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Relatively Complemented Lie Algebras

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Introduction

If L is a Lie algebra over a field Φ, we consider the lattice of all subalgebras of L. A Lie algebra will be called distributive, modular, upper semi-modular, lower semi-modular, complemented, or relatively complemented if its lattice of all subalgebras has the corresponding property. In [8], we investigated distributive, upper semi-modular, lower semi-modular, and modular Lie algebras. In this paper we continue the investigation of the relation between the structure of a Lie algebra and the structure of its lattice of all subalgebras and we concentrate on relatively complemented Lie algebras. We 1) characterize relatively complemented Lie algebras over algebraically closed fields of characteristic zero, 2) characterize relatively complemented Lie algebras over the field of real numbers, 3) study other properties of complemented and relatively complemented Lie algebras.

The Lie algebras considered in this paper will be finite dimensional, and will, unless otherwise stated, be over a field of characteristic zero.

In this paper, if L is a Lie algebra $[L, L]$ will be denoted by L', and $[L', L']$ by L''. Also the subalgebra of L generated by $e_1, e_2, \cdots, e_k$ will be denoted by $\{e_1, e_2, \cdots, e_k\}$.

SECTION 1. Preliminaries and Examples

Definition: A Lie algebra L, over a field of any characteristic, is called distributive, modular, upper semi-modular, lower semi-modular, complemented, or relatively complemented if its lattice of all subalgebras has the corresponding property.

Definition: Let L be a Lie algebra over a field of any characteristic. The Frattini subalgebra F of L is the intersection of all the maximal subalgebras of L.

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DEFINITION: Let L be a Lie algebra, over a field of any characteristic, and M a subset of L. Let $\{M\}$ denote the subalgebra of L generated by M. $x \in L$ is called a non-generator of L if whenever $L=\{x, M\}$ then $L=\{M\}$.

It is known that if $L \neq \{0\}$ is a Lie algebra over a field of any characteristic, then its Frattini subalgebra F is the set of all non-generators of L. Also, let L be a Lie algebra over a field of any characteristic, M a subset of L, F the Frattini subalgebra of L. The basis theorem asserts that if $\{M, F\}=L$, then $\{M\}=L$. Moreover, if L is a nilpotent Lie algebra over a field of any characteristic, then its Frattini subalgebra contains L'.

PROPOSITION 1.1 Let L be a simple three-dimensional Lie algebra. L is relatively complemented if and only if L is non-split. Moreover, if L is split, then L is complemented.

PROOF: It is known that L is non-split if and only if L has no two-dimensional subalgebra. Thus, if L is non-split it is relatively complemented.

Let L be split. Then there exists a basis e_1, e_2, e_3 for L such that $[e_1, e_2]=e_3$; $[e_1, e_3]=2e_1$; $[e_2, e_3]=-2e_2$.
We now define the following subalgebras of L:

$$L_1=\{e_1\}$$
$$L_2=L$$
$$M=\{e_1, e_3\}.$$

We assert that there exists no subalgebra N of L such that $M \cup N=L_2$ and $M \cap N=L_1$, for if such a subalgebra N exists then $N=\{e_1, a_2e_2+a_3e_3\}$, $a_2 \neq 0$. Now $[e_1, a_2e_2+a_3e_3]=2a_3e_1+a_2e_3$. Thus, N is a subalgebra if and only if there exists $\lambda, \mu \in \Phi$ such that $2a_3e_1+a_2e_3=\lambda e_1+\mu(a_2e_2+a_3e_3)$. Then $a_2=\mu a_3$; $2a_3=\lambda$; $\mu a_2=0$. Since $a_2 \neq 0$, $\mu=0$, which implies that $a_2=0$, a contradiction.

We now show that if L is split then it is complemented. First, every two-dimensional subalgebra of L has a complement, and if $c \neq 0, \in L$ then there exists an element $x \in L$ such that c and x generate L (see Lemma 1.1 in [8]). Hence, every one-dimensional subalgebra of L has a complement.

PROPOSITION 1.2 Let L be the four-dimensional Lie algebra over a field Φ with basis e_0, e_1, e_2, e_3 defined by $[e_1, e_2]=e_3$; $[e_2, e_3]=\alpha e_1$; $[e_3, e_1]=\beta e_2$; $[e_i, e_0]=0$, $i=1, 2, 3$, $\alpha \neq 0$, $\beta \neq 0, \in \Phi$. L is not relatively complemented.

