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On Some Properties of $\mathfrak{t}(n, \Phi)$ *and* $\mathfrak{st}(n, \Phi)$

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1. Introduction

It is known that every finite-dimensional Lie algebra L over a field Φ of arbitrary characteristic has a faithful finite-dimensional representation. If Φ is an algebraically closed field of characteristic 0, then every solvable subalgebra of $\mathfrak{gl}(n, \Phi)$ is isomorphic to a subalgebra of the Lie algebra $\mathfrak{t}(n, \Phi)$ of all the triangular matrices. Among solvable linear Lie algebras the following three Lie algebras are most familiar to us: $\mathfrak{t}(n, \Phi)$, the Lie algebra $\mathfrak{st}(n, \Phi)$ of all the triangular matrices of trace 0, and the Lie algebra $\mathfrak{n}(n, \Phi)$ of all the triangular matrices with 0's on the diagonal. B. Kostant orally informed the author that he had determined the structure of the first cohomology group $H^1(L, L)$ of $\mathfrak{n}(n, \Phi)$ and that the method of constructing an outer derivation which has been employed in the proof of Theorem 1 in [1] gives another way of finding all the nilpotent outer derivations of $\mathfrak{n}(n, \Phi)$.

It therefore seems to be an interesting problem to ask the structure of the first cohomology groups $H^1(L, L)$ of $\mathfrak{t}(n, \Phi)$ and $\mathfrak{st}(n, \Phi)$. In this paper we are concerned with this problem and show the following two theorems.

THEOREM 1. *Let L be $\mathfrak{st}(n, \Phi)$ with $n \geq 2$.*

(i) *If the characteristic of Φ is 0, or if the characteristic of Φ is $p \neq 0$ and $n \not\equiv 0$ (mod p), then $H^1(L, L) = (0)$.*

(ii) *If the characteristic of Φ is $p \neq 0$ and $n \equiv 0$ (mod p) and if $n \geq 5$, then* $\dim H^1(L, L) = n$.

THEOREM 2. *Let Φ be a field of arbitrary characteristic and let L be $\mathfrak{t}(n, \Phi)$ with $n \geq 2$. Then* $\dim H^1(L, L) = n$.

In Theorem 1 we exclude the case where the characteristic of Φ is $p \neq 0$, $n \equiv 0$ (mod p) and $n \leq 4$. The structure of the first cohomology group $H^1(L, L)$ of $\mathfrak{st}(n, \Phi)$ in this case will be determined in Section 5.

Throughout this paper, we shall denote by Φ a field of arbitrary characteristic unless otherwise stated, and by e_0 the identity matrix in $\mathfrak{gl}(n, \Phi)$.

2. Lemmas

Throughout Sections 2, 3, 4 and 5, we denote $\mathfrak{st}(n, \Phi)$ by L for the sake of simplicity and assume that $n \geq 3$ unless otherwise stated.

We choose a basis of L as follows.

e_k: the $(a_{ij}) \in L$ such that $a_{kk}=1$, $a_{k+1,k+1}=-1$ and all other $a_{ij}=0$.

$e_{k,k+l}$: the $(a_{ij}) \in L$ such that $a_{k,k+l}=1$ and all other $a_{ij}=0$.

$$(k=1, 2, \cdots, n-1, \quad l=1, 2, \cdots, n-k).$$

We put these elements of L in the following order:

$$(1) \qquad e_1, \cdots, e_{n-1};\ e_{12}, \cdots, e_{n-1,n};\ \cdots;\ e_{1,l+1}, \cdots, e_{n-l,n};\ \cdots;\ e_{1,n-1}, e_{2,n};\ e_{1n}.$$

Then we have

Lemma 1. *Let D be any derivation of L. Then*

$$De_k = \sum_{i=1}^{n-1} \lambda_k^i e_i + \sum_{i=1}^{n-1} \lambda_k^{i,i+1} e_{i,i+1} + \cdots + \sum_{i=1}^{n-l} \lambda_k^{i,i+l} e_{i,i+l} + \cdots + \lambda_k^{1n} e_{1n},$$

$$De_{k,k+l} = \sum_{i=1}^{n-l} \lambda_{k,k+l}^{i,i+l} e_{i,i+l} + \cdots + \lambda_{k,k+1}^{1n} e_{1n}$$

for $k=1, 2, \cdots, n-1$ *and* $l=1, 2, \cdots, n-k$.

Proof. This is immediate from the facts that

$$L^2 = (e_{12}, \cdots, e_{n-1,n};\ \cdots;\ e_{1n}),$$

$$(L^2)^l = (e_{1,l+1}, \cdots, e_{n-l,n};\ \cdots;\ e_{1n}) \qquad \text{for} \quad l=2, 3, \cdots, n-1$$

and that these are characteristic ideals of L.

We consider the following system of $n-1$ equations:

$$(2) \qquad \begin{cases} 2x_1 - x_2 = 0 \\ -x_1 + 2x_2 - x_3 = 0 \\ -x^2 + 2x_3 - x_4 = 0 \\ \quad \cdot \\ \quad \cdot \\ \quad \cdot \\ -x_{n-3} + 2x_{n-2} - x_{n-1} = 0 \\ -x_{n-2} + 2x_{n-1} = 0. \end{cases}$$

Then the determinant of the matrix of coefficients of (2) is n.

We need the following multiplication table:

$$
(3) \quad \begin{cases}
[e_1, e_{12}]=2e_{12}, \ [e_1, e_{23}]=-e_{23}; \\
[e_j, e_{j-1,j}]=-e_{j-1,j}, \ [e_j, e_{j,j+1}]=2e_{j,j+1}, \\
\qquad [e_j, e_{j+1,j+2}]=-e_{j+1,j+2}, \quad \text{for} \quad j=2, 3, \dots, n-2; \\
[e_{n-1}, e_{n-2,n-1}]=-e_{n-2,n-1}, [e_{n-1}, e_{n-1,n}]=2e_{n-1,n} \\
\text{and all other} \quad [e_k, e_{i,i+1}]=0.
\end{cases}
$$

LEMMA 2. *Let D be any derivation of L. Let De_k and $De_{k,k+l}$ be expressed as in Lemma 1. Then $\{\lambda_k^1, \lambda_k^2, \dots, \lambda_k^{n-1}\}$ for any $k=1, 2, \dots, n-1$ is a solution of the system (2). Except the case where $n=3$ and the characteristic of Φ is 3 and the case where $n=4$ and the characteristic of Φ is 2, we have*

$$\lambda_{k,k+1}^{i,i+1}=0 \qquad \textit{for} \quad i, k=1, 2, \dots, n-1 \quad \textit{and} \quad i \neq k.$$

PROOF. (i) The case where the characteristic of Φ is $\neq 2, 3$:
Apply D to $[e_1, e_{12}]=2e_{12}$. Then

$$2\lambda_1^1-\lambda_1^2=0 \quad \text{and} \quad \lambda_{12}^{i,i+1}=0 \qquad \text{for} \quad i \neq 1.$$

For $j \neq 1$, applying D to the products $[e_j, e_{12}]$, we obtain

$$2\lambda_j^1-\lambda_j^2=0.$$

For $k=2, 3, \dots, n-2$, apply D to $[e_k, e_{k,k+1}]=2e_{k,k+1}$. Then we have

$$-\lambda_k^{k-1}+2\lambda_k^k-\lambda_k^{k+1}=0 \quad \text{and} \quad \lambda_{k,k+1}^{i,i+1}=0 \qquad \text{for} \quad i \neq k.$$

For $j \neq k$, applying D to the products $[e_j, e_{k,k+1}]$, we obtain

$$-\lambda_j^{k-1}+2\lambda_j^k-\lambda_j^{k+1}=0.$$

Apply D to $[e_{n-1}, e_{n-1,n}]=2e_{n-1,n}$. Then

$$-\lambda_{n-1}^{n-2}+2\lambda_{n-1}^{n-1}=0 \quad \text{and} \quad \lambda_{n-1,n}^{i,i+1}=0 \qquad \text{for} \quad i \neq n-1.$$

For $j \neq n-1$, by applying D to the products $[e_j, e_{n-1,n}]$, we obtain

$$-\lambda_j^{n-2}+2\lambda_j^{n-1}=0.$$

(ii) The case where the characteristic of Φ is 2:
We first assume that $n \geq 5$. Apply D to $[e_2, e_{12}]=e_{12}$. Then

$$\lambda_2^2=0 \quad \text{and} \quad \lambda_{12}^{i,i+1}=0 \qquad \text{for} \quad i \neq 1, 3.$$

By applying D to $[e_4, e_{12}]=0$, we obtain

$$\lambda_4^2=0 \quad \text{and} \quad \lambda_{12}^{34}=0.$$

From the other products $[e_j, e_{12}]$, it follows that

$$\lambda_j^2=0 \qquad \text{for} \quad j\neq 2, 4.$$

Next apply D to $[e_1, e_{23}]=e_{23}$. Then

$$\lambda_1^1+\lambda_1^3=0 \quad \text{and} \quad \lambda_{23}^{i,i+1}=0 \qquad \text{for} \quad i\neq 2.$$

