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# *Chordal Limits of Holomorphic Functions at Plessner Points**)

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The purpose of this paper is to construct an example of a holomorphic function in the open unit disk $D$ of the complex plane that has a certain kind of boundary behavior at every point of the unit circle $\Gamma$. Before describing our example, we introduce some notation and terminology, and then discuss some results of Kurt Meier to which our work is closely related, in order to place it in its proper setting.

Let $f$ be a meromorphic function whose domain is $D$ and whose range is a subset of the Riemann sphere $\Omega$. We assume that the reader is familiar with some of the elementary notions of cluster set theory (see [5]). Thus, the cluster set of $f$ at a point $\zeta \in \Gamma$ is denoted by $C(f, \zeta)$. If $X$ is a chord at $\zeta$, then $C_X(f, \zeta)$ denotes the corresponding chordal cluster set of $f$ at $\zeta$. We say that $f$ has a chordal limit at $\zeta$ provided that there exists a chord $X$ at $\zeta$ and a value $\omega \in \Omega$ such that $C_X(f, \zeta)=\omega$; if, in particular, $X$ is the radius at $\zeta$, then $\omega$ is called the radial limit of $f$ at $\zeta$. We suppose that the reader knows what is meant when we say that $f$ has an angular limit at a point $\zeta \in \Gamma$.

We define the chordal principal cluster set of $f$ at a point $\zeta \in \Gamma$ as the set

$$\Pi_X(f, \zeta)\equiv\bigcap_X C_X(f, \zeta),$$

where $X$ ranges over the set of all chords at $\zeta$. The angular range of $f$ at $\zeta$ is defined to be the set $\Lambda(f, \zeta)$ of all values $\omega \in \Omega$ with the property that $f$ assumes the value $\omega$ in every Stolz angle at $\zeta$ arbitrarily close to $\zeta$.

We also take for granted that the reader knows what is meant by a Fatou point of $f$ and by a Plessner point of $f$. A Meier point of $f$ is defined to be a point $\zeta \in \Gamma$ at which

$$\Pi_X(f, \zeta)=C(f, \zeta)\subset\Omega,$$

where the symbol $\subset$ signifies proper inclusion. By an angular Picard point of $f$ we mean a point $\zeta \in \Gamma$ at which the set $\Omega-\Lambda(f, \zeta)$ contains at most two values. Finally, we define an alternative point of $f$ to be a point $\zeta \in \Gamma$ at which

$$\Pi_X(f, \zeta)\cup\Lambda(f, \zeta)=\Omega.$$

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When we say that almost every point of $\Gamma$ has a certain property, we mean that the exceptional set has Lebesgue measure zero; and when we say that nearly every point of $\Gamma$ has a certain property, we mean that the exceptional set is of first Baire category.

The classical theorem of Plessner (see [5, p. 70, Theorem 1]) is a metrical theorem and asserts the following.

THEOREM P. *If $f$ is meromorphic in $D$, then almost every point of $\Gamma$ is either a Plessner point or a Fatou point of $f$.*

Meier [4, p. 330, Theorem 5] has proved the following topological counterpart of Plessner's theorem (Meier works in a half-plane instead of a disk).

THEOREM M. *If $f$ is meromorphic in $D$, then nearly every point of $\Gamma$ is either a Plessner point or a Meier point of $f$.*

Meier [4, p. 329, Theorems 1 and 2] has also established the following two results.

THEOREM $\mathfrak{P}$. *If $f$ is meromorphic in $D$, then almost every point of $\Gamma$ is either an angular Picard point or an alternative point or a Fatou point of $f$.*

THEOREM $\mathfrak{M}$. *If $f$ is meromorphic in $D$, then nearly every point of $\Gamma$ is either an angular Picard point or an alternative point or a Meier point of $f$.*

If we observe that in general every angular Picard point is a Plessner point and every alternative point is a Plessner point, then it is apparent that Theorems $\mathfrak{P}$ and $\mathfrak{M}$ are refinements of Theorems P and M.