PROOF: If $\{e_1, e_2, e_3\}$ is split, then it is not relatively complemented. Thus, suppose $\{e_1, e_2, e_3\}$ is non-split. We then define the following subalgebras of L:

$$L_1=\{e_0+e_1\}$$
$$L_2=L$$
$$M=\{e_0, e_1\}$$

We assert that there exists no subalgebra N of L such that $M\cup N=L_2$ and $M\cap N=L_1$. Thus, if such a subalgebra N exists, it must be two or three-dimensional. Let $N=\{a, c, e_0+e_1\}$, where a and $c \in L$. Let $a=a_0e_0+a_1e_1+a_2e_2+a_3e_3$, and let $b=e_0+e_1$. We then have $[[a, b], [[a, b], b]]=-(a_2^2\alpha\beta+a_3^2\alpha\beta^2)e_1$. But this element $\in M\cap N$. It then follows that $a_2^2\alpha+a_3^2\alpha\beta=0$. Now $\{e_1, e_2, e_3\}$ is non-split if and only if $a_1^2\beta+a_2^2\alpha+a_3^2\alpha\beta=0$, $a_1, a_2, a_3 \in \Phi$, implies that $a_1=a_2=a_3=0$. Hence, we conclude that $a_2=a_3=0$. Thus, $a=a_0e_0+a_1e_1$. Since $a \in M\cap N$, we have $a_0=a_1$. Similarly, c is a scalar multiple of e_0+e_1. Hence, $M\cup N=M$. This completes the proof.

Definition: An $(n+1)$-dimensional $(n\geqslant 1)$ Lie algebra, over a field of any characteristic is called almost abelian if it has a basis $e_0, e_1, \cdots, e_n$ such that $[e_i, e_0]=e_i$ for $i\geqslant 1$, and $[e_i, e_j]=0$ for $i, j\geqslant 1$.

Proposition 1.3 Let L be an $(n+1)$-dimensional almost abelian or n-dimensional $(n\geqslant 1)$ abelian Lie algebra, over a field of any characteristic. Then L is relatively complemented.

Proof: In either case every subspace is a subalgebra.

SECTION 2. Relatively Complemented Lie Algebras

Theorem 2.1 If L is a complemented Lie algebra, over a field of any characteristic, then its Frattini subalgebra $F=\{0\}$.

Proof: Let M be the complement of F. Then $M\cup F=L$, and $M\cap F=\{0\}$. By the basis theorem, we have $(M)=L$. Since M is a subalgebra $(M)=M$, and hence $F=\{0\}$.

Theorem 2.2 If L is a complemented nilpotent Lie algebra, over a field of any characteristic, then L is abelian.

Proof: If L is nilpotent then its Frattini subalgebra contains L'.

Corollary 2.1 If L is a relatively complemented Lie algebra, over a field of any characteristic, then its nil radical is abelian.

Corollary 2.2 If L is a relatively complemented Lie algebra, over a field of any characteristic, then its Frattini subalgebra is abelian.

Proposition 2.1 If L is a relatively complemented Lie algebra, over an algebraically closed field, then L is solvable.

Proof: If L is non-solvable, then by Levi's Theorem $L=S\oplus L_1$, where L_1 is a semi-simple subalgebra of L and S is the radical of L. L_1 then contains a split simple three-dimensional subalgebra which is thus not relatively complemented.

We now turn to the characterization of relatively complemented Lie algebras over algebraically closed fields of characteristic zero. We first have the following

LEMMA 2.1 Let L be a Lie algebra of dimension $n+1$, $n \geqslant 1$, over a field of any characteristic, and let N be an abelian ideal of L of dimension n, and $x \in L$, $x \notin N$. If L is a relatively complemented Lie algebra then ad $x|_N$ is a diagonal transformation.

