping* ${}^t\mathcal{T}$ *is closed in* $H^{\frac{1}{\mu}}(R^{n+m})$;

(2) $\frac{1}{\nu^2(\xi')} = \int_{\Xi^m} \frac{|P(\xi',\tau)|^2}{\mu^2(\xi',\tau)}\, d\tau$ *is a temperate weight function*;

(3) *if* $f(\xi')\bar{P}(\xi) \in L^2_{\frac{1}{\mu^2}}(\Xi^{n+m})$, *where* $f(\xi')$ *is locally integrable, then*

$f \in L^2_{\frac{1}{\mu^2_{\tilde{P}}}}(\Xi^n)$.

If each of these conditions is satisfied, then $\nu(\xi') \sim \mu_{\tilde{P}}(\xi')$.

PROOF: Consider the transposed mapping ${}^t\mathcal{T}$ of $H^{\frac{1}{\mu_{\tilde{P}}}}(R^n)$ into $H^{\frac{1}{\mu}}(R^{n+m})$. We note that

$$\widehat{{}^t\mathcal{T} v}(\xi) = \hat{v}(\xi')\bar{P}(\xi)\,, \qquad v \in H^{\frac{1}{\mu_{\tilde{P}}}}(R^n)\ .$$

Indeed, it is sufficient to verify this relation when $v \in \mathcal{D}(R^n)$. Let f be any element of $\mathcal{D}(R^{n+m})$. Then the relations

$$<\mathcal{T}f, \bar{v}> = \left(\frac{1}{2\pi}\right)^n \int_{\Xi^n} \widehat{P(D)f(x', 0)}(\xi')\overline{\hat{v}(\xi')}d\xi'$$

$$= \left(\frac{1}{2\pi}\right)^{n+m} \int_{\Xi^n} \left(\int_{\Xi^m} P(\xi)\hat{f}(\xi)d\tau\right)\overline{\hat{v}(\xi')}d\xi'$$

$$= \left(\frac{1}{2\pi}\right)^{n+m} \int_{\Xi^{n+m}} \overline{\hat{v}(\xi')}P(\xi)\hat{f}(\xi')d\xi$$

and

$$<\overline{{}^t\mathcal{T}v}, f> = \left(\frac{1}{2\pi}\right)^{n+m} \int_{\Xi^{n+m}} \overline{\widehat{{}^t\mathcal{T}v}(\xi)}\hat{f}(\xi')d\xi$$

show our assertion.

The mapping ${}^t\mathcal{T}$ is injective. In fact, let ${}^t\mathcal{T}v = 0$, that is, $\widehat{{}^t\mathcal{T}v}(\xi) = 0$, then

$$\int_{\Xi^n} |\hat{v}(\xi')|^2 \left(\int_{\Xi^m} \frac{|P(\xi', \tau)|^2}{\mu^2(\xi', \tau)} d\tau\right) d\xi' = 0\ .$$

Since the polynominal $P(\xi', \tau)$ is non-trivial, $\int_{\Xi^m} \frac{|P(\xi', \tau)|^2}{\mu^2(\xi', \tau)} d\tau$ does not identically vanish in any relatively compact open subset of Ξ^n. Thus $\hat{v}(\xi') = 0$ *a.e.* in Ξ^n, which implies that $v = 0$.

Consequently the mapping $\mathcal{T}$ is an epimorphism if and only if the range of ${}^t\mathcal{T}$ is closed in $H^{\frac{1}{\mu}}(R^{n+m})$.

Suppose the range of ${}^t\mathcal{T}$ is closed, then there is a constant $C > 0$ such that $\|v\|_{\frac{1}{\mu_{\tilde{P}}}} \leq C\|{}^t\mathcal{T}v\|_{\frac{1}{\mu}}$ for every $v \in H^{\frac{1}{\mu_{\tilde{P}}}}(R^n)$. That is,

$$\int_{\Xi^n} |\hat{v}(\xi')|^2 \frac{1}{\mu^2_{\tilde{P}}(\xi')} d\xi' \leq C^2 \int_{\Xi^{n+m}} \frac{|\hat{v}(\xi')|^2 |P(\xi)|^2}{\mu^2(\xi)} d\xi$$

$$= C^2 \int_{\Xi^n} |\hat{v}(\xi')|^2 \left(\int_{\Xi^m} \frac{|P(\xi', \tau)|^2}{\mu^2(\xi', \tau)} d\tau\right) d\xi'\ .$$

Consequently

$$\frac{1}{\mu_{\tilde{P}}^2(\xi')} \leqq C^2 \int_{\Xi^m} \frac{|P(\xi',\tau)|^2}{\mu^2(\xi',\tau)} d\tau .$$

Since trivially $\int_{\Xi^m} \frac{|P(\xi',\tau)|^2}{\mu^2(\xi',\tau)} d\tau \leqq \frac{1}{\mu_{\tilde{P}}^2(\xi')}$, we have

$$\frac{1}{C} \frac{1}{\mu_{\tilde{P}}(\xi')} \leqq \frac{1}{\nu(\xi')} \leqq \frac{1}{\mu_{\tilde{P}}(\xi')} .$$

Consequently $\nu(\xi')$ is a temperate weight function equivalent to $\mu_{\tilde{P}}(\xi')$. Thus (1) implies (2).