From the other products $[e_j, e_{23}]$, it follows that

$$\lambda_j^1+\lambda_j^3=0 \qquad \text{for} \quad j\neq 1.$$

Apply D to $[e_2, e_{34}]=e_{34}$. Then

$$\lambda_2^2+\lambda_2^4=0 \quad \text{and} \quad \lambda_{34}^{i,i+1}=0 \qquad \text{for} \quad i\neq 1, 3.$$

By applying D to $[e_4, e_{34}]=e_{34}$, we obtain

$$\lambda_4^2+\lambda_4^4=0 \quad \text{and} \quad \lambda_{34}^{12}=0.$$

From the other products $[e_j, e_{34}]$, it follows that

$$\lambda_j^2+\lambda_j^4=0 \qquad \text{for} \quad j\neq 2, 4.$$

For $k=4, 5, \cdots, n-2$, apply D to $[e_{k-1}, e_{k,k+1}]=e_{k,k+1}$. Then

$$\lambda_{k-1}^{k-1}+\lambda_{k-1}^{k+1}=0 \quad \text{and} \quad \lambda_{k,k+1}^{i,i+1}=0 \qquad \text{for} \quad i\neq k-2, k.$$

By applying D to $[e_{k-3}, e_{k,k+1}]=0$, we obtain

$$\lambda_{k-3}^{k-1}+\lambda_{k-3}^{k+1}=0 \quad \text{and} \quad \lambda_{k,k+1}^{k-2,k-1}=0.$$

From the other products $[e_j, e_{k,k+1}]$, it follows that

$$\lambda_j^{k-1}+\lambda_j^{k+1}=0 \qquad \text{for} \quad j\neq k-3, k.$$

For $k=n-1$, apply D to $[e_{n-2}, e_{n-1,n}]=e_{n-1,n}$. Then

$$\lambda_{n-2}^{n-2}=0 \quad \text{and} \quad \lambda_{n-1,n}^{i,i+1}=0 \qquad \text{for} \quad i\neq n-3, n-1.$$

By applying D to $[e_{n-4}, e_{n-1,n}]=0$, we obtain

$$\lambda_{n-4}^{n-2}=0 \quad \text{and} \quad \lambda_{n-1,n}^{n-3,n-2}=0.$$

From the other products $[e_j, e_{n-1,n}]$, it follows that

$$\lambda_j^{n-2}=0 \qquad \text{for} \quad j\neq n-4, n-2.$$

Thus we see that the statement is proved for $n\geq 5$.

By employing a similar method, in the case where $n=3$ the statement is immediately proved and in the case where $n=4$ it is proved that $\{\lambda_k^1, \lambda_k^2, \lambda_k^3\}$, $k=1, 2, 3$, is a solution of the system (2) of equations.

(iii) The case where the characteristic of Φ is 3:
We first assume that $n \geq 4$. Apply D to $[e_1, e_{12}]=2e_{12}$ and we obtain

$$2\lambda_1^1-\lambda_1^2=0 \quad \text{and} \quad \lambda_{12}^{i,i+1}=0 \qquad \text{for} \quad i \neq 1, 2.$$

By applying D to $[e_3, e_{12}]=0$, we have

$$2\lambda_3^1-\lambda_3^2=0 \quad \text{and} \quad \lambda_{12}^{23}=0.$$

From the other products $[e_j, e_{12}]$, it follows that

$$2\lambda_j^1-\lambda_j^2=0 \qquad \text{for} \quad j \neq 1, 3.$$

Now let $k=2, 3, \cdots, n-2$. Apply D to $[e_k, e_{k,k+1}]=2e_{k,k+1}$. Then

$$-\lambda_k^{k-1}+2\lambda_k^k-\lambda_k^{k+1}=0 \quad \text{and} \quad \lambda_{k,k+1}^{i,i+1}=0 \qquad \text{for} \quad i \neq k-1, k, k+1.$$

By applying D to $[e_{k-1}, e_{k,k+1}]=[e_{k+1}, e_{k,k+1}]=-e_{k,k+1}$, we obtain

$$-\lambda_i^{k-1}+2\lambda_i^k-\lambda_i^{k+1}=0 \quad \text{and} \quad \lambda_{k,k+1}^{i,i+1}=0 \qquad \text{for} \quad i=k-1, k+1.$$

From the other products $[e_j, e_{k,k+1}]$, it follows that

$$-\lambda_j^{k-1}+2\lambda_j^k-\lambda_j^{k+1}=0 \qquad \text{for} \;\; j \neq k-1, k, k+1.$$

Finally, apply D to $[e_{n-1}, e_{n-1,n}]=2e_{n-1,n}$. Then

$$-\lambda_{n-1}^{n-2}+2\lambda_{n-1}^{n-1}=0 \quad \text{and} \quad \lambda_{n-1,n}^{i,i+1}=0 \qquad \text{for} \quad i \neq n-2, n-1.$$

By applying D to $[e_{n-3}, e_{n-1,n}]=0$, we obtain

$$-\lambda_{n-3}^{n-2}+2\lambda_{n-3}^{n-1}=0 \quad \text{and} \quad \lambda_{n-1,n}^{n-2,n-1}=0.$$

From the other products $[e_j, e_{n-1,n}]$, it follows that

$$-\lambda_j^{n-2}+2\lambda_j^{n-1}=0 \qquad \text{for} \quad j \neq n-1, n-3.$$

Hence the statement is proved for $n \geq 4$. In the case where $n=3$, it is immediate that $\{\lambda_k^1, \lambda_k^2\}$ for any $k=1, 2$ is a solution of the system (2) of equations.

Thus we see that in any case $\{\lambda_k^1, \lambda_k^2, \cdots, \lambda_k^{n-1}\}$ for $k=1, 2, \cdots, n-1$ satisfies the system (2) of equations, and that except the two cases indicated in the statement of the lemma

$$\lambda_{k,k+1}^{i,i+1}=0 \qquad \text{for} \quad i, k=1, 2, \cdots, n-1 \;\; \text{and} \;\; i \neq k.$$

LEMMA 3. *Let D be a derivation of L and let j be one of the integers* 2, 3, $\cdots, n-1$. *Assume that*

$$De_{k,k+1}=\sum_{i=1}^{n-j}\lambda_{k,k+1}^{i,i+j}e_{i,i+j}+\cdots+\lambda_{k,k+1}^{1n}e_{1n} \qquad \textit{for} \quad k=1, 2, \cdots, n-1.$$

Then for $l=1, 2, \cdots, n-j$

$$De_{k,k+l}=\sum_{i=1}^{n-j-l+1}\lambda_{k,k+l}^{i,i+j+l-1}e_{i,i+j+l-1}+\cdots+\lambda_{k,k+l}^{1n}e_{1n}$$

and for $l=n-j+1, \cdots, n-1$

$$De_{k,k+l}=0, \qquad k=1, 2, \cdots, n-l.$$

PROOF. We prove the lemma by induction on l. The case where $l=1$ is trivial. Assume that $l\geq 2$ and that the formula holds for $De_{k,k+l-1}$. For any $k=1, 2, \cdots, n-l$,

$$e_{k,k+l}=[e_{k,k+l-1}, e_{k+l-1,k+l}].$$

Hence if $l=2, 3, \cdots, n-j$,

$$\begin{aligned} De_{k,k+l}=&[\sum_{i=1}^{n-j-l+2}\lambda_{k,k+l-1}^{i,i+j+l-2}e_{i,i+j+l-2}+\cdots+\lambda_{k,k+l-1}^{1n}e_{1n}, e_{k+l-1,k+l}] \\ &+[e_{k,k+l-1}, \sum_{i=1}^{n-j}\lambda_{k+l-1,k+l}^{i,i+j}e_{i,i+j}+\cdots+\lambda_{k+l-1,k+l}^{1n}e_{1n}] \\ &\in (e_{1,j+l}, \cdots, e_{n-j-l+1,n}; \cdots; e_{1n}). \end{aligned}$$

If $l=n-j+1$,

$$\begin{aligned} De_{k,k+l}=&[\lambda_{k,k+l-1}^{1n}e_{1n}, e_{k+l-1,k+l}] \\ &+[e_{k,k+l-1}, \sum_{i=1}^{n-j}\lambda_{k+l-1,k+l}^{i,i+j}e_{i,i+j}+\cdots+\lambda_{k+l-1,k+l}^{1n}e_{1n}] \\ =&0. \end{aligned}$$

If $l=n-j+2, \cdots, n-1$,

$$\begin{aligned} De_{k,k+l}=&[e_{k,k+l-1}, \sum_{i=1}^{n-j}\lambda_{k+l-1,k+l}^{i,i+j}e_{i,i+j}+\cdots+\lambda_{k+l-1,k+l}^{1n}e_{1n}] \\ =&0. \end{aligned}$$

Thus the formula holds for $De_{k,k+l}$. This completes the proof.