Meier also presents three illuminating examples which, in the light of Theorems $\mathfrak{P}$ and $\mathfrak{M}$, serve as existence theorems for the possible abundance of alternative points or of angular Picard points.

The first example is that of the elliptic modular function $M(z)$, for which the following properties hold:

(i) $M$ is holomorphic in $D$,

(ii) $M$ omits the values $0, 1, \infty$,

(iii) $C(M, \zeta)=\Omega$ for every $\zeta \in \Gamma$,

(iv) only enumerably many points of $\Gamma$ are Fatou points of $M$.

It follows from (ii) that no point of $\Gamma$ is an angular Picard point of $M$; (iii) implies that no point of $\Gamma$ is a Meier point of $M$; (iv) implies that almost every point of $\Gamma$ is not a Fatou point of $M$. Hence, Theorems $\mathfrak{P}$ and $\mathfrak{M}$ imply the following:

*Almost every and nearly every point of $\Gamma$ is an alternative point of $M(z)$.*

The second example is that of a function $\Phi(z)$ constructed by Lusin and Privalov, for which the following properties hold:

(i) $\Phi$ is holomorphic in $D$, and not identically constant,

(ii) at almost every point of $\Gamma$, $\Phi$ has the radial limit 0.

These properties imply that almost every point of $\Gamma$ is not an alternative point of $\Phi$; and they also imply, in view of Privalov's uniqueness theorem (see [5, p. 72, Theorem 2]), that almost every point of $\Gamma$ is not a Fatou point of $\Phi$. Hence, Theorem $\mathfrak{P}$ implies the following:

*Almost every point of $\Gamma$ is an angular Picard point of $\Phi(z)$.*

The third example is that of a function $\Psi(z)$ for which the following properties hold:

(i) $\Psi$ is holomorphic in $D$,

(ii) $\Psi$ is unbounded at every point of $\Gamma$,

(iii) $\Psi$ has a finite chordal limit at every point of $\Gamma$.

It follows from (i) and (iii) that no point of $\Gamma$ is an alternative point of $\Psi$, and (ii) and (iii) imply that no point of $\Gamma$ is a Meier point of $\Psi$. Hence, Theorem $\mathfrak{M}$ implies the following:

*Nearly every point of $\Gamma$ is an angular Picard point of $\Psi(z)$.*

A comparison of the italicized statements following the second and third examples with that following the first immediately suggests the problem of ascertaining whether or not there exists a holomorphic function in $D$ for which almost every and nearly every point of $\Gamma$ is an angular Picard point. Our example solves this problem.

We shall in fact establish the existence of a function $\Upsilon(z)$ for which the following properties hold:

(i) $\Upsilon$ is holomorphic in $D$,

(ii) every point of $\Gamma$ is a Plessner point of $\Upsilon$,

(iii) 0 is a chordal limit of $\Upsilon$ at every point of $\Gamma$.

(With regard to (iii), it will be shown that the limit in question is actually uniform with respect to $\zeta \in \Gamma$.) It follows from (ii) that no point of $\Gamma$ is a Fatou point of $\Upsilon$ and no point of $\Gamma$ is a Meier point of $\Upsilon$, and (i) and (iii) imply that no point of $\Gamma$ is an alternative point of $\Upsilon$. Hence, Theorems $\mathfrak{P}$ and $\mathfrak{M}$ imply the following:

*Almost every and nearly every point of $\Gamma$ is an angular Picard point of $\Upsilon(z)$.*

This example also serves to carry further a remark made in [1, p. 1072]. Suppose that $f(z)$ is a meromorphic function in $D$. Let us call a point $\zeta \in \Gamma$ a saturated Plessner point of $f$ provided that $C_X(f, \zeta)=\Omega$ for every chord $X$ at $\zeta$. The remark in question is that a Plessner point need not be a saturated Plessner point; in fact, for the function $\Phi(z)$ described in the second example above, almost every point of $\Gamma$ is a Plessner point but not a saturated Plessner point. We may now evidently assert the following:

*Every point of $\Gamma$ is a Plessner point but not a saturated Plessner point of $\Upsilon(z)$.*

Let us turn to the construction of the function $\Upsilon(z)$.