Suppose $\nu(\xi')$ is a temperate weight function. First we show that $\nu(\xi') \sim \mu_{\tilde{P}}(\xi')$. For any $\eta \in \Xi^{n+m}$ with $|\eta| \leqq 1$, we can find positive constants C_1, C_2 such that

$$\frac{C_1}{\nu^2(\xi')} \geqq \frac{1}{\nu^2(\xi'+\eta')} = \int_{\Xi^m} \frac{|P(\xi+\eta)|^2}{\mu^2(\xi+\eta)} d\tau \geqq C_2 \int_{\Xi^m} \frac{|P(\xi+\eta)|^2}{\mu^2(\xi)} d\tau .$$

Taking into account the formula $P(\xi+\eta) = \sum_{|\alpha| \leqq 0} \frac{\eta^\alpha}{\alpha!} P^{(\alpha)}(\xi)$, we have for a positive constant C_3

$$\frac{1}{\nu^2(\xi')} \geqq C_3 \int_{\Xi^m} \frac{|P^{(\alpha)}(\xi',\tau)|^2}{\mu^2(\xi',\tau)} d\tau .$$

It follows therefore that $\nu(\xi') \sim \mu_{\tilde{P}}(\xi')$. Now let $f(\xi')$ be a locally integrable function such that $f(\xi')\tilde{P}(\xi) \in L^2_{\frac{1}{\mu^2}}$. Then

$$\int_{\Xi^n} |f(\xi')|^2 \frac{1}{\nu^2(\xi')} d\xi' = \int_{\Xi^{n+m}} \frac{|f(\xi')|^2 |P(\xi',\tau)|^2}{\mu^2(\xi',\tau)} d\xi < \infty .$$

This together with the relation $\nu(\xi') \sim \mu_{\tilde{P}}(\xi')$ shows that $f \in L^2_{\frac{1}{\mu_{\tilde{P}}}}$. Thus (2) implies (3).

Suppose (3) holds. We note that the integral $\int_{\Xi^m} \frac{|P(\xi',\tau)|^2}{\mu^2(\xi',\tau)} d\tau$ does not vanish in Ξ^n. Let ξ_0' be any point in Ξ^n, and B the closed unit ball with center ξ_0'. Consider the set E of all integrable functions $f(\xi')$ such that $\operatorname{supp} f \subset B$ and

$$\int_{\Xi^n} |f(\xi')|^2 \left(\int_{\Xi^m} \frac{|P(\xi',\tau)|^2}{\mu^2(\xi',\tau)} d\tau \right) d\xi' < \infty .$$

E is a Banach space with the norm $\|f\|_E$:

$$\|f\|_E^2=\Big[\int_B |f(\xi')|\, d\xi'\Big]^2+\int_B |f(\xi')|^2\Big(\int_{\Xi^m}\frac{|P(\xi',\tau)|^2}{\mu^2(\xi',\tau)}\,d\tau\Big)d\xi' \ .$$

Owing to the closed graph theorem the injection $E\to H^{\frac{1}{\mu_{\bar P}}}(R^n)$ is continuous. Let B_ε be the closed ball with center ξ_0' and radius ε, $0<\varepsilon<1$. Taking $f=\chi_{B_\varepsilon}$, the characteristic function of B_ε, we have for a positive constant C

$$C|B_\varepsilon|\leqq|B_\varepsilon|^2+\int_{B_\varepsilon}\Big(\int_{\Xi^m}\frac{|P(\xi',\tau)|^2}{\mu^2(\xi',\tau)}\,d\tau\Big)d\xi'$$

$$C\leqq|B_\varepsilon|+\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon}\Big(\int_{\Xi^m}\frac{|P(\xi',\tau)|^2}{\mu^2(\xi',\tau)}\,d\tau\Big)d\xi' \ ,$$

where $|B_\varepsilon|$ stands for the Lebesgue measure of B_ε. Now, passing to the limit $\varepsilon\to 0$, we have

$$\int_{\Xi^m}\frac{|P(\xi_0',\tau)|^2}{\mu^2(\xi_0',\tau)}\,d\tau\geqq C \ .$$

Let us now show that the range of ${}^t\mathcal{T}$ is closed in $H^{\frac{1}{\mu}}(R^{n+m})$. Let $\{v^j(\xi')\}$ be any sequence of $H^{\frac{1}{\mu_{\bar P}}}(R^n)$ such that $\{{}^t\mathcal{T}v^j\}$ converges in $H^{\frac{1}{\mu}}(R^{n+m})$ to u. Then $\hat v^j(\xi')\bar P(\xi)$ converges in $L^2_{\frac{1}{\mu^2}}(\Xi^{n+m})$ to $\hat u$. It follows that $\hat v^j(\xi')\Big[\int_{\Xi^m}\frac{|P(\xi',\tau)|^2}{\mu^2(\xi',\tau)}\,d\tau\Big]^{\frac{1}{2}}$ is a Cauchy sequence in $L^2(\Xi^n)$. Since $\int_{\Xi^m}\frac{|P(\xi',\tau)|^2}{\mu^2(\xi',\tau)}\,d\tau>0$, we see that $\hat v^j(\xi')$ converges in $L^1_{loc}(\Xi^n)$ to a function $f(\xi')$. Consequently we can write $\hat u(\xi)=f(\xi')\bar P(\xi)$. The condition (3) implies that $f\in L^2_{\frac{1}{\mu^2_{\bar P}}}(\Xi^n)$, so that u belongs to the range of ${}^t\mathcal{T}$. Therefore (3) implies (1).

Thus the proof is complete.

Remark 2. If $P(D)$ is a polynomial in D_t and

$$\frac{1}{\nu^2(\xi')}=\int_{\Xi^m}\frac{|P(\tau)|^2}{\mu^2(\xi',\tau)}\,d\tau<\infty \ ,$$

then it is clear that $\nu(\xi')$ is a temperate weight function. By virtue of Proposition 2, the mapping $u\to P(D_t)u(x',0)$ of $H^{\mu}(R^{n+m})$ into $H^{\mu_{\bar P}}(R^n)$ is an epimorphism.