LEMMA 4. *Let D be a derivation of L. Assume that*

$$De_{k}=\sum_{i=1}^{n-1}\lambda_{k}^{i,i+1}e_{i,i+1}+\cdots+\lambda_{k}^{1n}e_{1n},$$

$$De_{k,k+1}=\sum_{i=1}^{n-2}\lambda_{k,k+1}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{k,k+1}^{1n}e_{1n}$$

$$\textit{for} \quad k=1, 2, \cdots, n-1.$$

Then there exists an inner derivation $\operatorname{ad} x$ *such that* $D'=D+\operatorname{ad} x$ *has the fol-*

lowing form for e_k *and* $e_{k,k+1}$:

$$D'e_k=\sum_{i=1}^{n-2}\mu_k^{i,i+2}e_{i,i+2}+\cdots+\mu_k^{1n}e_{1n},$$

$$D'e_{k,k+1}=\sum_{i=1}^{n-2}\mu_{k,k+1}^{i,i+2}e_{i,i+2}+\cdots+\mu_{k,k+1}^{1n}e_{1n}$$

$$\textit{for}\quad k=1, 2, \cdots, n-1.$$

PROOF. (i) The case where the characteristic of Φ is $\neq 2$: We put

$$D'=D+\tfrac{1}{2}\sum_{i=1}^{n-1}\lambda_i^{i,i+1}\operatorname{ad}e_{i,i+1}.$$

Then we can write

$$(4)\qquad \begin{cases} D'e_k=\sum_{i=1}^{n-1}\mu_k^{i,i+1}e_{i,i+1}+\cdots+\mu_k^{1n}e_{1n}, \\ D'e_{k,k+1}=\sum_{i=1}^{n-2}\mu_{k,k+1}^{i,i+2}e_{i,i+2}+\cdots+\mu_k^{1n}e_{1n} \\ \qquad\qquad \text{for}\quad k=1, 2, \cdots, n-1. \end{cases}$$

We assert that $\mu_k^{k,k+1}=0$ for $k=1, 2, \cdots, n-1$. In fact,

$$D'e_k=\sum_{i=1}^{n-1}\lambda_k^{i,i+1}e_{i,i+1}+\cdots+\lambda_k^{1n}e_{1n}+\tfrac{1}{2}\sum_{i=1}^{n-1}\lambda_i^{i,i+1}[e_{i,i+1}, e_k]$$

and therefore by (3)

$$\mu_k^{k,k+1}=\lambda_k^{k,k+1}+\tfrac{1}{2}(-2\lambda_k^{k,k+1})=0,$$

as was asserted. Applying D' to $[e_1, e_2]=\cdots=[e_1, e_{n-1}]=0$, we have

$$\mu_1^{i,i+1}=0 \qquad \text{for}\quad i=1, 2, \cdots, n-1$$

and

$$\mu_m^{12}=\mu_m^{23}=0 \qquad \text{for}\quad m=2, 3, \cdots, n-1.$$

Assume that $k\geq 2$ and that we have

$$\mu_l^{i,i+1}=0 \qquad \text{for}\quad l=1, 2, \cdots, k-1 \quad \text{and}\quad i=1, 2, \cdots, n-1$$

and

$$\mu_m^{12}=\mu_m^{23}=\cdots=\mu_m^{k,k+1}=0 \qquad \text{for } m=k, k+1, \cdots, n-1.$$

Then by applying D' to $[e_k, e_{k+1}]=\cdots=[e_k, e_{n-1}]=0$, we obtain

$$\mu_k^{i,i+1}=0 \qquad \text{for}\quad i=k+1, k+2, \cdots, n-1$$

and

$$\mu_m^{k+1,k+2}=0 \qquad \text{for}\quad m=k+2, k+3, \cdots, n-1.$$

Hence by induction we see that

$$\mu_k^{i,i+1}=0 \qquad \text{for} \quad i,\ k=1,\ 2,\ \cdots,\ n-1.$$

(ii) The case where the characteristic of Φ is 2 and n is odd: We put

$$D'=D+(\lambda_2^{12}\operatorname{ad}e_{12}+\lambda_1^{23}\operatorname{ad}e_{23})+\cdots+(\lambda_{n-1}^{n-2,n-1}\operatorname{ad}e_{n-2,n-1}+\lambda_{n-2}^{n-1,n}\operatorname{ad}e_{n-1,n}).$$

Then we can express $D'e_k$ and $D'e_{k,k+1}$ in the form (4). Since

$$\begin{aligned}D'e_k=&\sum_{i=1}^{n-1}\lambda_k^{i,i+1}e_{i,i+1}+\cdots+\lambda_k^{1n}e_{1n}\\&+(\lambda_2^{12}[e_{12},\ e_k]+\lambda_1^{23}[e_{23},\ e_k])+\cdots+(\lambda_{n-1}^{n-2,n-1}[e_{n-2,n-1},\ e_k]\\&+\lambda_{n+2}^{n-1,n}[e_{n-1,n},\ e_k]),\end{aligned}$$

by making use of (3) it is immediate that

$$\mu_2^{12}=\mu_1^{23}=\cdots=\mu_{n-1}^{n-2,n-1}=\mu_{n-2}^{n-1,n}=0.$$

Applying D' to $[e_1,\ e_2]=\cdots=[e_1,\ e_{n-1}]=0$, we obtain

$$\mu_1^{i,i+1}=0 \qquad \text{for} \quad i=1,\ 2,\ \cdots,\ n-1$$

and

$$\mu_m^{23}=0 \qquad \text{for} \quad m=2,\ 3,\ \cdots,\ n-1.$$

Next apply D' to $[e_2,\ e_3]=\cdots=[e_2,\ e_{n-1}]=0$. Then

$$\mu_2^{i,i+1}=0 \qquad \text{for} \quad i=1,\ 2,\ \cdots,\ n-1$$

and

$$\mu_m^{12}=\mu_m^{34}=0 \qquad \text{for} \quad m=3,\ 4,\ \cdots,\ n-1.$$

Now, as in the proof of the first case, by induction we have

$$\mu_k^{i,i+1}=0 \qquad \text{for} \quad i,\ k=1,\ 2,\ \cdots,\ n-1.$$

(iii) The case where the characteristic of Φ is 2 and n is even: Put

$$\begin{aligned}D'=D+(\lambda_2^{12}\operatorname{ad}e_{12}+\lambda_1^{23}\operatorname{ad}e_{23})+\cdots+(\lambda_{n-2}^{n-3,n-2}\operatorname{ad}e_{n-3,n-2}\\+\lambda_{n-3}^{n-2,n-1}\operatorname{ad}e_{n-2,n-1})+\lambda_{n-2}^{n-1,n}\operatorname{ad}e_{n-1,n},\end{aligned}$$

and write $D'e_k$ and $D'e_{k,k+1}$ in the form (4). Then it is immediate by (3) that

$$\mu_2^{12}=\mu_1^{23}=\cdots=\mu_{n-2}^{n-3,n-2}=\mu_{n-3}^{n-2,n-1}=\mu_{n-2}^{n-1,n}=0.$$

If $n=4$, apply D' to $[e_1,\ e_2]=[e_1,\ e_3]=0$. Then

$$\mu_1^{12}=\mu_1^{34}=0,\quad \mu_2^{23}=\mu_3^{23}=0.$$

By applying D' to $[e_2, e_3]=0$, we obtain

$$\mu_3^{12}=\mu_3^{34}=0.$$

Since $\mu_1^{23}=\mu_2^{12}=\mu_2^{34}=0$ as seen above,

$$\mu_k^{i,i+1}=0 \qquad \text{for} \quad i, k=1, 2, 3.$$

Now we consider the case where $n\geq 6$. As in the previous case, we obtain

$$\mu_k^{i,i+1}=0 \qquad \text{for} \quad k=1, 2, \cdots, n-3 \quad \text{and} \quad i=1, 2, \cdots, n-1$$

and
$$\mu_m^{12}=\mu_m^{23}=\cdots=\mu_m^{n-2,n-1}=0 \qquad \text{for } m=n-2,\ n-1.$$

As seen above, $\mu_{n-2}^{n-1,n}=0$ and, by applying D' to $[e_{n-2}, e_{n-1}]=0$, we obtain $\mu_{n-1}^{n-1,n}=0$. Therefore

$$\mu_k^{i,i+1}=0 \qquad \text{for} \quad i, k=1, 2, \cdots, n-1.$$

Thus the proof is complete.