PROOF: Let $f(z)$ be the minimal polynomial of ad $x|_N$. Factoring $f(z)$ into its irreducible factors, we have $f(z)=p_1(z)^{e_1}p_2(z)^{e_2}\cdots p_k(z)^{e_k}$, where $p_i(z)$ are monic irreducible polynomials. If at least one of the $p_i(z)$, $(i=1, 2, \cdots, k)$ say $p_j(z)$, is of degree $r>1$, we write $p_j(z)=c_0+c_1z+c_2z^2+\cdots\cdots+c_{r-1}z^{r-1}+z^r$. Then there exists a basis $f_1, f_2, \cdots, f_n$ of N such that ad $x|_N$ can be represented by a matrix in rational canonical form. We thus have a subalgebra A of L, generated by $f_1, f_2, \cdots, f_r, x$, where

$$
\begin{aligned}
&[f_1, x]=f_2\\
&[f_2, x]=f_3\\
&\quad\vdots\\
&[f_{r-1}, x]=f_r\\
&[f_r, x] \ =c_0f_1+c_1f_2+\cdots+c_{r-1}f_r\\
&[f_i, f_j] \ =0 \qquad i, j=1, 2, \cdots, r.
\end{aligned}
$$

We now show that A is not relatively complemented. For define the following subalgebras of A:

$$
\begin{aligned}
&L_1=\{f_1, f_2, \cdots, f_{r-1}\}\\
&L_2=A=\{f_1, f_2, \cdots, f_r, x\}\\
&M=\{f_1, f_2, \cdots, f_r\}.
\end{aligned}
$$

If there exists a subalgebra N of A such that $M\cup N=L_2$ and $M\cap N=L_1$, then $N=\{f_1, f_2, \cdots, f_{r-1}, a\}$, where $a=a_rf_r+a_{r+1}x$, $a_{r+1}\neq 0$. Since N is a subalgebra, there exist $\lambda_1, \lambda_2, \cdots, \lambda_{r-1}, \mu \in \Phi$ such that $[f_{r-1}, a]=a_{r+1}f_r=\lambda_1f_1+\lambda_2f_2+\cdots+\lambda_{r-1}f_{r-1}+\mu a$. We then have $\mu a_r=a_{r+1}$ and $\mu a_{r+1}=0$, which imply that $a_{r+1}=0$. Thus, $f(z)=(z-\alpha)^{k_1}(z-\beta)^{k_2}\cdots$ where $\alpha, \beta, \cdots$ are the eigenvalues of ad $x|_N$. We next show that all $k_i=1$. For suppose some $k_i>1$. Then there exists a basis $f_1, f_2, \cdots, f_n$ of N such that ad $x|_N$ can be represented by a matrix in Jordan normal form. Thus, there exists a subalgebra B of L with basis f_1, f_2, f_3 such that $[f_1, f_2]=0$; $[f_1, f_3]=f_2+\alpha f_1$; $[f_2, f_3]=\alpha f_2$. B is not relatively complemented, for consider the following subalgebras of B:

$$L_3=\{f_3\}$$
$$L_4=B$$
$$M_1=\{f_2, f_3\}.$$

If there exists a subalgebra N_1 of B such that $M_1\cup N_1=L_4$, $M_1\cap N_1=L_3$, then $N_1=\{f_3, a\}$, where $a=f_1+a_2f_2$. Since N_1 is a subalgebra, there exist $\lambda, \mu \in \Phi$ such that $[a, f_3]=\alpha f_1+(1+\alpha a_2)f_2=\lambda a+\mu f_3$. Thus, we have the the following system of inconsistent equations:

$$\lambda=\alpha$$
$$\lambda a_2=1+a_2\alpha$$
$$\mu=0.$$

Hence, all $k_i=1$, which implies that $\operatorname{ad} x|_N$ is a diagonal transformation. This completes the proof.

THEOREM 2.3 Let L be a solvable Lie algebra. Then L is relatively complemented if and only if L is abelian or almost abelian.

PROOF: If L is abelian or almost abelian, then by Proposition 1.3 L is relatively complemented.

Conversely, suppose that L is relatively complemented. If L is not abelian, let N be its nil radical. N is then abelian. Now let $x \in L$, $x \notin N$, and consider $\operatorname{ad} x|_N$. Since N is an ideal, $\{x, N\}$ is a subalgebra satisfying the hypotheses of the lemma and thus, if $f(z)$ is the minimal polynomial of $\operatorname{ad} x|_N$ then $f(z)=(z-\alpha)(z-\beta)\cdots$ where $\alpha, \beta, \cdots$ are the distinct eigenvalues of ad $x|_N$. Let the decomposition of N into its eigenspaces relative to $\operatorname{ad} x|_N$ be $N=N_\alpha+N_\beta+\cdots$. Then there exist $e_\alpha \in N_\alpha$, $e_\beta \in N$ such that $[e_\alpha, x]=\alpha e_\alpha$; $[e_\beta, x]=\beta e_\beta$ and $[e_\alpha, e_\beta]=0$. Thus, $\{x, e_\alpha, e_\beta\}$ is a three-dimensional subalgebra A of L. If $\alpha=0$ and $\beta\neq 0$, we then show that A is not relatively complemented. For consider the following subalgebras of A:

$$L_1=\{e_\alpha+e_\beta\}$$
$$L_2=A$$
$$M=\{e_\alpha+e_\beta, e_\beta\}.$$

If there exists a subalgebra N of A such that $M\cup N=L_2$, $M\cap N=L_1$, then $N=\{e_\alpha+e_\beta, a\}$, where $a=a_1e_\beta+a_2x+a_3e_\alpha$. Since N is a subalgebra, there exist $\lambda, \mu \in \Phi$ such that $[e_\alpha+e_\beta, a]=\beta a_2e_\beta=\lambda(e_\alpha+e_\beta)+\mu a$. We then have the following system of equations:

$$\lambda+\mu a_1=\beta a_2$$
$$\mu a_2=0$$
$$\lambda+\mu a_3=0,$$

which implies that $a_2=0$, and thus, $M\cup N\neq L_2$.

If all the eigenvalues are zero, then the subspace $M=\{x, N\}$ is an abelian ideal of L, contradicting the maximal nilpotency of N.

We next show that dim $(L/N)=1$. Let $e_1\in L$, $e_1\notin N$, and assume that we can find $e_2\in L$, $e_2\notin\{e_1, N\}$. Now if $x\in L$, $x\notin N$, then all the matrices representing ad $x|_N$ can be taken in simultaneous triangular from, by Lie's Theorem. Thus, if α, β, $\cdots$ are weights then $\alpha(a)\neq 0$ if $a\neq 0$ where $a\in L$, $a\notin N$. Now $\alpha(\alpha(e_1)e_2-\alpha(e_2)e_1)=0$, and thus $\alpha(e_1)e_2-\alpha(e_2)e_1=0$, contradicting the hypothesis that e_1 and e_2 are linearly independent.

Let $e_0, e_1, \cdots, e_n$ be a basis of L. We then have $[e_i, e_0]=\alpha_i e_i$, $\alpha_i\neq 0$, $i\geqslant 1$ and $[e_i, e_j]=0$, $i, j\geqslant 1$.

We now show that all the α_i are equal. Suppose that, say $\alpha_2\neq 1$ and $\alpha_1=1$. Consider the three-dimensional Lie algebra $\{e_0, e_1, e_2\}=B$. We assert that B is not relatively complemented. For consider the following subalgebras of B:

$$L_1=\{e_1+e_2\}$$

$$L_2=B$$

$$M=\{e_1, e_2\}.$$

If there exists a subalgebra N of B such that $M\cup N=L_2$ and $M\cap N=L_1$, then $N=\{e_1+e_2, a\}$, where $a=a_0e_0+a_1e_1+a_2e_2$. Since N is a subalgebra, there exist λ, $\mu\in\Phi$ such that $[e_1+e_2, a]=a_0e_1+a_0\alpha_2e_2=\lambda(e_1+e_2)+\mu a$. We then have the following system of equations:

$$\mu a_0=0$$

$$\lambda+\mu a_1=a_0$$

$$\lambda+\mu a_2=a_0\alpha_2.$$

Now $\mu=0$ implies that $\alpha_2=1$, and $a_0=0$ implies that $M\cup N\neq L_2$. Hence, all α_i are equal and thus L is almost abelian. This completes the proof of Theorem 2.3.

COROLLARY 2.3 Let L be a Lie algebra over an algebraically closed field. Then L is relatively complemented if and only if L is abelian or almost abelian.

We now turn to an investigation of simple Lie algebras over the field of real numbers.

THEOREM 2.4 If L is a non-compact real simple Lie algebra, then L is not relatively complemented.

PROOF: Let L be a non-compact real simple Lie algebra of rank 1. Let L^C be the complexification of L, and let $h_1, h_2, \cdots, h_l, e_\alpha, e_{-\alpha}, e_\beta, e_{-\beta}, \cdots$ be a basis of L^C. The compact form L_u of L^C has a basis $ih_1, ih_2, \cdots, ih_l, (e_\alpha + e_{-\alpha})$, $i(e_\alpha - e_{-\alpha})$ α ranging over the positive roots. We now use Cartan's Theorem ([5], p. 227) for obtaining all real forms of a given complex semi-simple Lie algebra.