Remark 3. If the differential operator $P(D)$ is elliptic or more generally hypoelliptic ([1], p. 75, p. 100), then the mapping $u\to P(D)u(x',0)$ of $H^{\mu}(R^{n+m})$ into $H^{\mu_{\bar P}}(R^n)$ is an epimorphism. In fact, there exist constants C, K such that

$$|P^{(\alpha)}(\xi)|\leqq C|P(\xi)| \quad \text{for} \quad |\xi|>K \ .$$

If $P(\xi)=\sum_{|\alpha''|\geqq 0}\frac{\tau^{\alpha''}}{\alpha''!}P^{(\alpha'')}(\xi', 0)=0$, there is a $P^{(\alpha'')}(\xi', 0)$, not identically vanishing. We can therefore find $\sigma_j \in \Xi^m$, $1\leqq j\leqq s$, such that, for any $|\xi|\leqq K$, we have for a constant C_0

$$|P^{(\alpha)}(\xi)|\leqq C_0\big(|P(\xi)|+|P(\xi',\tau+\sigma_1)+\cdots+|P(\xi',\tau+\sigma_s)|\big) .$$

Consequently we have for a constant C'

$$|P^{(\alpha)}(\xi)|\leqq C'\big(|P(\xi)|+|P(\xi',\tau+\sigma_1)|+\cdots+|P(\xi',\tau+\sigma_s)|\big), \quad \xi \in \Xi^{n+m} .$$

Since $\mu(\xi)$ is a temperate weight function, we have for some constant C_j

$$\int_{\Xi^m}\frac{|P(\xi',\tau+\sigma_j)|^2}{\mu^2(\xi',\tau)}\,d\tau=\int_{\Xi^m}\frac{|P(\xi',\tau)|^2}{\mu^2(\xi',\tau-\sigma_j)}\,d\tau\leqq C_j\int_{\Xi^m}\frac{|P(\xi',\tau)|^2}{\mu^2(\xi',\tau)}\,d\tau .$$

Consequently we have for a constant C''

$$\int_{\Xi^m}\frac{|P^{(\alpha)}(\xi',\tau)|^2}{\mu^2(\xi',\tau)}\,d\tau\leqq C''\int_{\Xi^m}\frac{|P(\xi',\tau)|^2}{\mu^2(\xi',\tau)}\,d\tau .$$

By virtue of Proposition 2, the trace mapping $\mathfrak{T}$ is an epimorphism.

REMARK 4. Let $n=m=1$. If we put $\mu(\xi',\tau)=1+\tau^2$ and $P(D)=D_{x'}D_t$. Then we have $P(\xi',\tau)=\xi'\tau$, $\tilde{P}^2(\xi',\tau)=(1+\xi'^2)(1+\tau^2)$ and $\mu_{\tilde{P}}\sim(1+\xi'^2)^{-\frac{1}{2}}$. On the other hand

$$\int_{-\infty}^{\infty}\frac{|P(\xi',\tau)|^2}{\mu^2(\xi',\tau)}\,d\tau=\xi'^2\int_{-\infty}^{\infty}\frac{\tau^2}{(1+\tau^2)^2}\,d\tau .$$

This is not a temperate weight function. Thus the mapping $u\to D_{x'}D_t u(x',0)$ of $H^{\mu}(R^2)$ into $H^{\mu_{\tilde{P}}}(R^1)$ is not an epimorphism.

3. Trace Theorems. Let $\mu(\xi)=\mu(\xi',\tau)=\mu(\xi_1,\cdots,\xi_n,\tau)$ be a temperate weight function in Ξ^{n+1}. We assume that

$$\frac{1}{\nu_l^2(\xi')}=\int_{-\infty}^{\infty}\frac{\tau^{2l}}{\mu^2(\xi',\tau)}\,d\tau<\infty ,$$

where l is a non-negative integer. We put

$$\frac{1}{\nu_p^2(\xi')}=\int_{-\infty}^{\infty}\frac{\tau^{2p}}{\mu^2(\xi',\tau)}\,d\tau \qquad \text{for} \quad 0\leqq p\leqq l .$$

Let us consider the trace mapping $\mathfrak{T}$:

$$u(x',t)\to\big(u(x',0),\ D_t u(x',0),\ \cdots,\ D_t^l u(x',0)\big)$$

of $H^{\mu}(R^{n+1})$ into $\prod_{p=0}^{l} H^{\nu_p}(R^n)$. In this section, we discuss the conditions in order that $\mathfrak{T}$ may be an epimorphism.

THEOREM 1. *A necessary and sufficient condition in order that the trace mapping $\mathfrak{T}$ of $H^{\mu}(R^{n+1})$ into $\boldsymbol{H}=\prod_{p=0}^{l} H^{\nu_p}(R^n)$ may be an epimorphism is that each of the following conditions is satisfied:*

(1) *the range of the transposed mapping ${}^t\mathfrak{T}$ is closed in $H^{\frac{1}{\mu}}(R^{n+1})$;*

(2) *there is a positive constant C such that*

$$\begin{vmatrix} \kappa_0 & \kappa_1 & \cdots\kappa_l \\ \kappa_1 & \kappa_2 & \cdots\kappa_{l+1} \\ & \cdots & \\ \kappa_l & \kappa_{l+1} & \cdots\kappa_{2l} \end{vmatrix} \geqq C\kappa_0\kappa_2\cdots\kappa_{2l}, \quad \text{where} \quad \kappa_p(\xi')=\int_{-\infty}^{\infty}\frac{\tau^p}{\mu^2(\xi',\tau)}d\tau\,;$$

(3) *if $u\in H^{\frac{1}{\mu}}(R^{n+1})$ and $\hat{u}(\xi)=f_0(\xi')+f_1(\xi')\tau+\cdots+f_l(\xi')\tau^l$, then $f_p(\xi')\in L^2_{\frac{1}{\nu_p^2}}(\Xi^n)$ for $p=0, 1, \cdots, l$;*

