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A Remark on Vector Fields on Lens Spaces

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§1. Introduction

Let M be a C^∞-manifold. The (continuous) vector field v on M is a cross-section of the tangent bundle of M, and k-field on M is a set of k vector fields $v_1, \cdots, v_k$ such that the k vectors $v_1(x), \cdots, v_k(x)$ are linearly independent for each point $x \in M$. We denote by $span(M)$ the maximal number of k where M admits a k-field.

In this note, it is remarked that $span(L^n(p))$, of the $(2n+1)$-dimensional mod p lens space $L^n(p)$, is given partially by the following

Proposition. *Let* $n+1=m2^t$ (m: odd), $t+1=c+4d$ $(0 \leqq c \leqq 3)$

(i) *If* $c=0$, *then* $2t+1 \leqq span(L^n(p)) \leqq 2t+2$ $(=span(S^{2n+1}))$.

(ii) *If* $c=1, 2$, *then* $span(L^n(p))=2t+1$ $(=span(S^{2n+1}))$.

(iii) *If* $c=3$, *then* $2t+1 \leqq span(L^n(p)) \leqq 2t+3$ $(=span(S^{2n+1}))$.

Here the lens space $L^n(p)$ $(p>1)$ is the quotient space S^{2n+1}/Γ of the unit sphere S^{2n+1} by the topological transformation group $\Gamma=\{1, \gamma, \cdots, \gamma^{p-1}\}$ defined by

$$\gamma \cdot (z_0, z_1, \cdots, z_n)=(e^{2\pi i/p}z_0, e^{2\pi i/p}z_1, \cdots, e^{2\pi i/p}z_n)$$
$$((z_0, z_1, \cdots, z_n) \in S^{2n+1} \subset C^{n+1}).$$

We notice that the above proposition holds in the following form for the case $p=2$:

$$span(L^n(2))=span(S^{2n+1}).$$

This follows easily from the fact that $L^n(2)$ is the $(2n+1)$-dimensional real projective space RP^{2n+1}, and

$$span(RP^n)=span(S^n),$$

which is an immediate consequence of the fact that S^n has a linear k-field, $k=span(S^n)$.

Also, we notice that there is a lens space such that

$$span(L^n(p))<span(S^{2n+1}),$$

since $span(L^3(3))=5$(cf. §3) and $span(S^7)=7$.

§2. Proofs of Proposition

Since a k-field on $L^n(p)$ defines clearly a k-field on S^{2n+1}, we have $span\ (L^n(p))\leqq span(S^{2n+1})$.

Also we have

$$span(L^n(p))\geqq 2t+1$$

for the integer t of the proposition, and these and the results of $span(S^{2n+1})$, determined by J. F. Adams [1], show the proposition.

The above relation means that $L^n(p)$ admits a $(2t+1)$-field, and this is proved as the corollary of the results of B. Eckmann [2] as follows. By 6 of [2], it is shown that tehre exist $2t+1$ unitary matrices $A_1, \dots, A_{2t+1}\in U(n+1)$ such that

$$A_k^2=-E,\ A_kA_l+A_lA_k=0 \qquad (k,\ l=1,\ \dots,\ 2t+1;\ k\neq l),$$

where E is the unit matrix. For an arbitrary element $u\in S^{2n+1}\subset C^{n+1}$, the first equation shows that

$$<u,\ A_k(u)>=<A_k(u),\ A_k^2(u)>=<A_k(u),\ -u>=-\overline{<u,\ A_k(u)>},$$

and so the real part of the inner product $<u,\ A_k(u)>$ is zero. Hence $A_k(u)$ $(u\in S^{2n+1})$ is a vector field on S^{2n+1}, and this defines a vector field on $L^n(p)$ since $A_k(\gamma\cdot u)=\gamma\cdot A_k(u)$.

Also, the second equation shows that the real part of $<A_l(u),\ A_k(u)>$ $(k\neq l)$ is zero for any $u\in S^{2n+1}$, and so $A_1, \cdots, A_{2t+1}$ define a $(2t+1)$-field on $L^n(p)$.

§3. Remarks

(a) $span(L^3(3))=5$ is proved as follows. By (iii) of the proposition, we have $5\leqq span(L^3(3))\leqq 7$. Assume that $span(L^3(3))\geqq 6$, then $L^3(3)$ admits a 7-field since $L^3(3)$ is an orientable 7-manifold. This means that $L^3(3)$ is parallelizable and so is immersible in the real 8-space R^8. But it is shown that $L^3(3)$ is not immersible in R^9 by Theorem 6 of [3], we have $span(L^3(3))=5$.

(b) In relation to the above, it is known that $span(L^{11}(p))\geqq 6$ (p: odd) by Corollary 1.4 of [4].

(c) For the lens space $L(p;\ l_1, \cdots, l_n)=S^{2n+1}/\Gamma'$ such that Γ' is generated by γ':

$\gamma'\cdot(z_0,\ z_1,\ \dots,\ z_n)=(e^{2\pi i/p}z_0,\ e^{2\pi il_1/p}z_1,\ \cdots,\ e^{2\pi il_n/p}z_n)$, the above proposition holds if $1=l_1=\dots=l_{2^t-1}$, $l_{2^t}=\dots=l_{2.2^t-1}$, $\cdots$, $l_{(m-1).2^t}=\cdots=l_{m.2^t-1}$, where m is the integer in the proposition. This is easily seen from the form of the above A_k.

References

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