Let S be an involutive automorphism of L_u. Then S is either an inner or an outer automorphism. We first consider the case where S is an inner automorphism of L_u and let $L = L_1 \oplus L_{-1}$ be the decomposition of L with respect to the eigenvalues ± 1 of S. Then the Cartan subalgebra is contained in L_1. Now if $e_\alpha + e_{-\alpha}$ and $i(e_\alpha - e_{-\alpha}) \epsilon L_{-1}$, then ih_α, $i(e_\alpha + e_{-\alpha})$ and $-(e_\alpha - e_{-\alpha})$ belong to the real form of L^C. Since $[e_\alpha, e_{-\alpha}] = h_\alpha$, $[e_\alpha, h_\alpha] = 2e_\alpha$, $[e_{-\alpha}, h_\alpha] = -2e_{-\alpha}$, we have that $\{ih_\alpha, i(e_\alpha + e_{-\alpha}), -(e_\alpha - e_{-\alpha})\}$ is a split simple three-dimensional subalgebra of the real form of L^C, and hence the real form is not relatively complemented.

We now let S be an outer automorphism of L_u. Only five kinds of real forms are obtained by outer automorphisms (see Sugiura's paper, [9] p. 414). These are $(A_n\mathrm{I})$ $(n \geqslant 2)$, $(A\mathrm{II})$, $(D_n\mathrm{I}b)$ $(n \geqslant 3)$, $(E\mathrm{I})$ and $(E\mathrm{IV})$.

Now $(A_n\mathrm{I})$ is the normal real form of A_n, ([9], p. 397), and $(E\mathrm{I})$ is the normal real form of E_6 ([9] p. 417). Hence, both $(A_n\mathrm{I})$ and $(E\mathrm{I})$ are split and thus not relatively complemented.

Now $(A\mathrm{II})$ is also a real form of A_n, and from [9], p. 398 we see that $(A\mathrm{II})$ is $SL(m, Q)$, where Q is the field of quaternions, and $2m = n+1$. Since $SL(m, Q) \supset SL(m, R)$ and $SL(m, R)$ is split, we conclude that $(A\mathrm{II})$ is not relatively complemented.

Next we turn to $(D_n\mathrm{Ib})$ $n \geqslant 3$, and again follow Sugiura ([9] pp. 401–406). Let $0 \leqslant m \leqslant n$, m odd. Then G_m is a real form of $(D_n\mathrm{I}b)$, where

$$G_m = \left\{ \begin{pmatrix} A & B & D \\ C & -{}^tA & F \\ {}^tF & {}^tD & L \end{pmatrix} ; \begin{array}{l} A \in gl(m, R),\ B, C \in o(m),\ L \in o(p) \\ D, F \text{ are real } m \times p \text{ matrices} \end{array} \right\}$$

and $p + 2m = 2n$.
We also have that $G_m = K + P$ is a Cartan decomposition of G_m, where

$$K = \left\{ \begin{pmatrix} A & B & D \\ B & -{}^tA & -D \\ -{}^tD & {}^tD & L \end{pmatrix} ; \begin{array}{l} A, B \in o(m),\ L \in o(p) \\ D \text{ is a real } m \times p \text{ matrix.} \end{array} \right\}$$

$$P=\left\{\begin{pmatrix} A & B & D\\ -B & -A & D\\ {}^tD & {}^tD & 0\end{pmatrix};\ \begin{array}{l} A \text{ is a real symmetric } m\times m \text{ matrix},\\ B\in o(m)\\ D \text{ is a real } m\times p \text{ matrix.}\end{array}\right\}$$

