J. Sci. Hiroshima Univ. Ser. A-I
31 (1967), 173-190

On Fixations and Reciprocal Images of Currents

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(Received September 20, 1967)

Let Ω be a non-empty open subset of an N-dimensional Euclidean space R^N. The investigations have been made in our previous paper [1] about the multiplication between distributions defined on Ω. The multiplicative product of S, $T \in \mathcal{D}'(\Omega)$ is the section of $S(x) \otimes T(x-y)$ for $y=0$, if it exists, which will be denoted by $S \cdot T$ instead of $S \bigcirc T$ throughout this paper. $S \cdot T$ will then be in a certain sense the section of $S(x) \otimes T(y)$ for $x=y$. In this paper a distribution is understood as a current of degree 0 and of even kind.

Our main purpose of this paper is to introduce the notion of the section of a current on a submanifold so as to make it possible to generalize the multiplicative product of distributions to the exterior product of currents. We consider here two kinds of sections; one in a narrow sense, and the other in a wider sense. Accordingly we may discuss the exterior product of currents in either sense. Owing to these notions we can give an approach to define a reciprocal image of a current under a C^∞ map. Of course, a C^∞ map need not admit a reciprocal image of every current. A detailed discussion thereof confined to distributions was given in [2], where we introduced the concept of "admissible map". The section of a current on a submanifold $M_0 \subset M$ will be, as we shall see in this paper, the reciprocal image of the current under the injection $j: M_0 \to M$. This leads us to the study of Stokes' formula for currents, an attempt to generalize the formula $\int_a^b S'(x)dx = S(b) - S(a)$, where S is a one-dimensional distribution with values at a and b.

In what follows we shall call a current of even kind simply a current whenever no confusion may occur, however, we shall underline the letter denoting a current of odd kind. We note that a current on $\Omega \subset R^N$ is a form whose coefficients are distributions on Ω.

The presentation of the material is arranged as follows: In Section 1 we shall introduce the notion of the section of a current defined on $\Omega \subset R^N$ and show that it is invariant under diffeomorphisms. In Section 2 we shall study the section of a current on a submanifold $M_0 \subset M$ and the exterior product between currents of any kind. We shall consider, in Section 3, the reciprocal image of a given current $T \in \overset{p}{\mathcal{D}}'(M)$, which we define as follows: Let N' and N be the dimensions of manifolds M' and M respectively. For a C^∞ map ξ of M' into M, the direct image $\xi \underline{\beta}$ of every $\underline{\beta}$ of $\overset{N'-p}{\underline{\mathcal{D}}}(M')$ is an odd $(N-p)$-current.

If the exterior product $\xi\beta\wedge T$ exists for every β, we can show that the linear map $\beta\to\int\xi\beta\wedge T$ is continuous, then the current ξ^*T determined by the equation $<\beta, \xi^*T>=\int\xi\beta\wedge T$ is called the reciprocal image of T under the map ξ. The same is true of odd currents, if the map ξ is oriented. Taking ξ for the injection j of a submanifold M_0 into M, we show that j^*T exists if and only if the section $T|M_0$ of T on M_0 exists, and that if this is the case, $j^*T=T|M_0$. Stokes' formula is shown. In Section 4 we show that the trace map on a submanifold M_0 coincides with the fixation to M_0 for a space of currents with certain conditions. The final section is devoted to some considerations about an admissible map, which is defined as the map admitting a reciprocal image of every current, and the section closes with some statements refining the results of [2].

1. The section of a current defined on an open subset of R^N

Let Ω be a non-empty open subset of $R^N=R^n\times R^m$. A point of $R^n_x\times R^m_y$ will be denoted by (x, y), where $x=(x_1, \cdots, x_n)$ and $y=(y_1, \cdots, y_m)$. Let

$$\Omega_{x_0}=\{y;\ (x_0, y)\in\Omega\},$$

where we suppose $\Omega_{x_0}\neq\emptyset$. We shall often use the symbol $T(x, y)$ for a distribution $T\in\mathcal{D}'(\Omega)$.

If there exists a distribution $S\in\mathcal{D}'(\Omega_{x_0})$ such that

$$(*)\qquad \lim_{\lambda\to+0}T(x_0+\lambda x,y)=S(y)\qquad (=1_x\otimes S(y)\text{ more precisely}),$$

namely

$$\lim_{\lambda\to+0}<T,\ \frac{1}{\lambda^n}\phi\Big(\frac{x-x_0}{\lambda}\Big)\psi(y)>\ =\ <S,\psi>\int\phi(x)dx$$

for any $\phi\in\mathcal{D}(R^n)$, $\psi\in\mathcal{D}(\Omega_{x_0})$, then according to S. Łojasiewicz [3, p. 15] we shall say that $x=x_0$ can be fixed in $T(x, y)$ and that S is the section of T for $x=x_0$ with notation $T(x_0, y)$.