(4) *if $u\in H^{\frac{1}{\mu}}(R^{n+1})$ and $\hat{u}(\xi)=f_0(\xi')+f_1(\xi')\tau+\cdots+f_l(\xi')\tau^l$, then $\hat{u}\left(\xi',\frac{\tau}{2}\right)\in L^2_{\frac{1}{\mu^2}}(\Xi^{n+1})$.*

PROOF: Consider the transposed mapping ${}^t\mathfrak{T}$ of $\boldsymbol{H}'=\prod_{p=0}^{l} H^{\frac{1}{\nu_p}}(R^n)$ into $H^{\frac{1}{\mu}}(R^{n+1})$. Then

$$\widehat{{}^t\mathfrak{T}\vec{v}}(\xi)=\sum_{p=0}^{l}\hat{v}_p(\xi')\tau^p,\ \vec{v}=\{v_0(x'),\cdots,v_l(x')\}\in\boldsymbol{H}'\ .$$

Indeed, it is sufficient to verify this relation when $v_p\in\mathcal{D}(R^n)$, $p=0, 1, \cdots, l$. Let f be any element of $\mathcal{D}(R^{n+1})$. Then the relations

$$\begin{aligned} <\mathfrak{T}f,\vec{v}> &=\left(\frac{1}{2\pi}\right)^n\sum_{p=0}^{l}\int_{\Xi^n}\widehat{D_t^pf(x',0)}(\xi')\overline{\hat{v}_p(\xi')}d\xi' \\ &=\left(\frac{1}{2\pi}\right)^{n+1}\sum_{p=0}^{l}\int_{\Xi^n}\{\int_{-\infty}^{\infty}\hat{f}(\xi)\tau^p d\tau\}\overline{\hat{v}_p(\xi')}d\xi' \\ &=\left(\frac{1}{2\pi}\right)^{n+1}\sum_{p=0}^{l}\int_{\Xi^{n+1}}\overline{\hat{v}_p(\xi')\tau^p}\hat{f}(\xi')d\xi \end{aligned}$$

and

$$<\overline{{}^t\mathfrak{T}\vec{v}},f>=\left(\frac{1}{2\pi}\right)^{n+1}\int_{\Xi^{n+1}}\overline{\widehat{{}^t\mathfrak{T}\vec{v}}(\xi)}\hat{f}(\xi)d\xi$$

show our assertion.

The mapping ${}^t\mathfrak{T}$ is injective. In fact, let ${}^t\mathfrak{T}\vec{v}=0$, that is, $\sum_{p=0}^{l}\hat{v}_p(\xi')\tau^p=0$. Since $\{\tau^p\}$, $p=0, 1, \cdots, l$, is linearly independent, it follows that $\vec{v}=0$.

Consequently the mapping $\mathfrak{T}$ is an epimorphism if and only if the range of ${}^t\mathfrak{T}$ is closed in $H^{\frac{1}{\mu}}(R^{n+1})$.

Suppose the range of ${}^t\mathfrak{T}$ is closed, there is a constant $C>0$ such that $\|\vec{v}\|_{\boldsymbol{H}'}\leqq C\|{}^t\mathfrak{T}\vec{v}\|_{\frac{1}{\mu}}$ for every $\vec{v}\in\boldsymbol{H}'$. That is,

$$\sum_{p=0}^{l}\int_{\Xi^n}\frac{|\hat{v}_p(\xi')|^2}{\nu_p^2(\xi')}d\xi'\leqq C^2\int_{\Xi^{n+1}}\frac{|\sum_{p=0}^{l}\hat{v}_p(\xi')\tau^p|^2}{\mu^2(\xi',\tau)}d\xi$$

$$=C^2\sum_{p,q=0}^{l}\int_{\Xi^{n+1}}\frac{\hat{v}_p(\xi')\overline{\hat{v}_q(\xi')}\tau^{p+q}}{\mu^2(\xi',\tau)}d\xi\ .$$

If we put $g_p(\xi')=\dfrac{\hat{v}_p(\xi')}{\nu_p(\xi')}\in L^2$, then

$$\sum_{p=0}^{l}\int_{\Xi^n}|g_p(\xi')|^2d\xi'\leqq C^2\sum_{p,q=0}^{l}\int_{\Xi^n}g_p(\xi')\overline{g_q(\xi')}\kappa_{p+q}(\xi')\nu_p(\xi')\nu_q(\xi')d\xi'\ .$$

This inequality holds for any $g_p\in L^2(\Xi^n)$. Let ξ_0' be a point in Ξ^n, and B_ε the closed ball with center ξ_0' and radius $\varepsilon>0$. Taking $g_p=a_p\chi_{B_\varepsilon}$, where a_p is a real number and χ_{B_ε} the characteristic function of B_ε,

$$|B_\varepsilon|\sum_{p=0}^{l}|a_p|^2\leqq C^2\sum_{p,q=0}^{l}a_pa_q\int_{B_\varepsilon}\kappa_{p+q}(\xi')\nu_p(\xi')\nu_q(\xi')d\xi'\ .$$

Now passing to the limit $\varepsilon\to 0$, we have

$$\sum_{p=0}^{l}|a_p|^2\leqq C^2\sum_{p,q=0}^{l}a_pa_q\kappa_{p+q}(\xi_0')\nu_p(\xi_0')\nu_q(\xi_0')\ .$$

Therefore we have for a positive constant C'

$$\det|\kappa_{p+q}(\xi')\nu_p(\xi')\nu_q(\xi')|\geqq C',\qquad \xi'\in\Xi^n\ .$$

That is, $\det|\kappa_{p+q}|\geqq C'\kappa_0\kappa_2\cdots\kappa_{2l}$. Thus (1) implies (2).