We now define the following elements in G_m:

$$e_1=\begin{array}{cc} & \begin{array}{ccc} \quad m \quad & \quad m \quad & \quad p \quad \end{array}\\ \begin{array}{c} m\\ \\ m\\ \\ p\end{array} & \left(\begin{array}{c|c|c} \begin{matrix} 0 & 1 & 0 & \cdots & 0\\ -1 & 0 & \cdots & \cdots & 0\\ \vdots & & & & \\ 0 & \cdots & \cdots & \cdots & 0\end{matrix} & 0 & 0\\ \hline 0 & \begin{matrix} 0 & 1 & 0 & \cdots & 0\\ -1 & 0 & \cdots & \cdots & 0\\ \vdots & & & & \\ 0 & \cdots & \cdots & \cdots & 0\end{matrix} & 0\\ \hline 0 & 0 & 0\end{array}\right)\end{array}\in K$$

$$e_2=\begin{array}{cc} & \begin{array}{ccc} \quad m \quad & \quad m \quad & \quad p \quad \end{array}\\ \begin{array}{c} m\\ \\ m\\ \\ p\end{array} & \left(\begin{array}{c|c|c} \begin{matrix} 1 & 0 & 0 & \cdots & 0\\ 0 & -1 & 0 & \cdots & 0\\ \vdots & & & & \\ 0 & \cdots & \cdots & \cdots & 0\end{matrix} & 0 & 0\\ \hline 0 & \begin{matrix} -1 & 0 & \cdots & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ \vdots & & & & \\ 0 & \cdots & \cdots & \cdots & 0\end{matrix} & 0\\ \hline 0 & 0 & 0\end{array}\right)\end{array}\in P$$

$$e_3=\begin{array}{cc} & \begin{array}{ccc} \quad m \quad & \quad m \quad & \quad p \quad \end{array}\\ \begin{array}{c} m\\ \\ m\\ \\ p\end{array} & \left(\begin{array}{c|c|c} \begin{matrix} 0 & 1 & 0 & \cdots & 0\\ 1 & 0 & \cdots & \cdots & 0\\ \vdots & & & & \\ 0 & \cdots & \cdots & \cdots & 0\end{matrix} & 0 & 0\\ \hline 0 & \begin{matrix} 0 & -1 & 0 & \cdots & 0\\ -1 & 0 & 0 & \cdots & 0\\ \vdots & & & & \\ 0 & \cdots & \cdots & \cdots & 0\end{matrix} & 0\\ \hline 0 & 0 & 0\end{array}\right)\end{array}\in P.$$

Since $\{e_1, e_2, e_3\}$ is a split three-dimensional simple subalgebra of G_m, we conclude that G_m is not relatively complemented. Thus, $(D_n\mathrm{I}b)$, $n \geqslant 3$, is not relatively complemented.

Next, we turn to $(E\ \mathrm{IV})$. We use the results of Gantmacher, ([5] p. 246), on the chief outer automorphisms Z of E_6. In our notation E_6 is L^C, and $(E\ \mathrm{IV})$ is L. Now the roots of L are:

$$\phi_p - \phi_q,\ \pm(\phi_p+\phi_q+\phi_s),\ \pm\sum_1^6 \phi_p,\ \text{ where } p, q, s=1, 2, \cdots, 6 \quad p<q<s.$$

The vectors $e_\alpha \in L^C$ corresponding to the roots

$$\phi_p-\phi_q,\ \phi_p+\phi_q+\phi_s,\ -(\phi_p+\phi_q+\phi_s),\ \sum_1^6\phi_p,\ -\sum_1^6\phi_p$$

are denoted by

$$e_{pq},\ e_{pqs},\ e'_{pqs},\ e_o,\ e'_o.$$

Let p_1 be the index which is conjugate to p, i.e., the index which together with p forms one of the pairs (1, 2), (3, 4), (5, 6).

The chief outer automorphism Z of L^C which gives L acts on e_{pq} as follows:

$$Ze_{pq}=(-1)^{p-q+1}e_{q_1p_1}.$$

We then have:

$$\begin{aligned} Ze_{14}&=e_{32} \qquad & Ze_{32}&=e_{14} \\ Ze_{23}&=e_{41} & Zh_{14}&=-h_{23} \\ Ze_{41}&=e_{23} & Zh_{23}&=-h_{14}. \end{aligned}$$

Thus, $Z(h_{14}+h_{23})=-(h_{14}+h_{23})$.