If $z_1$, $z_2$ are points of $D$, let $\rho(z_1, z_2)$ denote the non-Euclidean hyperbolic distance between them (see [3, p. 343]). Denote by $K_0$ the circle $|z|=\frac{1}{2}$. Suppose that $\frac{1}{2}<r<1$, and that $K$ denotes the circle $|z|=r$. Then it is readily seen that, given any positive number $\varepsilon$, there exists an $r'$ satisfying $r<r'<1$ such that if $Q$ is any rectilinear segment extending from a point of $K_0$ to a point of $\Gamma$, and if we denote the point of intersection of $Q$ with $K$ by $z$ and the point of intersection of $Q$ with $K'=\{z: |z|=r'\}$ by $z'$, then $\rho(z, z')<\varepsilon$. It follows that there exists a sequence $\{r_n\}$, where

$$\frac{1}{2}<r_1<r_2<\cdots<r_n<r_{n+1}<\cdots\to 1,$$

such that, given any $\varepsilon>0$, there exists a natural number $n_0=n_0(\varepsilon)$ with the property that if we draw any rectilinear segment from a point of $K_0$ to a point of $\Gamma$, and if we denote the point of intersection of this segment with the circle

$$K_n=\{z: |z|=r_n\} \qquad (n=1, 2, 3, \cdots)$$

by $z_n$, then $\rho(z_n, z_{n+1})<\frac{\varepsilon}{2}$ for every $n>n_0$.

Let $S$ be the set of all points on the circle $K_0$ of the form

$$z=\frac{1}{2}e^{\pm i(0.t_1t_2t_3\ldots)\pi},$$

where $0.t_1t_2t_3\cdots$ is a ternary fraction in which each $t_j$ is either 0 or 2. For every $z \in S$, let $X_z$ be the chord extending from the point $z$ to the point

$$\zeta=e^{\pm i(0.b_1b_2b_3\ldots)\pi} \in \Gamma,$$

where $0.b_1b_2b_3\cdots$ is the binary fraction such that, for $j=1, 2, 3, \cdot$

$$b_j=\begin{cases} 0 & \text{if } t_j=0, \\ 1 & \text{if } t_j=2 \end{cases}$$

(we consider the point $z$ as belonging to $X_z$, the point $\zeta$ as not). The set $S$ is a perfect nowhere dense subset of $K_0$. Let $A_1, A_2, A_3, \cdots$ be the enumerably many open subarcs of $K_0$ that are complementary to $S$. Let $z_{m1}$, $z_{m2}$ be the left and right end points of $A_m$ as viewed by an observer at the origin, and let $\mu_{m0}$ be the midpoint of the arc $A_m$. Then to every point $\zeta$ of $\Gamma$, with the exception of an enumerable everywhere dense subset $V$ of $\Gamma$, there corresponds exactly one $z \in S$ such that $X_z$ is a chord at $\zeta$. On the other hand, to every point $\zeta \in V$ there correspond exactly two points, $z_{m1}$, $z_{m2}$, in $S$ such that $X_{z_{m1}}$ and $X_{z_{m2}}$ are chords at $\zeta$; let $X_\zeta^*$ denote the chord at $\zeta$ extending from $\mu_{m0}$ to $\zeta$, and let $\mu_{mn}$ be the point of intersection of $X_\zeta^*$ and $K_n (n=1, 2, 3, \cdots)$. The region whose boundary is

$$A_m \cup X_{z_{m1}} \cup X_{z_{m2}} \cup \{\zeta\} \tag{1}$$

will be called $\Delta_\zeta$.