To prove the next Iemma, we need the following multiplication tables where $j=2, 3, \cdots, n-1$. For $n>2j+1$,

$$(5)_{\mathrm{I}} \quad \left\{\begin{array}{l}
[e_1, e_{1,j+1}]=e_{1,j+1}, \quad [e_1, e_{2,j+2}]=-e_{2,j+2}, \\
\qquad\qquad\vdots \\
[e_{j-1}, e_{j-1,2j-1}]=e_{j-1,2j-1}, \quad [e_{j-1}, e_{j,2j}]=-e_{j,2j}, \\
[e_j, e_{1,j+1}]=e_{1,j+1}, \quad [e_j, e_{j,2j}]=e_{j,2j}, \quad [e_j, e_{j+1,2j+1}]=-e_{j+1,2j+1}, \\
[e_{j+1}, e_{1,j+1}]=-e_{1,j+1}, \quad [e_{j+1}, e_{2,j+2}]=e_{2,j+2}, \quad [e_{j+1}, e_{j+1,2j+1}]=e_{j+1,2j+1}, \\
\qquad [e_{j+1}, e_{j+2,2j+2}]=-e_{j+2,2j+2}, \\
\qquad\qquad\vdots \\
[e_{n-j-1}, e_{n-2j-1,n-j-1}]=-e_{n-2j-1,n-j-1}, \quad [e_{n-j-1}, e_{n-2j,n-j}]=e_{n-2j,n-j}, \\
\qquad [e_{n-j-1}, e_{n-j-1,n-1}]=e_{n-j-1,n-1}, \quad [e_{n-j-1}, e_{n-j,n}]=-e_{n-j,n}, \\
[e_{n-j}, e_{n-2j,n-j}]=-e_{n-2j,n-j}, \quad [e_{n-j}, e_{n-2j+1,n-j+1}]=e_{n-2j+1,n-j+1}, \\
\qquad [e_{n-j}, e_{n-j,n}]=e_{n-j,n}, \\
[e_{n-j+1}, e_{n-2j+1,n-j+1}]=-e_{n-2j+1,n-j+1}, \quad [e_{n-j+1}, e_{n-2j+2,n-j+2}]=e_{n-2j+2,n-j+2}, \\
\\
[e_{n-1}, e_{n-j-1,n-1}]=-e_{n-j-1,n-1}, \quad [e_{n-1}, e_{n-j,n}]=e_{n-j,n}, \\
\text{and all other products} \quad [e_k, e_{i,i+j}]=0.
\end{array}\right.$$

For $n=2j+1$,

$$(5)_{\mathrm{II}}\begin{cases}[e_1,\ e_{1,j+1}]=e_{1,j+1},\quad [e_1,\ e_{2,j+2}]=-e_{2,j+2},\\ \qquad\vdots\\ [e_{j-1},\ e_{j-1,2j-1}]=e_{j-1,2j-1},\quad [e_{j-1},\ e_{j,2j}]=-e_{j,2j},\\ [e_j,\ e_{1,j+1}]=e_{1,j+1},\quad [e_j,\ e_{j,2j}]=e_{j,2j},\quad [e_j,\ e_{j+1,2j+1}]=-e_{j+1,2j+1},\\ [e_{n-j},\ e_{1,j+1}]=-e_{1,j+1},\quad [e_{n-j},\ e_{2,j+2}]=e_{2,j+2},\quad [e_{n-j},\ e_{n-j,n}]=e_{n-j,n},\\ [e_{n-j+1},\ e_{2,j+2}]=-e_{2,j+2},\quad [e_{n-j+1},\ e_{3,j+3}]=e_{3,j+3},\\ \qquad\vdots\\ [e_{n-1},\ e_{n-j-1,n-1}]=-e_{n-j-1,n-1},\quad [e_{n-1},\ e_{n-j,n}]=e_{n-j,n}\\ \text{and all other products}\quad [e_k,\ e_{i,i+j}]=0.\end{cases}$$

For $n=2j$,

$$(5)_{\mathrm{III}}\begin{cases}[e_1,\ e_{1,j+1}]=e_{1,j+1},\quad [e_1,\ e_{2,j+2}]=-e_{2,j+2},\\ \qquad\vdots\\ [e_{j-1},\ e_{j-1,2j-1}]=e_{j-1,2j-1},\quad [e_{j-1},\ e_{j,2j}]=-e_{j,2j},\\ [e_{n-j},\ e_{1,j+1}]=e_{1,j+1},\quad [e_{n-j},\ e_{n-j,n}]=e_{n-j,n},\\ [e_{n-j+1},\ e_{1,j+1}]=-e_{1,j+1},\quad [e_{n-j+1},\ e_{2,j+2}]=e_{2,j+2},\\ \qquad\vdots\\ [e_{n-1},\ e_{n-j-1,n-1}]=-e_{n-j+1,n-1},\quad [e_{n-1},\ e_{n-j,n}]=e_{n-j,n}\\ \text{and all other products}\quad [e_k,\ e_{i,i+j}]=0.\end{cases}$$

For $j+1<n<2j$,

$$(5)_{\mathrm{IV}}\begin{cases}[e_1,\ e_{1,j+1}]=e_{1,j+1},\quad [e_1,\ e_{2,j+2}]=-e_{2,j+2},\\ \qquad\vdots\\ [e_{n-j-1},\ e_{n-j-1,n-1}]=e_{n-j-1,n-1},\quad [e_{n-j-1},\ e_{n-j,n}]=-e_{n-j,n},\\ [e_{n-j},e_{n-j,n}]=e_{n-j,n},\\ [e_j,\ e_{1,j+1}]=e_{1,j+1},\\ [e_{j+1},\ e_{1,j+1}]=-e_{1,j+1},\quad [e_{j+1},\ e_{2,j+2}]=e_{2,j+2},\\ \qquad\vdots\\ [e_{n-1},\ e_{n-j-1,n-1}]=-e_{n-j-1,n-1},\quad [e_{n-1},\ e_{n-j,n}]=e_{n-j,n}\\ \text{and all other products}\quad [e_k,\ e_{i,i+j}]=0.\end{cases}$$

For $n=j+1$,

$$(5)_{\mathrm{V}} \begin{cases} [e_1, e_{1n}]=e_{1n}, \\ [e_{n-1}, e_{1n}]=e_{1n}, \\ \text{and all other products} \quad [e_k, e_{i,i+j}]=0. \end{cases}$$

LEMMA 5. *Let D be a derivation of L and let j be one of the integers $2, 3, \cdots, n-1$. Assume that*

$$De_k=\sum_{i=1}^{n-j}\lambda_k^{i,i+j}e_{i,i+j}+\cdots+\lambda_k^{1n}e_{1n},$$

$$De_{k,k+1}=\sum_{i=1}^{n-j}\lambda_{k,k+1}^{i,i+j}e_{i,i+j}+\cdots+\lambda_{k,k+1}^{1n}e_{1n}$$

$$\textit{for} \quad k=1, 2, \cdots, n-1.$$

Then except the case where $n=4$, the characteristic of Φ is 2 and $j=3$, we have

$$\lambda_{k,k+1}^{i,i+j}=0 \qquad \textit{for} \quad k=1, 2, \cdots, n-1 \quad \textit{and} \quad i=1, 2, \cdots, n-j.$$

PROOF. (i) The case where the characteristic of Φ is $\neq 2, 3$;

As shown in the table (5),

$$[e_k, e_{i,i+j}]=\alpha(k, i, j)e_{i,i+j}$$

$$\text{for} \quad k=1, 2, \cdots, n-1 \quad \text{and} \quad i=1, 2, \cdots, n-j$$

where $\alpha(k, i, j)=0$ or 1 or -1. For $k=1, 2, \cdots, n-1$, applying D to $[e_k, e_{k,k+1}]=2e_{k,k+1}$, we obtain

$$\begin{aligned} 2(&\sum_{i=1}^{n-j}\lambda_{k,k+1}^{i,i+j}e_{i,i+j}+\cdots+\lambda_{k,k+1}^{1n}e_{1n}) \\ &=[\sum_{i=1}^{n-j}\lambda_k^{i,i+j}e_{i,i+j}+\cdots+\lambda_k^{1n}e_{1n}, e_{k,k+1}] \\ &\quad+[e_k, \sum_{i=1}^{n-j}\lambda_{k,k+1}^{i,i+j}e_{i,i+j}+\cdots+\lambda_{k,k+1}^{1n}e_{1n}] \\ &=\sum_{i=1}^{n-j}\alpha(k, i, j)\lambda_{k,k+1}^{i,i+j}e_{i,i+j}+\sum_{i=1}^{n-j-1}\mu_{k,k+1}^{i,i+j+1}e_{i,i+j+1}+\cdots+\mu_{k,k+1}^{1n}e_{1n}. \end{aligned}$$

It follows that

$$\lambda_{k,k+1}^{i,i+j}=0 \qquad \text{for} \quad k=1, 2, \cdots, n-1 \quad \text{and} \quad i=1, 2, \cdots, n-j.$$

(ii) The case where the characteristic of Φ is 2:

First we assume that $n>2j+1$. We divide the proof into several cases according to the value of k.