Now Z induces an outer automorphism Z in L_u, and we have

$$\begin{aligned} &Z\{i(h_{14}+h_{23})\}=-i(h_{14}+h_{23}) \\ &Z\{(e_{14}+e_{23})+(e_{41}+e_{32})\}=(e_{14}+e_{23})+(e_{41}+e_{32}) \\ &Z\{i(e_{14}+e_{23})-i(e_{41}+e_{32})\}=i(e_{41}+e_{32})-i(e_{14}+e_{23}). \end{aligned}$$

If $L_u=L_1\oplus L_{-1}$, we have that $i(h_{14}+h_{23})$ and $i(e_{14}+e_{23})-i(e_{41}+e_{32}) \in L_{-1}$, whereas $(e_{14}+e_{23})+(e_{41}+e_{32}) \in L_1$. Using Cartan's Theorem, we conclude that the following elements $e_1, e_2, e_3 \in L$,

$$\begin{aligned} e_1&=(e_{14}+e_{23})+(e_{41}+e_{32}) \\ e_2&=-(e_{14}+e_{23})+(e_{41}+e_{32}) \\ e_3&=h_{14}+h_{23}. \end{aligned}$$

Now $\{e_1, e_2, e_3\}$ is a split simple three-dimensional subalgebra of L, and, hence, L is not relatively complemented. This completes the proof of Theorem 2.4.

LEMMA 2.2 The compact classical simple Lie algebras of types A_2 and B_2 are not relatively complemented.

PROOF: Each of these classical simple Lie algebras contains a four-dimensional subalgebra L. L then has a basis $\{e_0, e_1, e_2, e_3\}$ such that $L=\{e_1, e_2, e_3\}\oplus\{e_0\}$ where $[e_1, e_2]=e_3$; $[e_2, e_3]=\alpha e_1$; $[e_3, e_1]=\beta e_2$; $[e_i, e_0]=0$, for $i=1, 2, 3$, $\alpha\neq 0$ $\beta\neq 0$, $\in \Phi$. The result then follows from Proposition 1.2.

THEOREM 2.5 If L is a compact real simple Lie algebra of rank >1, then L is not relatively complemented.

PROOF: Lemma 2.2 implies that the compact classical simple Lie algebras of types A_n and B_n are not relatively complemented. Now since $A_2\subset G_2$ and $G_2\subset F_4\subset E_6\subset E_7\subset E_8$, we conclude that the exceptional compact Lie algebras are not relatively complemented. Moreover, since $A_3\subset D_4$ and $B_2\subset C_3$, the classical compact Lie algebras of types C_n and D_n are not relatively complemented.

THEOREM 2.6 A Lie algebra L over the field of real numbers is relatively complemented if and only if L is abelian, almost abelian or compact simple of rank one.

PROOF: The sufficiency follows from Propositions 1.1 and 1.3.

Now let L be relatively complemented. If L is solvable then L is abelian or almost abelian. If L is non-solvable, then by Levi's Theorem $L=S\oplus L_1$, where S is the radical of L and L_1 is a semisimple subalgebra of L. It then follows that $L_1$is a compact simple Lie algebra of rank one. Thus, L_1 has a basis e_1, e_2, e_3 such that $[e_1, e_2]=e_3$; $[e_2, e_3]=e_1$; $[e_3, e_1]=e_2$. We now show that $S=0$.

Since S is solvable and relatively complemented, it is abelian or almost abelian. If S is abelian, let $x\in L_1$, $x\notin S$. Note that $\{x, S\}$ is a subalgebra of L and consider $\mathrm{ad}\, x\,|_S$. From Lemma 2.1 it follows that $\mathrm{ad}\, x\,|_S$ is a diagonal transformation. For $x\in L_1$, $x\notin S$, define $\phi(x)=\mathrm{ad}\, x\,|_S$. Then ϕ is a representation of L_1, and since L_1 is compact all the eigenvalues of ϕ are pure imaginary and it then follows that $\phi=0$, which implies that $S=$center of L.

If $S\neq 0$, let $e_4\neq 0$, $\in S$, and consider the subalgebra $B=\{e_1, e_2, e_3\}\oplus\{e_4\}$ of L. By Proposition 1.2, B is not relatively complemented. Hence, $S=0$.

Now suppose S is almost abelian. Let N be the nil radical of S and consider $N\oplus L_1$. It then follows that $N\oplus L_1$ is a subalgebra of L and N is its radical. Thus, N is abelian and we can then apply the preceding proof to conclude that $N=0$, a contradiction. This completes the proof of Theorem 2.6.

COROLLARY 2.4 Let L be a Lie algebra over an algebraically closed field or the field of real numbers. L is relatively complemented if and only if L is upper semi-modular.

PROOF: It is shown in [8] (Theorem 2.4) that L is upper semi-modular if and only if L is abelian, almost abelian or special simple ([8], p. 152).

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