Recently R. Shiraishi has shown in [6, p. 91] that the condition $(*)$ is equivalent to

$$\lim_{k\to\infty}<T(x,\ y),\ \rho_k(x-x_0)>\ =S(y)$$

for every restricted δ-sequence $\{\rho_k\}$ in $\mathcal{D}(R^n)$, that is, every sequence of non-negative functions $\rho_k\in\mathcal{D}(R^n)$ with the following conditions:

(i) $\operatorname{Supp}\rho_k$ converges to $\{0\}$ as $k\to\infty$.

(ii) $\int\rho_k(x)dx$ converges to 1 as $k\to\infty$.

(iii) $\int |x|^{|p|} |D^p \rho_k(x)| dx \leqq K_p$, a constant independent of k.

We note that a sequence $\{\rho_k\}$ satisfying the conditions (i) and (ii), is called a δ-sequence. For simplicity we assume $x_0 = 0$.

Consider a diffeomorphism

$$\chi: \begin{cases} x' = \xi(x, y), & \xi = (\xi_1, \cdots, \xi_n) \\ y' = \eta(x, y), & \eta = (\eta_1, \cdots, \eta_m) \end{cases}$$

of Ω onto an open subset $\Omega' \subset R^n_{x'} \times R^m_{y'}$, which refers $x=0$ in Ω to $x'=0$ in Ω'. The Jacobian of the map χ will be denoted by J_χ. For any $T \in \mathcal{D}'(\Omega)$ and $S \in \mathcal{D}'(\Omega_0)$ we define $T' \in \mathcal{D}'(\Omega')$ and $S' \in \mathcal{D}(\Omega'_0)$ as follows:

$$\begin{aligned} &< T'(x', y'), \phi(x', y')> \\ &\qquad = < T(x, y), |J_\chi(x, y)| \phi(\xi(x, y), \eta(x, y))>, \qquad \phi \in \mathcal{D}(\Omega') \end{aligned}$$

and

$$< S'(y'), \psi(y')> = < S(y), |J_{\eta_0}| \psi(\eta_0(y))>, \qquad \psi \in \mathcal{D}(\Omega'_0),$$

where $\eta_0(y) = \eta(0, y)$ is the diffeomorphism of Ω_0 onto Ω'_0. We shall first show the following

Lemma 1. *Let $T \in \mathcal{D}'(\Omega)$ and let k be a real number. If there exists a distribution $S \in \mathcal{D}'(\Omega_0)$ such that*

$$\lim_{\lambda \to +0} \lambda^k T(\lambda x, y) = S(y),$$

then $\lim_{\lambda \to +0} \lambda^k T'(\lambda x', y')$ exists and is equal to $S'(y')$.

Proof. It is sufficient to show that

$$\lim_{\lambda \to +0} < \lambda^k T'(\lambda x', y'), \phi_1(x')\phi_2(y')> = < S'(y'), \phi_2(y')> \int \phi_1(x') dx'$$

for any $\phi_1 \in \mathcal{D}(R^n)$ and $\phi_2 \in \mathcal{D}(\Omega'_0)$.

Since J_χ does not vanish and $\xi(0, y) \equiv 0$, we must have $\left.\dfrac{d\xi}{dx} \dfrac{d\eta}{dy}\right|_{x=0} \neq 0$, where $\dfrac{d\xi}{dx}$ and $\dfrac{d\eta}{dy}$ stand for the Jacobians $\dfrac{\partial(\xi_1, \cdots, \xi_n)}{\partial(x_1, \cdots, x_n)}$ and $\dfrac{\partial(\eta_1, \cdots, \eta_m)}{\partial(y_1, \cdots, y_m)}$ respectively. For any given compact set $K \subset \Omega_0$, $\dfrac{d\xi}{dx}$ does not vanish for $y \in K$ and for sufficiently small $|x|$. We can therefore find positive constants c, c_1 and ε satisfying the condition:

(i) $\qquad c|x| \leqq |\xi(x, y)| \leqq c_1|x|, \quad (x, y) \in B_\varepsilon \times K \subset \Omega,$

where B_ε stands for the ball in R^n with center 0 and radius ε. Now we have for sufficiently small λ

$$<\lambda^k T'(\lambda x', y'), \phi_1(x')\phi_2(y')> = <\lambda^k T'(x', y'), \frac{1}{\lambda^n}\phi_1\Big(\frac{x'}{\lambda}\Big)\phi_2(y')>$$

$$= <\lambda^k T(x, y), \frac{1}{\lambda^n}|J_X(x, y)|\phi_1\Big(\frac{\xi(x, y)}{\lambda}\Big)\phi_2(\eta(x, y))>$$

$$= <\lambda^k |J_X(\lambda x, y)|\, T(\lambda x, y), \phi_1\Big(\frac{\xi(\lambda x, y)}{\lambda}\Big)\phi_2(\eta(\lambda x, y))>.$$

If we put $\psi_\lambda=\phi_1\Big(\frac{\xi(\lambda x, y)}{\lambda}\Big)\phi_2(\eta(\lambda x, y))$, then $\{\psi_\lambda\}$ will be uniformly bounded in $\mathcal{D}(R^n\times\Omega_0)$ for sufficiently small λ. Indeed, let K' be any compact subset of Ω_0' such that

(ii) $$\eta_0^{-1}(K')\subset K^0\subset K.$$

We choose a positive constant δ so that

(iii) $$\chi^{-1}(B_\delta\times K')\subset B_\varepsilon\times K.$$

Let $\phi_1\in\mathcal{D}_{B_a}$, $a>0$, and $\phi_2\in\mathcal{D}_{K'}$. It then follows from these properties (i), (ii) and (iii) that $\operatorname{supp}\psi_\lambda(x, y)$, $0<\lambda\leqq\frac{\delta}{a}$, must be contained in a fixed compact set. In view of the fact that $\xi(0, y)\equiv 0$, we see that $|D_y^p\xi_j(x, y)|=O(|x|)$ uniformly for $y\in K$ as $|x|\to 0$ and so $|D_y^p\xi_j(\lambda x, y)|=O(\lambda|x|)$, whence the set $\{D_x^pD_y^q\psi_\lambda\}_\lambda$ is uniformly bounded for $0<\lambda\leqq\frac{\delta}{a}$. Therefore $\{\psi_\lambda\}_{0<\lambda\leqq\frac{\delta}{a}}$ is bounded in $\mathcal{D}(R^n)\times\mathcal{D}(\Omega_0)$.

Thus we have

$$\lim_{\lambda\to+0}<\lambda^k|J_X(\lambda x, y)|\,T(\lambda x, y), \psi_\lambda(x, y)>$$

$$=<|J_X(0, y)|\,S(y), \phi_2(\eta(0, y))\int\phi_1\Big(\sum_j\frac{\partial\xi}{\partial x_j}(0, y)x_j\Big)dx>$$

and

$$\int\phi_1\Big(\sum_j\frac{\partial\xi}{\partial x_j}(0, y)x_j\Big)dx=\int\phi_1(x')\frac{1}{|\Delta(y)|}dx',$$

where $\Delta(y)$ is the Jacobian of the map $x\to x'=\sum_j\frac{\partial\xi}{\partial x_j}(0, y)x_j$.

Since $\xi_j(0, y)\equiv 0$ as already remarked, we obtain

$$J_X(0, y)=J_{\eta_0}(y)\Delta(y).$$

Thus we have

$$\lim_{\lambda\to+0} < \lambda^k T'(\lambda x', y'), \phi_1(x')\phi_2(y') > = < S(y), |J_{\eta_0}(y)| \phi_2(\eta_0(y)) > \int \phi_1(x')dx'$$

$$= < S'(y'), \phi_2(y') > \int \phi_1(x')dx',$$

which completes the proof.

Let δ be the Dirac measure concentrated at origin and let $\phi \in \mathcal{D}(R^n)$ with $\phi \geqq 0$ and $\int \phi dx = 1$. We put $\phi_\lambda(x) = \frac{1}{\lambda^n}\phi\left(\frac{x}{\lambda}\right)$. For our purpose later on we shall show

Lemma 2. *For any real number k, the condition*

$$\lim_{\lambda\to+0} \lambda^k T(\lambda x, y) = S(y)$$

is equivalent to

$$\lim_{\lambda\to+0} \lambda^k \delta(x+\lambda u) T(x, y) = \delta_x \otimes S(y)$$

or

$$\lim_{\lambda\to+0} \lambda^k \check{\phi}_\lambda(x) T(x, y) = \delta_x \otimes S(y).$$