Suppose (2) holds. Let u be any element of $H^{\frac{1}{\mu}}(R^{n+1})$ such that $\hat{u}=f_0(\xi')+f_1(\xi')\tau+\cdots+f_l(\xi')\tau^l$. If we put $\hat{w}(\xi)=\dfrac{\hat{u}(\xi)}{\mu^2(\xi)}$, then $w\in H^\mu(R^{n+1})$, $h_p=\widehat{D_t^pw(x',0)}\in L^2_{\nu_p^2}(\Xi^n)$, and

$$h_p=\widehat{D_t^pw(x',0)}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tau^p\hat{w}(\xi)d\tau$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tau^p\frac{\sum_{q=0}^{l}f_q(\xi')\tau^q}{\mu^2(\xi)}\,d\tau=\frac{1}{2\pi}\sum_{q=0}^{l}f_q(\xi')\kappa_{p+q}(\xi')\ .$$

Thus we have

$$\sum_{q=0}^{l}\frac{f_q(\xi')}{\nu_q(\xi')}\kappa_{p+q}(\xi')\nu_p(\xi')\nu_q(\xi')=2\pi h_p(\xi')\nu_p(\xi')\in L^2\ .$$

Since $\det|\kappa_{p+q}\nu_p\nu_q|\geqq C_2>0$ and $|\kappa_{p+q}\nu_p\nu_q|\leqq 1$, we have $\frac{f_q(\xi')}{\nu_q(\xi')}\in L^2$, that is, $f_q\in L^2_{\frac{1}{\nu_q^2}}$. Thus (2) implies (3).

Suppose (3) holds. Let u be any element of $H^{\frac{1}{\mu}}(R^{n+1})$ such that $\hat{u}(\xi)=\sum_{p=0}^{l}f_p(\xi')\tau^p$. By our assumption, $f_p(\xi')\in L^2_{\frac{1}{\nu_p^2}}(\Xi^n)$, that is, $f_p(\xi')\tau^p\in L^2_{\frac{1}{\mu^2}}(\Xi^{n+1})$. Therefore $\hat{u}\left(\xi',\frac{\tau}{2}\right)\in L^2_{\frac{1}{\mu^2}}(\Xi^{n+1})$. Thus (3) implies (4).

Suppose (4) holds. We shall show that the range of ${}^t\mathfrak{T}$ is closed in $H^{\frac{1}{\mu}}(R^{n+1})$. Let $\{\vec{v}^j\}$ be any sequence of $\boldsymbol{H}'$ such that $\{{}^t\mathfrak{T}\vec{v}^j\}$ converges in $H^{\frac{1}{\mu}}(R^{n+1})$ to u. That is, $\sum_{p=0}^{l}\hat{v}_p^j(\xi')\tau^p$ converges in $L^2_{\frac{1}{\mu^2}}(\Xi^{n+1})$ to $\hat{u}$. Since $\mu(\xi)$ is continuous and positive, it converges in L^1_{loc} to $\hat{u}$. We can write $\hat{u}=\sum_{p=0}^{l}\hat{v}_p(\xi')\tau^p$ with $\hat{v}_p(\xi')\in L^1_{loc}$. Indeed, $\sum_{p=0}^{l}\int_0^1(\hat{v}_p^j(\xi')\tau^p-\hat{u}(\xi))\tau^q d\tau\to 0$ in L^1_{loc} as $j\to\infty$ for $q=0,1,\cdots,l$. Therefore $\sum_{p=0}^{l}\hat{v}_p^j(\xi')\int_{-\infty}^{\infty}\tau^{p+q}d\tau$ converges in L^1_{loc}. Since $\det|\int_0^1\tau^{p+q}d\tau|>0$, $\hat{v}_p^k$ converges in L^1_{loc} to a $\hat{v}_p$ and we can write $\hat{u}=\sum_{p=0}^{l}\hat{v}_p(\xi')\tau^p$. From our assumption it follows that $\hat{u}\left(\xi',\frac{\tau}{2^j}\right)=\sum_{p=0}^{l}\hat{v}_p(\xi')\left(\frac{\tau}{2^j}\right)^p\in L^2_{\frac{1}{\mu^2}}(\Xi^{n+1})$ for $j=0,1,\cdots,l$. Therefore $\hat{v}_p(\xi')\tau^p\in L^2_{\frac{1}{\mu^2}}(\Xi^{n+1})$, that is, $v_p\in H^{\frac{1}{\nu_p}}(R^n)$, $p=0,1,\cdots,l$, so that u belongs to the range of ${}^t\mathfrak{T}$. Therefore (4) implies (1).

Thus the proof is complete.

When $l=0$, the mapping $\mathfrak{T}$ is always an epimorphism since the condition (2) of Theorem 1 is satisfied.

REMARK 5. The mapping $\mathfrak{T}$ is not always an epimorphism. Let $n=1$. Consider the differential operator $P(D)=(D_{x'}-D_t)^2$. Put $\mu(\xi)=\tilde{P}^2(\xi)\sim 1+(\xi'-\tau)^2$. Here we can take $l=1$.

$$\kappa_0=\int_{-\infty}^{\infty}\frac{1}{\mu^2}\,d\tau=\int_{-\infty}^{\infty}\frac{d\tau}{(1+\tau^2)^2}=C_0\ ,$$

$$\kappa_1=\int_{-\infty}^{\infty}\frac{\tau}{\mu^2}\,d\tau=\int_{-\infty}^{\infty}\frac{\xi'-\tau}{(1+\tau^2)^2}\,d\tau=\xi'\int_{-\infty}^{\infty}\frac{d\tau}{(1+\tau^2)^2}=C_0\xi'\ ,$$

$$\kappa_2=\int_{-\infty}^{\infty}\frac{\tau^2}{\mu^2}\,d\tau=\int_{-\infty}^{\infty}\frac{\tau^2}{(1+\tau^2)^2}\,d\tau+\xi'^2\int_{-\infty}^{\infty}\frac{d\tau}{(1+\tau^2)^2}=C_1+C_0\xi'^2 .$$

Then $\begin{vmatrix}\kappa_0 & \kappa_1\\ \kappa_1 & \kappa_2\end{vmatrix}=C_0C_1,\ \kappa_0\kappa_2=C_0C_1+C_0^2\xi'^2$. Therefore the condition (2) of Theorem 1 is not satisfied, so that the mapping $\mathfrak{T}$ is not an epimorphism.