$k=1, 2, \cdots, j-2$: By applying D to $[e_{k+1}, e_{k,k+1}]=e_{k,k+1}$, we obtain

$$\lambda_{k,k+1}^{i,i+j}=0 \qquad \text{for} \quad i\neq k+1, k+2.$$

From $[e_k, e_{k,k+1}]=0$, it follows that

$$\lambda_{k,k+1}^{k+1,k+j+1}=0.$$

From $[e_{k+2}, e_{k,k+1}]=0$, it follows that

$$\lambda_{k,k+1}^{k+2,k+j+2}=0.$$

$k=j-1$: By applying D to $[e_{k+1}, e_{k,k+1}]=e_{k,k+1}$, we obtain

$$\lambda_{k,k+1}^{i,i+j}=0 \qquad \text{for} \quad i\neq 1,\ k+1,\ k+2.$$

From $[e_k, e_{k,k+1}]=0$, it follows that

$$\lambda_{k,k+1}^{k+1,k+j+1}=0.$$

From $[e_{k+2}, e_{k,k+1}]=0$, it follows that

$$\lambda_{k,k+1}^{1,j+1}=\lambda_{k,k+1}^{k+2,k+j+2}=0.$$

$k=j$: By applying D to $[e_{k-1}, e_{k,k+1}]=e_{k,k+1}$, we have

$$\lambda_{k,k+1}^{i,i+j}=0 \qquad \text{for} \quad i\neq k-1,\ k.$$

From $[e_k, e_{k,k+1}]=0$, it follows that

$$\lambda_{k,k+1}^{k,k+j+1}=0.$$

From $[e_{2k-1}, e_{k,k+1}]=0$, it follows that

$$\lambda_{k,k+1}^{k-1,k+j-1}=0.$$

$k=j+1$: By applying D to $[e_{k-1}, e_{k,k+1}]=e_{k,k+1}$, we obtain

$$\lambda_{k,k+1}^{i,i+j}=0 \qquad \text{for} \quad i\neq 1,\ k-1,\ k.$$

From $[e_k, e_{k,k+1}]=0$, it follows that

$$\lambda_{k,k+1}^{1,j+1}=\lambda_{k,k+1}^{k,k+j}=0.$$

From $[e_{2j}, e_{k,k+1}]=0$, it follows that

$$\lambda_{k,k+1}^{k-1,k+j-1}=0.$$

$k=j+2, \cdots, n-j$: By applying D to $[e_{k-1}, e_{k,k+1}]=e_{k,k+1}$ we have

$$\lambda_{k,k+1}^{i,i+j}=0 \qquad \text{for} \quad i\neq k-j-1,\ k-j,\ k-1,\ k.$$

From $[e_k, e_{k,k+1}]=0$, it follows that

$$\lambda_{k,k+1}^{k-j,k}=\lambda_{k,k+1}^{k,k+j}=0.$$

From $[e_{k-2}, e_{k,k+1}]=0$, it follows that

$$\lambda_{k,k+1}^{k-j-1,k-1}=\lambda_{k,k+1}^{k-1,k+j-1}=0.$$

$k=n-j+1$: By applying D to $[e_{k-1}, e_{k,k+1}]=e_{k,k+1}$, we see

$$\lambda_{k,k+1}^{i,i+j}=0 \qquad \text{for} \quad i\neq k-j-1,\ k-j,\ n-j.$$

From $[e_k, e_{k,k+1}]=0$, it follows that

$$\lambda_{k,k+1}^{k-j,k}=0.$$

From $[e_{k-2}, e_{k,k+1}]=0$, it follows that

$$\lambda_{k,k+1}^{k-j-1,k-1}=\lambda_{k,k+1}^{n-j,n}=0.$$

$k=n-j+2, \cdots, n-1$: By applying D to $[e_{k-1}, e_{k,k+1}]=e_{k,k+1}$, we have

$$\lambda_{k,k+1}^{i,i+j}=0 \qquad \text{for} \quad i\neq k-j-1,\, k-j.$$

From $[e_k, e_{k,k+1}]=0$, it follows that

$$\lambda_{k,k+1}^{k-j,k}=0.$$

From $[e_{k-2}, e_{k,k+1}]=0$, it follows that

$$\lambda_{k,k+1}^{k-j-1,k-1}=0.$$

By a similar method we can show the assertion of the lemma in the case $n=2j+1$, the case $n=2j$ and the case $j+1<n<2j$ respectively by using the multiplication tables $(5)_{\text{II}}$, $(5)_{\text{III}}$ and $(5)_{\text{IV}}$. Therefore we omit the proof for these cases.

Now we consider the remaining case $n=j+1$. If $n=3$, apply D to $[e_1, e_{12}]=[e_2, e_{23}]=0$. Then we have

$$\lambda_{12}^{13}=\lambda_{23}^{13}=0.$$

For $n\geq 5$, applying D to $[e_1, e_{12}]=[e_{n-1}, e_{n-1,n}]=0$, we obtain

$$\lambda_{12}^{1n}=\lambda_{n-1,n}^{1n}=0.$$

From $[e_3, e_{23}]=e_{23}, \cdots, [e_{n-2}, e_{n-3,n-2}]=e_{n-3,n-2}$, it follows that

$$\lambda_{23}^{1n}=\cdots=\lambda_{n-3,n-2}^{1n}=0.$$

From $[e_{n-3}, e_{n-2,n-1}]=e_{n-2,n-1}$, it follows that

$$\lambda_{n-2,n-1}^{1n}=0.$$

Thus in the case where the characteristic of Φ is 2, we have shown the assertion of the lemma where the case $n=4$ is excluded.

(iii) The case where the characteristic of Φ is 3:
First we consider the case where $n>2j+1$.
For $k=1, 2, \cdots, j$, applying D to $[e_k, e_{k,k+1}]=2e_{k,k+1}$ we have

$$\lambda_{k,k+1}^{i,i+j}=0 \qquad \text{for} \quad i\neq k+1.$$

From $[e_{k+1}, e_{k,k+1}]=-e_{k,k+1}$, it follows that

$$\lambda_{k,k+1}^{k+1,k+j+1}=0.$$

For $k=j+1, \cdots, n-j-1$, applying D to $[e_k, e_{k,k+1}]=2e_{k,k+1}$ we obtain

$$\lambda_{k,k+1}^{i,i+j}=0 \qquad \text{for} \quad i\neq k-j,\ k+1.$$

From $[e_{k+1}, e_{k,k+1}]=-e_{k,k+1}$, it follows that

$$\lambda_{k,k+1}^{k-j,k}=\lambda_{k,k+1}^{k+1,k+j+1}=0.$$

For $k=n-j, \cdots, n-1$, applying D to $[e_k, e_{k,k+1}]=2e_{k,k+1}$ we have

$$\lambda_{k,k+1}^{i,i+j}=0 \qquad \text{for} \quad i\neq k-j.$$

From $[e_{k-1}, e_{k,k+1}]=-e_{k,k+1}$, it follows that

$$\lambda_{k,k+1}^{k-j,k}=0.$$

By a similar method we can show the assertion for the case $n=2j+1$, the case $n=2j$, the case $j+1<n<2j$ and the case $n=j+1$ respectively by using the tables $(5)_{\mathrm{II}}$, $(5)_{\mathrm{III}}$, $(5)_{\mathrm{IV}}$ and $(5)_{\mathrm{V}}$. So we omit the proof for these cases.

LEMMA 6. *Let D be a derivation of L and let j be one of the integers* 2, 3, $\cdots$, $n-1$. *Assume that for* $k=1, 2, \cdots, n-1$

$$De_k=\sum_{i=1}^{n-j}\lambda_k^{i,i+j}e_{i,i+j}+\cdots+\lambda_k^{1n}e_{1n},$$

$$De_{k,k+1}=\begin{cases}\displaystyle\sum_{i=1}^{n-j-1}\lambda_{k,k+1}^{i,i+j+1}e_{i,i+j+1}+\cdots+\lambda_{k,k+1}^{1n}e_{1n} & \text{if} \quad j\neq n-1,\\ 0 & \text{if} \quad j=n-1.\end{cases}$$

If we put $D'=D+\sum_{i=1}^{n-j}\lambda_i^{i,i+j}\operatorname{ad}e_{i,i+j}$, *then for* $j\neq n-1$

$$D'e_k=\sum_{i=1}^{n-j-1}\mu_k^{i,i+j+1}e_{i,i+j+1}+\cdots+\mu_k^{1n}e_{1n},$$

$$D'e_{k,k+1}=\sum_{i=1}^{n-j-1}\mu_{k,k+1}^{i,i+j+1}e_{i,i+j+1}+\cdots+\mu_{k,k+1}^{1n}e_{1n}$$

and for $j=n-1$

$$D'e_k=D'e_{k,k+1}=0, \qquad k=1, 2, \cdots, n-1.$$

PROOF. It is immediate that D' has the same form as that of D for e_k and $e_{k,k+1}$. Therefore we put

$$D'e_k=\sum_{i=1}^{n-j}\mu_k^{i,i+j}e_{i,i+j}+\cdots+\mu_k^{1n}e_{1n},$$

$$D'e_{k,k+1}=\sum_{i=1}^{n-j-1}\mu_{k,k+1}^{i,i+j+1}e_{i,i+j+1}+\cdots+\mu_{k,k+1}^{1n}e_{1n}$$

$$\text{for} \quad k=1, 2, \cdots, n-1.$$

Then we have

$$\mu_k^{k,k+j}=0 \qquad \text{for } k=1, 2, \cdots, n-j.$$

In fact,

$$D'e_k=De_k+\sum_{i=1}^{n-j}\lambda_i^{i+j}[e_{i,i+j}, e_k]$$

and therefore

$$\mu_k^{k,k+j}=\lambda_k^{k,k+j}+(-\lambda_k^{k,k+j})=0.$$

By applying D' to $[e_1, e_2]=\cdots=[e_1, e_{n-j-1}]=0$, we have

$$\mu_1^{i,i+j}=0 \qquad \text{for} \quad i=1, 2, \cdots, n-j$$

and $$\mu_m^{1,j+1}=\mu_m^{2,j+2}=0 \qquad \text{for} \quad m=2, 3, \cdots, n-j-1.$$