On each arc $A_m$, let $z'_{mj}$ $(j=1, 2)$ be a point between $\mu_{m0}$ and $z_{mj}$, and let $B_{mj}$ be a circular arc extending from $z'_{mj}$ to $\zeta$ that lies (except for the points $z'_{mj}$ and $\zeta$) in $\Delta_\zeta$ and is tangent to $X_{z_{mj}}$ at the point $\zeta$.

Now let $m$ be an arbitrary natural number, and let $\zeta$ be the point of $V$ associated with $A_m$. For every $n=0, 1, 2, \cdots$, we carry out the following construction. Take $n+3$ numbers $r_n^{(j)} (j=1, 2, 3, \cdots, n+3)$ such that

$$r_n < r_n^{(1)} < r_n^{(2)} < \cdots < r_n^{(n+2)} < r_n^{(n+3)} < r_{n+1}.$$

For $j=1, 2, \cdots, n+1$, let $\lambda_{mn}^{(j)}$ be the point of intersection of the chord $X_\zeta^*$ with the circle $|z|=r_n^{(j)}$. Let $\sigma_{mn}$ be the intersection of the arc $B_{m1}$ with the circle $|z|=r_n^{(n+2)}$, and let $\tau_{mn}$ be the intersection of the arc $B_{m2}$ with the circle $|z|=r_n^{(n+3)}$. Designate as $J_m$ the arc consisting of the rectilinear segments extending from $\mu_{mn}$ to $\lambda_{mn}^{(n+1)}$, from $\lambda_{mn}^{(n+1)}$ to $\sigma_{mn}$, from $\sigma_{mn}$ to $\tau_{mn}$, and from $\tau_{mn}$ to $\mu_{m,n+1}$ $(n=0, 1, 2, \cdots)$.

Denote by $c_k$ $(k=0, 1, 2, \cdots)$ the enumerable set of complex numbers whose real and imaginary parts are both rational, and agree that $c_0=0$. We define a function $g(z)$ on $J_m$ as follows. We let

$$g(\mu_{mn}) = c_0 \qquad (n=0, 1, 2, \cdots), \tag{2}$$

$$g(\lambda_{mn}^{(j)}) = c_j \qquad (n=0, 1, 2, \cdots;\ j=1, 2, \cdots, n+1), \tag{3}$$

$$g(\sigma_{mn}) = g(\tau_{mn}) = c_{n+1} \qquad (n=0, 1, 2, \cdots), \tag{4}$$

and we define $g(z)$ to be linear (possibly identically constant) on each one of the (open) segments that go to make up $J_m$. Evidently $g(z)$ as defined in this way is continuous on $J_m$.

For every $z \in S$, define $g(z)$ on $X_z$ to be identically equal to zero. Finally, join each $z \in S$ as well as each $\mu_{m0}$ to the origin by means of a rectilinear segment, and take $g(z)$ to be identically equal to zero on each such segment.

Denote by $T$ the set on which $g(z)$ has thus been defined. Then $T$ is a tress [2, p. 186, Definition 1] having the property described in [2, p. 190, Corollary 2], and $g(z)$ is continuous on $T$. It follows from the proof of [2, p. 190, Corollary 2] that there exists a function $\Upsilon(z)$, holomorphic in $D$, such that

$$\lim_{\substack{|z|\to 1\\ z\in T}} (\Upsilon(z)-g(z))=0. \tag{5}$$

Clearly then, for every $z \in S$,

$$\lim_{\substack{|z|\to 1\\ z\in X_z}} \Upsilon(z)=0,$$

so that 0 is a chordal limit of $\Upsilon$ at every point of $\Gamma$. It remains to be shown that every point of $\Gamma$ is a Plessner point of $\Upsilon$.