COROLLARY. If $\mu(\xi)$ is a temperate weight function in Ξ^{n+1} such that

$$\mu(\xi', 2\tau)\geqq C\mu(\xi',\tau)$$

for a constant C, the trace mapping $\mathfrak{T}$ is an epimorphism.

PROOF: It is known that $H^{\mu}(R^{n+1})\subset H^{\nu}(R^{n+1})$ if and only if $\nu\leqq C\mu$, C being a constant ([5], p. 33). One can easily verify that the condition $\mu(\xi', 2\tau)\geqq C\mu(\xi',\tau)$ is equivalent to saying that $\hat{u}\left(\xi,\frac{\tau}{2}\right)$ belongs to $L^2_{\frac{1}{\mu^2}}(\Xi^{n+1})$ for every $u\in H^{\frac{1}{\mu}}(R^{n+1})$. It follows therefore from the condition (4) of the preceding proposition that $\mathfrak{T}$ is an epimorphism.

PROPOSITION 3. *Let $\vec{f}=\{f_0(x'),\cdots,f_l(x')\}$ be an arbitrary element of $\prod_{p=0}^{l}H^{\nu_p}(R^n)$ and $\psi\in\mathcal{D}(R^1)$ be equal to 1 in a neighbourhood of 0. Suppose there exist a positive continuous $\lambda_p(\xi')$ in Ξ^n and a slowly increasing continuous function $\Phi_p(\tau)$ in Ξ^1 for $p=0,1,\cdots,l$ such that*

$$\mu(\xi',\lambda_p\tau)\leqq\lambda_p^{p+\frac{1}{2}}(\xi')\nu_p(\xi')\Phi_p(\tau) .$$

If we put

$$\hat{u}_{x'}(\xi',t)=\sum_{p=0}^{l}\hat{f}_p(\xi')\frac{(it)^p}{p\,!}\psi(\lambda_pt),$$

then u belongs to $H^{\mu}(R^{n+1})$ and $D_t^pu(x',0)=f_p(x')$ for $p=0,1,\cdots,l$.

PROOF: We can write

$$\hat{u}(\xi',\tau)=\sum_{p=0}^{l}\frac{(-1)^p}{p\,!}\hat{f}_p(\xi')\left(\frac{d}{d\tau}\right)^p\int_{-\infty}^{\infty}\psi(\lambda_pt)e^{-it\tau}dt$$

$$=\sum_{p=0}^{l}\frac{(-i)^p}{p\,!}\hat{f}_p(\xi')\frac{1}{\lambda_p^{p+1}}\hat{\psi}^{(p)}\left(\frac{\tau}{\lambda_p}\right) .$$

After a change of variable $\tau\to\lambda_p\tau$ and using the fact that $\int_{-\infty}^{\infty}|\hat{\psi}^{(p)}(\tau)|^2|\Phi_p(\tau)|^2d\tau<\infty$, we have

$$\int_{\Xi^{n+1}} |\hat{u}(\xi)|^2 \mu^2(\xi) d\xi$$

$$\leqq (l+1) \sum_{p=0}^{l} \int_{\Xi^n} \frac{1}{p!} |\hat{f}_p(\xi')|^2 \frac{d\xi'}{\lambda_p^{2p+1}(\xi')} \int_{-\infty}^{\infty} |\hat{\psi}^{(p)}(\tau)|^2 \mu^2(\xi', \lambda_p \tau) d\tau$$

$$\leqq (l+1) \sum_{p=0}^{l} \int_{\Xi^n} \frac{1}{p!} |\hat{f}_p(\xi')|^2 \nu_p^2(\xi') d\xi' \int_{-\infty}^{\infty} |\hat{\psi}^{(p)}(\tau)|^2 |\Phi_p(\tau)|^2 d\tau < \infty ,$$

which implies $u \in H^{\mu}(R^{n+1})$. Clearly $D_t^p u(x', 0) = f_p(x')$ for $p = 0, 1, \cdots, l$. Thus the proof is complete.

Here we note that if λ_p exists, then

$$\left(\frac{\mu(\xi', 0)}{\nu_p(\xi')} \right)^{\frac{2}{2p+1}} \leqq C \lambda_p(\xi') ,$$

where C is a constant.

Example 1. Let $\mu(\xi)$ be written in the form

$$\mu(\xi', \tau) = \mu_1(\xi') + |\tau|^a \mu_2(\xi') ,$$

where $\mu_1(\xi')$, $\mu_2(\xi')$ are temperate weight function and a is a real number $> \frac{1}{2}$. Let l be the largest integer such that $l < a - \frac{1}{2}$. Then $\nu_p(\xi') \sim \mu_1^{1-\frac{1}{a}(p+\frac{1}{2})} \mu_2^{\frac{1}{a}(p+\frac{1}{2})}$, $0 \leqq p \leqq l$ and λ_p may be chosen as $\left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{a}}$. Putting

$$\hat{u}_{x'}(\xi', t) = \sum_{p=0}^{l} \hat{f}_p(\xi') \frac{(it)^p}{p!} \psi\left(\left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{a}} t \right), \quad \vec{f} \in \prod_{p=0}^{l} H^{\nu_p}(R^n) ,$$

then u belongs to $H^{\mu}(R^{n+1})$ and $D_t^p u(x', 0) = f_p(x')$ for $p = 0, 1, \cdots, l$.