For $m=n-j, \cdots, n-1$, apply D' to $[e_1, e_m]=0$ and we obtain

$$\mu_m^{1,j+1}=\mu_m^{2,j+2}=0.$$

Now assume that $2\leq k\leq n-j-1$ and that we have

$$\mu_l^{i,i+j}=0 \qquad \text{for} \quad l=1, 2, \cdots, k-1 \quad \text{and} \quad i=1, 2, \cdots, n-j$$

and $$\mu_m^{1,j-1}=\mu_m^{2,j+1}=\cdots=\mu_m^{k,k+j}=0 \qquad \text{for} \quad m=k, k+1, \cdots, n-1.$$

Then applying D' to $[e_k, e_{k+1}]=\cdots=[e_k, e_{n-j-1}]=0$, we have

$$\mu_k^{i,i+j}=0 \qquad \text{for} \quad i=k+1, \cdots, n-j$$

and $$\mu_m^{k+1,k+j+1}=0 \qquad \text{for} \quad m=k+2, \cdots, n-j-1.$$

For $m=n-j, \cdots, n-1$, apply D' to $[e_k, e_m]=0$ and we obtain

$$\mu_m^{k+1,k+j+1}=0.$$

Thus we conclude that

$$\mu_k^{i,i+j}=0 \qquad \text{for} \quad k=1, 2, \cdots, n-1 \quad \text{and} \quad i=1, 2, \cdots, n-j.$$

3. The first statement of Theorem 1

Throughout this section we assume that either the characteristic of Φ is 0, or the characteristic of Φ is $p\neq 0$ and $n\not\equiv 0 \pmod{p}$.

By our assumption on the characteristic of Φ, the system (2) of $n-1$ equations has the nonsingular matrix of coefficients. Therefore by virtue of Lemma 2 any derivation D of L has the following form:

$$
(6)\qquad \begin{cases} De_k=\sum_{i=1}^{n-1}\lambda_k^{i,i+1}e_{i,i+1}+\cdots+\lambda_k^{1n}e_{1n}, \\ De_{k,k+1}=\lambda_{k,k+1}^{k,k+1}e_{k,k+1}+\sum_{i=1}^{n-2}\lambda_{k,k+1}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{k,k+1}^{1n}e_{1n} \\ \qquad\qquad\qquad\text{for}\quad k=1,2,\cdots,n-1. \end{cases}
$$

LEMMA 7. *Let D be any derivation of L. Then there exist $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ in Φ such that*

$$D'=D-\sum_{i=1}^{n-1}\alpha_i \operatorname{ad} e_i$$

has the following form for $e_{k,k+1}$:

$$D'e_{k,k+1}=\sum_{i=1}^{n-2}\lambda_{k,k+1}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{k,k+1}^{1n}e_{1n}$$

$$\textit{for}\quad k=1,2,\cdots,n-1.$$

PROOF. By the remark preceding the lemma,

$$De_{k,k+1}=\lambda_{k,k+1}^{k,k+1}e_{k,k+1}+\sum_{i=1}^{n-2}\lambda_{k,k+1}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{k,k+1}^{1n}e_{1n}$$

$$\text{for}\quad k=1,2,\cdots,n-1.$$

We now consider the following system of $n-1$ equations:

$$
\begin{cases} 2x_1-x_2 = \lambda_{12}^{12} \\ -x_1+2x_2-x_3 = \lambda_{23}^{23} \\ \quad\cdot \\ \quad\quad\cdot \\ \quad\quad\quad\cdot \\ -x_{n-3}+2x_{n-2}-x_{n-1}=\lambda_{n-2,n-1}^{n-2,n-1} \\ -x_{n-2}+2x_{n-1}=\lambda_{n-1,n}^{n-1,n} \end{cases}
$$

Since the matrix of coefficients of the system is nonsingular, the system has a unique solution, which we denote by $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$. With these α_i's we define D' as in the statement. Then

$$
D'e_{k,k+1}=De_{k,k+1}-\sum_{i=1}^{n-1}\alpha_i[e_i, e_{k,k+1}]
$$

$$
\begin{cases} (\lambda_{12}^{12}-2\alpha_1+\alpha_2)e_{12}+\sum_{i=1}^{n-2}\lambda_{12}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{12}^{1n}e_{1n} \qquad \text{for}\quad k=1, \\ (\lambda_{k,k+1}^{k,k+1}+\alpha_{k-1}-2\alpha_k+\alpha_{k+1})e_{k,k+1}+\sum_{i=1}^{n-2}\lambda_{k,k+1}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{k,k+1}^{1n}e_{1n} \end{cases}
$$

$$
= \begin{cases} & \text{for} \quad k=2, 3, \cdots, n-2, \\ (\lambda^{n-1,n}_{n-1,n}+\alpha_{n-2}-2\alpha_{n-1})e_{n-1,n}+\sum_{i=1}^{n-2}\lambda^{i,i+2}_{n-1,n}e_{i,i+2}+\cdots+\lambda^{1n}_{n-1,n}e_{1n} & \\ & \text{for} \quad k=n-1. \end{cases}
$$

$$
= \sum_{i=1}^{n-2}\lambda^{i,i+2}_{k,k+1}e_{i,i+2}+\cdots+\lambda^{1n}_{k,k+1}e_{1n}.
$$

Proof of the First Statement of Theorem 1:

In the case where $n=2$, the characteristic of Φ is $\neq 2$. Therefore L is a 2-dimensional non-abelian solvable Lie algebra. It is known that L has then no outer derivation, that is, $H^1(L, L)=(0)$.

We therefore assume that $n \geq 3$. Let D be any derivation of L. Then D has the form (6) for e_k and $e_{k,k+1}$, $k=1, 2, \cdots, n-1$. By virtue of Lemma 7, adding an inner derivation to D, we may suppose that

$$
\lambda^{k,k+1}_{k,k+1}=0 \qquad \text{for} \quad k=1, 2, \cdots, n-1.
$$

Owing to Lemma 4, by adding an inner derivation to D, we may furthermore suppose that

$$
\lambda^{i,i+1}_{k}=0 \qquad \text{for} \quad i, k=1, 2, \cdots, n-1.
$$

By making use of Lemmas 5 and 6, we can proceed by induction to conclude that after replacing D by the sum of D and a suitable inner derivation we have

$$
De_k=De_{k,k+1}=0 \qquad \text{for} \quad k=1, 2, \cdots, n-1.
$$

But Lemma 3 then tells us that $D=0$. This shows that the first given D is an inner derivation and we have $H^1(L, L)=(0)$.

4. The second statement of Theorem 1

Throughout this section, we assume that the characteristic of Φ is $p\neq 0$ and that $n\equiv 0 \pmod{p}$.

The matrix of coefficients of the system (2) of $n-1$ equations is singular but has rank $n-2$. Therefore any solution of (2) is of the form:

$$
x_1=\beta,\ x_2=2\beta,\ \cdots,\ x_{n-1}=(n-1)\beta,
$$

where β is an element of Φ.

By virtue of Lemma 2 any derivation D of L has the following form for e_k and $e_{k,k+1}$.

$$(7)\quad \begin{cases} De_k=\sum_{i=1}^{n-1} i\beta_k e_i+\sum_{i=1}^{n-1}\lambda_k^{i,i+1}e_{i,i+1}+\cdots+\lambda_k^{1n}e_{1n} \qquad \text{for} \quad k=1, 2, \cdots, n-1. \\ \text{For} \quad n\geq 5, \\ De_{k,k+1}=\lambda_{k,k+1}^{k,k+1}e_{k,k+1}+\sum_{i=1}^{n-1}\lambda_{k,k+1}^{i,i+1}e_{i,i+1}+\cdots+\lambda_{k,k+1}^{1n}e_{1n} \quad \text{for } k=1, 2, \cdots, n-1. \end{cases}$$

LEMMA 8. *The center of L is spanned by the identity matrix e_0.*

PROOF. The trace of e_0 is 0 and therefore $e_0 \in L$. e_0 evidently belongs to the center of L.