Suppose first that $\zeta \in \Gamma - V$. Let $\Delta$ be a Stolz angle at $\zeta$. We have to show that, in every neighborhood of $\zeta$, $\Upsilon$ comes arbitrarily close in $\Delta$ to every point of $\Omega$. To do this, it suffices to show that given any $\varepsilon>0$, any $\delta>0$, and any nonnegative integer $k$, there is a point $z \in \Delta$ with

$$|z-\zeta|<\delta \quad\text{and}\quad |f(z)-c_k|<\varepsilon. \tag{6}$$

Since $\zeta \notin V$, there exists a unique point $z \in S$ such that $X_z$ is a chord at $\zeta$. The angle $\Delta$ contains points that lie either to the left of $X_z$ or to the right of $X_z$. Assume, for the sake of definiteness, that there are points of $\Delta$ that lie to the left of $X_z$. Assume further that $\varepsilon'$, with $0<\varepsilon'<\varepsilon$, is less than the smaller of the following two positive numbers: the non-Euclidean hyperbolic distance between the sides of $\Delta$; the non-Euclidean hyperbolic distance between the left side of $\Delta$ and the chord $X_z$. Since the set $V$ is everywhere dense on $\Gamma$, there exists a sequence of points $\zeta_p \in V$ tending to $\zeta$ from the left as $p\to\infty$. Let $J_{m_p}$ denote the arc at $\zeta_p$ that was constructed above, and observe that $\mu_{m_p 0}$ tends to $z$ from the left as $p\to\infty$. Consequently, for all sufficiently large $p$, the arc $J_{m_p}$ intersects $\Delta$ in a subarc of $J_{m_p}$. Let $n_0(\varepsilon')$ be the natural number associated with the sequence $\{r_n\}$ at the beginning of the construction of $\Upsilon$. Determine a number $\delta'$, with $0<\delta'<\delta$, so small that the following conditions are satisfied:

(a) If $K_n$ intersects the disk $U_{\delta'}=\{z: |z-\zeta|<\delta'\}$, then $n>n_0(\varepsilon')$.

(b) If $K_n$ intersects the disk $U_{\delta'}$, then $n\geqq k$.

(c) If $1-\delta'<|z|<1$ and $z \in T$, then $|\Upsilon(z)-g(z)|<\varepsilon$.

Now let $\Delta(\delta')=\Delta\cap U_{\delta'}$. Because of the assumption made about $\varepsilon'$ relative to $\Delta$, and as a consequence of (a), it is possible to find a $p$ so large that the intersection of $J_{m_p}$ with $\Delta(\delta')$ contains a subarc of $J_{m_p}$ extending from a point $\mu_{m_p q}$ to $\lambda^{(n+1)}_{m_p q}$. It then follows from (b) that $\Delta(\delta')$ contains the point $\lambda^{(k)}_{m_p q}$ if $k>0$. In any case, in view of (2) and (3), $\Delta(\delta')$ contains a point $z$ at which $g(z)=c_k$, and due to (c) this implies that (6) is satisfied.

Suppose finally that $\zeta \in V$. Let $\Delta$ again be a Stolz angle at $\zeta$. Consider the region $\Delta_\zeta$ that was associated above (see (1)) with the point $\zeta$. If $\Delta$ intersects $\Delta_\zeta$, then it is clear from the definition of the arcs $B_{mj}$ ($j=1, 2$) that, for all sufficiently large values of $n$, $\Delta$ intersects the rectilinear segment extending from $\sigma_{mn}$ to $\tau_{mn}$. In view of (4), the definition of $g(z)$ on the aforementioned rectilinear segment, and (5), we see directly that

$$C_\Delta(\Upsilon, \zeta) = \Omega.$$

If, however, $\Delta$ does not intersect $\Delta_\zeta$, then $\Delta$ contains points that lie either to the left of $X_{z_{m1}}$ or to the right of $X_{z_{m2}}$, say the former. Then as before there exists a sequence of points $\zeta_p \in V$ tending to $\zeta$ from the left; and an argument analogous to the one given earlier leads once more to the conclusion that $\zeta$ is a Plessner point of $\Upsilon$.

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