In fact, we have

$$\frac{1}{\nu_p^2(\xi')} = \int_{-\infty}^{\infty} \frac{\tau^{2p}}{(\mu_1(\xi') + |\tau|^a \mu_2(\xi')^2)} d\tau$$

$$= \frac{1}{\mu_1^{2-\frac{1}{a}(2p+1)} \mu_2^{\frac{1}{a}(2p+1)}} \int_{-\infty}^{\infty} \frac{\tau^{2p}}{(1+|\tau|^a)^2} d\tau .$$

Hence $\nu_p \sim \mu_1^{1-\frac{1}{a}(p+\frac{1}{2})} \mu_2^{\frac{1}{a}(p+\frac{1}{2})}$. Putting $\lambda_p = \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{a}}$, we have

$$\mu(\xi', \lambda_p \tau) = \mu_1(\xi') + |\lambda_p \tau|^a \mu_2(\xi') = \mu_1(\xi')(1 + |\tau|^a)$$

$$\sim \lambda_p^{p+\frac{1}{2}}(\xi') \nu_p(\xi')(1 + |\tau|^a) .$$

Consequently our assertion follows from Proposition 3.

We note that the result also follows from theorems on Hilbert spaces due to Lions ([2], p. 422, p. 426).

EXAMPLE 2. Consider the temperate weight function μ given by the following formula [3]:

$$\mu(\xi, \eta)=\prod_{1}^{n}(1+|\xi|^{p_i}+|\eta|^{q_i}),$$

where $\xi=(\xi_1, \cdots, \xi_r)$, $\eta=(\eta_1, \cdots, \eta_s)$ and p_i, q_i are positive integers. We may assume that $\frac{p_1}{q_1}\leqq\frac{p_2}{q_2}\leqq\cdots\leqq\frac{p_n}{q_n}$. We use the notations: $\eta=(\eta', \tau)$, $\eta'=(\eta_1, \cdots, \eta_{s-1})$, $\tau=\eta_s$. Let $l=\sum_{1}^{n}q_i-1$, $q_0=0$. Calculation shows that for any p, $0\leqq p\leqq l$, we have

$$\nu_p(\xi, \eta')\sim(1+|\xi|^{\frac{p_m}{q_m}}+|\eta'|)^{\sum_1^m q_i-p-\frac{1}{2}}\prod_{m+1}^{n}(1+|\xi|^{p_i}+|\eta'|^{q_i}),$$

where m is chosen as

$$\sum_{0}^{m-1}q_i\leqq p\leqq\sum_{0}^{m}q_i-1 .$$

We can take $\lambda_p=(1+|\xi|^{\frac{p_m}{q_m}}+|\eta'|)$, because we have

$$\mu(\xi, \eta', \lambda_p\tau)\sim\prod_{1}^{n}(1+|\xi|^{p_i}+|\eta'|^{q_i}+|\lambda_p\tau|^{q_i})$$

$$\sim\lambda_p^{p+\frac{1}{2}}\nu_p(\xi, \eta')\frac{\prod_{1}^{n}(1+|\xi|^{p_i}+|\eta'|^{q_i}+|\lambda_p\tau|^{q_i})}{\lambda_p^{\sum_1^m q_i}\prod_{m+1}^{n}(1+|\xi|^{p_i}+|\eta'|^{q_i})}$$

$$\sim\lambda_p^{p+\frac{1}{2}}\nu_p(\xi, \eta')\prod_{1}^{m}\frac{1+|\xi|^{p_i}+|\eta'|^{q_i}+|\lambda_p\tau|^{q_i}}{\lambda_p^{q_i}}\times$$

$$\times\prod_{m+1}^{n}\frac{1+|\xi|^{p_i}+|\eta'|^{q_i}+|\lambda_p\tau|^{q_i}}{1+|\xi|^{p_i}+|\eta'|^{q_i}}$$

$$\leqq C\lambda_p^{p+\frac{1}{2}}\nu_p(\xi, \eta')(1+|\tau|)^{\sum_1^n q_i},$$

where C is a positive constant.

Suppose there exists j such that $\frac{p_j}{q_j}<\frac{p_{j+1}}{q_{j+1}}$. We can show that μ is not equivalent to a temperate weight function as considered in the preceding example. In fact, if the contrary is assumed, it will be equivalent to

$$\mu_0(\xi, \eta) = \prod_1^n (1 + |\xi|^{p_i} + |\eta'|^{q_i}) + |\tau|^{\sum_1^n q_i} .$$

In view of the inequality $\dfrac{\sum_1^n p_i}{\sum_1^n q_i} > \dfrac{\sum_1^j p_i}{\sum_1^j q_i}$, putting $\dfrac{1}{\alpha} = \dfrac{\sum_1^n p_i}{\sum_1^n q_i}$, we have $\alpha \sum_{j+1}^n p_i + \sum_1^j q_i > \alpha \sum_1^n p_i$, and

$$\frac{\mu(\xi, 0, \tau)}{\mu_0(\xi, 0, \tau)} \geq \frac{|\xi|^{\sum_{j+1}^n p_i} |\tau|^{\sum_1^j q_i}}{\prod_1^n (1 + |\xi|^{p_i}) + |\tau|^{\sum_1^n q_i}} .$$

Putting $|\xi| = |\tau|^{\alpha}$, and passing to the limit $\tau \to \infty$, we have

$$\lim_{\tau \to \infty} \frac{\mu(\xi, 0, \tau)}{\mu_0(\xi, 0, \tau)} = \infty ,$$

which is a contradiction.