Conversely, suppose that

$$e=\sum_{i=1}^{n-1}\lambda^i e_i+\sum_{i=1}^{n-1}\lambda^{i,i+1}e_{i,i+1}+\cdots+\lambda^{1n}e_{1n}$$

is an element of the center of L. By taking the products $[e, e_i]=0$, $i=1, 2, \cdots, n-1$, and by using the tables (3) and (5), we see that

$$\lambda^{i,i+j}=0 \qquad \text{for} \quad i=1, 2, \cdots, n-j \quad \text{and} \quad j=1, 2, \cdots, n-1.$$

From $[e, e_{12}]=0$, it follows that

$$2\lambda^1-\lambda^2=0.$$

For $k=2, 3, \cdots, n-2$, it follows from $[e, e_{k,k+1}]=0$ that

$$-\lambda^{k-1}+2\lambda^k-\lambda^{k+1}=0.$$

From $[e, e_{n-1,n}]=0$, it follows that

$$-\lambda^{n-2}+2\lambda^{n-1}=0.$$

Thus $\{\lambda^1, \lambda^2, \cdots, \lambda^{n-1}\}$ is a solution of the system (2) of equations. Therefore we can write

$$\lambda^i=i\beta, \qquad i=1, 2, \cdots, n-1$$

with some $\beta \in \Phi$. Hence

$$e=\beta(\sum_{i=1}^{n-1} ie_i)=\beta e_0.$$

LEMMA 9. *For any $k=1, 2, \cdots, n-1$, let D_k be the endomorphism of L sending e_k to e_0 and all other elements of a basis* (1) *to* 0. *Let D_{12} be the endomorphism of L sending e_{1k} to e_{1k} for $k=2, 3, \cdots, n$ and all other elements of a basis* (1) *to* 0. *Then D_k and D_{12} are outer derivations of L.*

PROOF. By Lemma 8 we see that D_k maps L into the center of L and L^2 into (0). Hence D_k is a derivation of L, which is outer since $e_0 \notin L^2$.

It is easy to verify that D_{12} is a derivation of L. It is furthemore outer. In fact, suppose that

$$D_{12}=\sum_{i=2}^{n-1}\lambda^{i}\,\mathrm{ad}\,e_i+\sum_{i=1}^{n-1}\lambda^{i,i+1}\,\mathrm{ad}\,e_{i,i+1}+\cdots+\lambda^{1n}\,\mathrm{ad}\,e_{1n}.$$

Applying D_{12} to e_k, $k=1, 2, \cdots, n-1$, by (3) and (5) we obtain

$$\lambda^{i,i+j}=0 \qquad \text{for} \quad i=1, 2, \cdots, n-j \quad \text{and} \quad j=1, 2, \cdots, n-1.$$

Hence $D_{12}=\sum_{i=2}^{n-1}\lambda^{i}\,\mathrm{ad}\,e_i$. Now apply D_{12} to $e_{12}, e_{23}, \cdots, e_{n-1,n}$. Then we see that $-\lambda^2=1$ and that $\lambda^2, \lambda^3, \cdots, \lambda^{n-1}$ satisfy the following system of equations.

$$(8)\qquad \begin{cases} -2x_2-x_3 & =0 \\ -x_2+2x_3-x_4 & =0 \\ \quad\cdots & \\ -x_{n-3}+2x_{n-2}-x_{n-1} & =0 \\ -x_{n-2}+2x_{n-1} & =0 \end{cases}$$

The system (8) has the nonsingular matrix of coefficients and therefore it has only the trivial solution. Hence

$$\lambda^2=\lambda^3=\cdots=\lambda^{n-1}=0.$$

This contradicts the fact that $-\lambda^2=1$. Therefore D_{12} is outer, as was asserted.

Lemma 10. *Let D be a derivation of L. Assume that*

$$De_k=\sum_{i=1}^{n-1}i\beta_k e_i+\sum_{i=1}^{n-1}\lambda_k^{i,i+1}e_{i,i+1}+\cdots+\lambda_k^{1n}e_{1n} \qquad \textit{for} \quad k=1, 2, \cdots, n-1.$$

If we put

$$D'=D-\sum_{i=1}^{n-1}\beta_i D_i$$

with D_i's the derivations defined in Lemma 9, then

$$D'e_k=\sum_{i=1}^{n-1}\lambda_k^{i,i+1}e_{i,i-1}+\cdots+\lambda_k^{1n}e_{1n} \qquad \textit{for} \quad k=1, 2, \cdots, n-1.$$

Proof. By Lemma 9 we have

$$D'e_k=\sum_{i=1}^{n-1}i\beta_k e_i+\sum_{i=1}^{n-1}\lambda_k^{i,i+1}e_{i,i+1}+\cdots+\lambda_k^{1n}e_{1n}-\Big(\sum_{i=1}^{n-1}\beta_i D_i\Big)e_k$$

$$=\beta_k e_0+\sum_{i=1}^{n-1}\lambda_k^{i,i+1}e_{i,i+1}+\cdots+\lambda_k^{1n}e_{1n}-\beta_k D_k e_k$$

$$=\sum_{i=1}^{n-1}\lambda_k^{i,i+1}e_{i,i+1}+\cdots+\lambda_k^{1n}e_{1n}.$$

LEMMA 11. *Let D be a derivation of L. Assume that*

$$De_k=\sum_{i=1}^{n-1}\lambda_k^{i,i+1}e_{i,i+1}+\cdots+\lambda_k^{1n}e_{1n},$$

$$De_{k,k+1}=\lambda_{k,k+1}^{k,k+1}e_{k,k+1}+\sum_{i=1}^{n-2}\lambda_{k,k+1}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{k,k+1}^{1n}e_{1n}$$

$$\text{for}\quad k=1,2,\cdots,n-1.$$

Then there exist $\alpha_2, \alpha_3, \cdots, \alpha_{n-1}$ in Φ such that

$$D'=D-(\lambda_{12}^{12}+\alpha_2)D_{12}-\sum_{i=2}^{n-1}\alpha_i\,\mathrm{ad}\,e_i$$

has the following form for $e_{k,k+1}$:

$$D'e_{k,k+1}=\sum_{i=1}^{n-2}\lambda_{k,k+1}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{k,k+1}^{1n}e_{1n}\qquad \text{for}\quad k=1,2,\cdots,n-1.$$

PROOF. We consider the system of $n-2$ equations

$$\begin{cases} 2x_2-x_3 = \lambda_{23}^{23} \\ -x_2+2x_3-x_4 = \lambda_{34}^{34} \\ \quad\cdot \\ \quad\quad\cdot \\ \quad\quad\quad\cdot \\ -x_{n-3}+2x_{n-2}-x_{n-1}=\lambda_{n-2,n-1}^{n-2,n-1} \\ -x_{n-2}+2x_{n-1}=\lambda_{n-1,n}^{n-1,n}. \end{cases}$$

This system has the nonsingular matrix of coefficients and therefore it has a solution. Denote a solution of the system by $\alpha_2, \alpha_3, \cdots, \alpha_{n-1}$ and define D' as in the statement. Then

$$D'e_{12}=De_{12}-(\lambda_{12}^{12}+\alpha_2)e_{12}-\sum_{i=2}^{n-1}\alpha_i[e_i, e_{12}]$$

$$=\big(\lambda_{12}^{12}-(\lambda_{12}^{12}+\alpha_2)+\alpha_2\big)e_{12}+\sum_{i=1}^{n-2}\lambda_{12}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{12}^{1n}e_{1n}$$

$$=\sum_{i=1}^{n-2}\lambda_{12}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{12}^{1n}e_{1n}.$$

$$D'e_{23}=De_{23}-\sum_{i=2}^{n-1}\alpha_i[e_i, e_{23}]$$

$$=(\lambda_{23}^{23}-2\alpha_2+\alpha_3)e_{23}+\sum_{i=1}^{n-2}\lambda_{23}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{23}^{1n}e_{1n}$$

$$=\sum_{i=1}^{n-2}\lambda_{23}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{23}^{1n}e_{1n}.$$

For $k=3, 4, \cdots, n-2$,

$$D'e_{k,k+1}=De_{k,k+1}-\sum_{i=2}^{n-1}\alpha_i[e_i, e_{k,k+1}]$$

$$=(\lambda_{k,k+1}^{k,k+1}+\alpha_{k-1}-2\alpha_k+\alpha_{k+1})e_{k,k+1}+\sum_{i=1}^{n-2}\lambda_{k,k+1}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{k,k+1}^{1n}e_{1n}$$

$$=\sum_{i=1}^{n-2}\lambda_{k,k+1}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{k,k+1}^{1n}e_{1n}.$$

$$D'e_{n-1,n}=De_{n-1,n}-\sum_{i=2}^{n-1}\alpha_i[e_i, e_{n-1,n}]$$

$$=(\lambda_{n-1,n}^{n-1,n}+\alpha_{n-2}-2\alpha_{n-1})e_{n-1,n}+\sum_{i=1}^{n-2}\lambda_{n-1,n}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{n-1,n}^{1n}e_{1n}$$

$$=\sum_{i=1}^{n-2}\lambda_{n-1,n}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{n-1,n}^{1n}e_{1n}.$$

Thus the proof is complete.

PROOF OF THE SECOND STATEMMENT OF THEOREM 1:

Any derivation D of L has the form (7) for e_k and $e_{k,k+1}$, $k=1, 2, \cdots, n-1$. Put

$$D'=D-\sum_{i=1}^{n-1}\beta_i D_i-(\lambda_{12}^{12}+\alpha_2)D_{12}-\sum_{i=2}^{n-1}\alpha_i \operatorname{ad} e_i,$$

where $\alpha_2, \alpha_3, \cdots, \alpha_{n-1}$ are the elements of Φ chosen in Lemma 11. Then by making use of Lemmas 10 and 11 we have

$$D'e_k=\sum_{i=1}^{n-1}\lambda_k^{i,i+1}e_{i,i+1}+\cdots+\lambda_k^{1n}e_{1n},$$

$$D'e_{k,k+1}=\sum_{i=1}^{n-2}\lambda_{k,k+1}^{i,i+2}e_{i,i+2}+\cdots+\lambda_{k,k+1}^{1n}e_{1n} \qquad \text{for} \quad k=1, 2, \cdots, n-1.$$

Now as in the proof of the first statement of Theorem 1, we can use Lemmas 3, 4, 5 and 6 to see that D' is an inner derivation of L. Therefore in order to see that $H^1(L, L)$ is of dimension n, it is sufficient for us to show that $D_1, D_2, \cdots, D_{n-1}$ and D_{12} are linearly independent modulo the inner derivations. Suppose that the derivation

(9) $$\sum_{i=1}^{n-1}\lambda^i D_i+\lambda D_{12}+\sum_{i=2}^{n-1}\mu^i \operatorname{ad} e_i+\sum_{i=1}^{n-1}\mu^{i,i+1}\operatorname{ad} e_{i,i+1}+\cdots+\mu^{1n}\operatorname{ad} e_{1n}$$

is identically 0, where all the λ's and μ's are in Φ. Applying the derivation

(9) to e_k, we obtain $\lambda^k=0$ for $k=1, 2, \cdots, n-1$. Apply the derivation (9) to e_{12}. Then $\lambda-\mu^2=0$. By applying the derivation (9) to $e_{23}, e_{34}, \cdots, e_{n-1,n}$, we see that $\mu^2, \mu^3, \cdots, \mu^{n-1}$ satisfy the system (8) of $n-2$ equations. It follows that $\mu^2=\mu^3=\cdots=\mu^{n-1}=0$. Therefore $\lambda=0$. It is now immediate that all the other μ's are 0. This completes the proof.

5. Remark to Theorem 1

In this section, we shall consider the three cases excluded in Theorem 1.

In the case where $n=2$ and the characteristic of Φ is 2, L is a 2-dimensional abelian Lie algebra. Hence $\dim H^1(L, L)=4$.

The case where $n=3$ and the characteristic of Φ is 3:
By virtue of Lemma 2 we see that

$$De_k=\beta_k(e_1-e_2)+\sum_{i=1}^{2}\lambda_k^{i,i+1}e_{i,i+1}+\lambda_k^{13}e_{13},$$

$$De_{k,k+1}=\sum_{i=1}^{2}\lambda_{k,k+1}^{i,i+1}e_{i,i+1}+\lambda_k^{13}e_{13} \qquad \text{for} \quad k=1, 2.$$

Let D_{12}^{23} (resp. D_{23}^{12}) be the endomorphism of L sending e_{12} (resp. e_{23}) to e_{23} (resp. e_{12}) and all other elements of a basis (1) to 0. Then these are outer derivations of L. With a slight modification of the reasoning in the preceding section, we can show that any derivation of L is a linear combination of D_1, D_2, D_{12}, D_{12}^{23}, D_{23}^{12} and an inner derivation. It is easy to see that these outer derivations are linearly independent modulo the inner derivations. Therefore we conclude that $\dim H^1(L, L)=5$.

The case where $n=4$ and the characteristic of Φ is 2:
By Lemma 2 and its proof, we see that

$$De_k=\beta_k(e_1+e_3)+\sum_{i=1}^{3}\lambda_k^{i,i+1}e_{i,i+1}+\sum_{i=1}^{2}\lambda_k^{i,i+2}e_{i,i+2}+\lambda_k^{14}e_{14} \qquad \text{for} \quad k=1, 2, 3,$$

$$De_{12}=(\lambda_{12}^{12}e_{12}+\lambda_{12}^{34}e_{34})+\sum_{i=1}^{2}\lambda_{12}^{i,i+2}e_{i,i+2}+\lambda_{12}^{14}e_{14},$$

$$De_{23}=\lambda_{23}^{23}e_{23}+\sum_{i=1}^{2}\lambda_{23}^{i,i+2}e_{i,i+2}+\lambda_{23}^{14}e_{14},$$

$$De_{34}=(\lambda_{34}^{12}e_{12}+\lambda_{34}^{34}e_{34})+\sum_{i=1}^{2}\lambda_{34}^{i,i+2}e_{i,i+2}+\lambda_{34}^{14}e_{14}.$$

Let D_{12}^{34} (resp. D_{34}^{12}) be the endomorphism of L sending e_{12} (resp. e_{34}) to e_{34} (resp. e_{12}), e_{13} (resp. e_{24}) to $-e_{24}$ (resp. $-e_{13}$) and all the other elements of a basis (1) to 0. Let D_{23}^{14} be the endomorphism of L sending e_{23} to e_{14} and all other elements of a basis (1) to 0. Then these are outer derivations of L. With a slight modification of the reasoning in the preceding section, we can show that any derivation of L is a linear combination of D_1, D_2, D_3, D_{12}, D_{12}^{34}, D_{34}^{12}, D_{23}^{14} and an inner derivation. It is easy to see that these outer derivations are linearly independent modulo the inner derivations. Thus we conclude that $\dim H^1(L, L)=7$.

6. Proof of Theorem 2

We can prove Theorem 2 in a quite similar manner as in the proof of the second statement of Theorem 1. Therefore we shall only write the outline of the proof.

Throughout this section, let Φ be a field of arbitrary characteristic and denote $\mathfrak{t}(n, \Phi)$ with $n \geq 2$ by L for the sake of simplicity.

We choose a basis of L as follows.

e_k: the $(a_{ij}) \in L$ such that $a_{kk}=1$ and all other $a_{ij}=0$,
for $k=1, 2, \cdots, n$.

$e_{k,k+l}$: the $(a_{ij}) \in L$ such that $a_{k,k+l}=1$ and all other $a_{ij}=0$,
for $k=1, 2, \cdots, n-1$ and $l=1, 2, \cdots, n-k$.

We put these elements of L in the following order:

$$e_1, \cdots, e_n; e_{12}, \cdots, e_{n-1,n}; \cdots; e_{1,n-1}, e_{2n}; e_{1n}. \tag{10}$$

Then, corresponding to Lemma 1, for any derivation D of L we have

$$De_k=\sum_{i=1}^{n}\lambda_k^i e_i+\sum_{i=1}^{n-1}\lambda_k^{i,i+1}e_{i,i+1}+\cdots+\lambda_k^{1n}e_{1n} \qquad \text{for} \quad k=1, 2, \cdots, n,$$

$$De_{k,k+l}=\sum_{i=1}^{n-l}\lambda_{k,k+l}^{i,i+l}e_{i,i+l}+\cdots+\lambda_{k,k+l}^{1n}e_{1n}$$

$$\text{for} \quad k=1, 2, \cdots, n-1 \quad \text{and} \quad l=1, 2, \cdots, n-k.$$

Corresponding to Lemma 2, we can show without any restriction on n and Φ that

$$\lambda_k^1=\lambda_k^2=\cdots=\lambda_k^n \qquad \text{for} \quad k=1, 2, \cdots, n$$

and

$$\lambda_{k,k+1}^{i,i+1}=0 \qquad \text{for} \quad i, k=1, 2, \cdots, n-1 \quad \text{and} \quad i \neq k.$$

The results corresponding to Lemmas 3, 4, 5 and 6 hold for a derivation of $L=\mathfrak{t}(n, \Phi)$ without any restriction on n and Φ. It is to be noted that in the proof of the result corresponding to Lemma 4 we only need to define D' as follows:

$$D' = D + \sum_{i=1}^{n-1} \lambda_i^{i,i+1} \operatorname{ad} e_{i,i+1}.$$

It is evident that the center of L is spanned by the identity matrix e_0. We define the derivation D_k, $k=1, 2, \cdots, n$, as in Lemma 9. Corresponding to (8), we consider the following system of $n-1$ equations in n indeterminates:

$$\begin{cases} x_1 - x_2 & = \lambda_{12}^{12} \\ \quad x_2 - x_3 & = \lambda_{23}^{23} \\ \qquad \cdot & \\ \qquad\quad \cdot & \\ \qquad\qquad \cdot & \\ \qquad\qquad x_{n-1} - x_n & = \lambda_{n-1,n}^{n-1,n}. \end{cases}$$

Then the system has a solution of the following type:

$$x_1 = 0, \ x_2 = \alpha_2, \ \cdots, \ x_n = \alpha_n.$$

Putting

$$D' = D - \sum_{i=1}^{n} \lambda_i^1 D_i - \sum_{i=2}^{n} \alpha_i \operatorname{ad} e_i,$$

we have

$$D' e_k = \sum_{i=1}^{n-1} \lambda_k^{i,i+1} e_{i,i+1} + \cdots + \lambda_k^{1n} e_{1n}, \qquad k = 1, 2, \cdots, n,$$

$$D' e_{k,k+1} = \sum_{i=1}^{n-2} \lambda_{k,k+1}^{i,i+2} e_{i,i+2} + \cdots + \lambda_{k,k+1}^{1n} e_{1n}, \qquad k = 1, 2, \cdots, n-1.$$

Now as in the proof of the second statement of Theorem 1, we see by making use of the results corresponding to Lemmas 3, 4, 5 and 6 that D' is an inner derivation of L and we can conclude that $\dim H^1(L, L) = n$.

Reference

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