Thus this example is a case to which we can not apply the results of Lions ([2], p. 422, p. 426).

4. Extension to the Case $m > 1$. Let $\mu(\xi) = \mu(\xi', \tau)$ be a temperate weight function defined in Ξ^{n+m}, where $\xi' = (\xi_1, \cdots, \xi_n)$ and $\tau = (\tau_1, \cdots, \tau_m)$. We shall assume that for a non-negative integer l

$$\int_{\Xi^m} \frac{\tau^{2l}}{\mu^2(\xi', \tau)} d\tau < \infty .$$

For any $p = (p_1, p_2, \cdots, p_m)$, p_j being a non-negative integer, such that $|p| \leq l$, we put

$$\frac{1}{\nu_p^2(\xi')} = \int_{\Xi^m} \frac{\tau^{2p}}{\mu^2(\xi', \tau)} d\tau .$$

Let us consider the trace mapping $\mathfrak{T}$:

$$u \in H^{\mu}(R^{n+m}) \to \{D_t^p u(x', 0)\}_{|p| \leq l} \in \prod_{|p| \leq l} H^{\nu_p}(R^n) .$$

The results established in Section 3 will remain valid for the mapping $\mathfrak{T}$ with necessary modifications. They can be proved along the same line as in Section 3, so we shall only enumerate them without proof.

THEOREM 1'. *A necessary and sufficient condition in order that the mapping* $\mathfrak{T}$ *may be an epimorphism is that each of the following conditions is satisfied:*

(1) *the range of the transposed mapping* ${}^t\mathfrak{T}$ *is closed in* $H^{\frac{1}{\mu}}(R^{n+m})$;

(2) *there exists a positive constant* C *such that* $\det|\kappa_{p+q}|\geqq C\prod_{|p|\leqq l}\kappa_{2p}$,

where $\kappa_p(\xi')=\int_{\Xi^m}\frac{\tau^p}{\mu^2(\xi',\tau)}d\tau$;

(3) *if* $u\in H^{\frac{1}{\mu}}(R^{n+m})$, and $\hat{u}(\xi)=\sum_{|p|\leqq l}f_p(\xi')\tau^p$, *then* $f_p\in L^2_{\frac{1}{\nu_p^2}}(\Xi^n)$ *for* $|p|\leqq l$;

(4) *if* $u\in H^{\frac{1}{\mu}}(R^{n+m})$ *and* $\hat{u}(\xi)=\sum_{|p|\leqq l}f_p(\xi')\tau^p$, *then*

$$\hat{u}\left(\xi',\tau_1,\cdots,\tau_{j-1},\frac{\tau_j}{2},\tau_{j+1},\cdots,\tau_m\right)\in L^2_{\frac{1}{\mu^2}}(\Xi^{n+m}),\ \text{for } j=1,2,\cdots,m.$$

COROLLARY. If $\mu(\xi',\tau_1,\cdots,\tau_{j-1},2\tau_j,\tau_{j+1},\cdots,\tau_m)\geqq C\mu(\xi)$, *C being a constant, for* $j=1,2,\cdots,m$, *then the mapping* $\mathfrak{T}$ *is an epimorphism.*

PROPOSITION 3'. *Let* $\vec{f}=\{f_p(x')\}_{|p|\leqq l}$ *be an arbitrary element of* $\prod_{|p|\leqq l}H^{\nu_p}(R^n)$ *and* $\psi\in\mathcal{D}(R^m)$ *be equal to* 1 *in a neighbourhood of* 0. *Suppose there exist a positive continuous* $\lambda_p(\xi')$ *in* Ξ^n *and a slowly increasing continuous function* $\Phi_p(\tau)$ *in* Ξ^m *for every* $|p|\leqq l$ *such that*

$$\mu(\xi',\lambda_p\tau)\leqq\lambda_p^{|p|+\frac{m}{2}}(\xi')\nu_p(\xi')\Phi_p(\tau)\ .$$

Then, if we put

$$\hat{u}_{x'}(\xi',t)=\sum_{|p|\leqq l}\hat{f}_p(\xi')\frac{(it)^p}{p\,!}\psi(\lambda_p t)\,,$$

then u *belongs to* $H^{\mu}(R^{n+m})$ *and* $D_t^p u(x',0)=f_p(x')$ *for* $|p|\leqq l$.

EXAMPLE 3. *Let* $\mu(\xi)$ *be written, as in Example* 1, *in the form*

$$\mu(\xi',\tau)=\mu_1(\xi')+|\tau|^a\mu_2(\xi')\,,$$

where $\mu_1(\xi')$, $\mu_2(\xi')$ are temperate weight function, and a is a real number $>\frac{m}{2}$ and $\tau=(\tau_1,\cdots,\tau_m)$. Let l be the largest integer such that $l<a-\frac{m}{2}$. Then we shall have $\nu_p(\xi')\sim\mu_1^{1-\frac{1}{a}(|p|+\frac{m}{2})}\mu_2^{\frac{1}{a}(|p|+\frac{m}{2})}$, $|p|\leqq l$, and λ_p may be chosen as $\left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{a}}$, which is independent of p. Putting $\hat{u}_{x'}(\xi',t)=\sum_{|p|\leqq l}\hat{f}_p(\xi')\frac{(it)^p}{p\,!}\psi\left(\left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{a}}t\right)$, $\vec{f}\in\prod_{|p|\leqq l}H^{\nu_p}$, we can see that u belongs to $H^{\mu}(R^{n+m})$ and $D_t^p u(x',0)=f_p(x')$ for $|p|\leqq l$. In fact, these assertions may be verified

as in Example 1.

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