Proof. It is clear that the last two conditions are equivalent. Now, suppose that $\lim_{\lambda\to+0} \lambda^k T(\lambda x, y)$ exists and equals $S(y)$. If $\psi_1(x) \in \mathcal{D}(R^n)$ and $\psi_2(y) \in \mathcal{D}(\Omega_0)$, then since we can write for sufficiently small λ

$$< \lambda^k \check{\phi}_\lambda(x) T(x, y), \psi_1(x)\psi_2(y) > = < \lambda^k T(x, y), \check{\phi}_\lambda(x)\psi_1(x)\psi_2(y) >$$

$$= < \lambda^k T(\lambda x, y), \check{\phi}(x)\psi_1(0)\psi_2(y) > + < \lambda^k T(\lambda x, y), \check{\phi}(x)(\psi_1(\lambda x) - \psi_1(0))\psi_2(y) >,$$

it follows that

$$\lim_{\lambda\to+0} < \lambda^k \check{\phi}_\lambda(x) T(x, y), \psi_1(x)\psi_2(y) > = < 1_x \otimes S(y), \check{\phi}(x)\psi_1(0)\psi_2(y) >$$

$$= \psi_1(0) < S(y), \psi_2(y) >.$$

This implies that the limit $\lim_{\lambda\to+0} \lambda^k \check{\phi}_\lambda(x) T(x, y)$ exists and equals $\delta_x \otimes S(y)$.

Conversely, suppose that $\lim_{\lambda\to+0} \lambda^k \check{\phi}_\lambda(x) T(x, y)$ exists and equals $\delta_x \otimes S(y)$. If we take $\psi_1(x) \in \mathcal{D}(R^n)$ to be 1 near the origin, then we have for any $\psi_2(y) \in \mathcal{D}(\Omega_0)$

$$\lim_{\lambda\to+0} < \lambda^k T(\lambda x, y), \phi(x)\psi_2(y) > = \lim_{\lambda\to+0} < \lambda^k T(x, y), \phi_\lambda(x)\psi_2(y) >$$

$$= \lim_{\lambda\to+0} < \lambda^k \phi_\lambda T(x, y), \psi_1(x)\psi_2(y) >$$

$$= < \delta_x \otimes S(y), \psi_1(x)\psi_2(y) >$$

$$= < S(y), \psi_2(y) >,$$

which completes the proof.

Now, let $\overset{p}{T}$, $0 \leqq p \leqq N$, be a p-current defined on $\Omega \subset R^N = R^n_x \times R^m_y$, which is understood as a form with distributional coefficients:

$$\overset{p}{T}(x, y) = \sum_{I,K} T_{I,K}\, dx_I \wedge dy_K, \qquad T_{I,K} \in \mathcal{D}'(\Omega),$$

where $I = \{i_1, \cdots, i_s\}$ and $K = \{k_1, \cdots, k_t\}$ with $s+t=p$ are strictly increasing multi-indices between 1 and n and between 1 and m respectively and

$$dx_I \wedge dy_K = dx_{i_1} \wedge \cdots \wedge dx_{i_s} \wedge dy_{k_1} \wedge \cdots \wedge dy_{k_t}.$$

Furthermore we shall write $T_K = T_{I,K}$ for $|K| = p$. We have for any positive real number λ

$$\overset{p}{T}(\lambda x, y) = \sum_{I,K} \lambda^{|I|} T_{I,K}(\lambda x, y)\, dx_I \wedge dy_K,$$

where $|I|$ stands for the number of the components of I.

DEFINITION 1. Let $\overset{p}{T}$ be a p-current on $\Omega \subset R^n \times R^m$. If the limit $\lim_{\lambda \to +0} T(\lambda x, y)$ exists and does not depend on x, then we say that $x=0$ can be fixed in $T(x, y)$ and that the limit is the *section* of T for $x=0$ with notation $T(0, y)$.

This definition means that the distributional limits

$$\lim_{\lambda \to +0} T_{I,K}(\lambda x, y) = T_K(0, y) \qquad \text{for} \quad |I| = 0,$$

$$\lim_{\lambda \to +0} \lambda^s T_{I,K}(\lambda x, y) = 0 \qquad \text{for} \quad |I| = s > 0$$

exist and

$$\overset{p}{T}(0, y) = \sum_K T_K(0, y)\, dy_K.$$

If T happens to be a distribution on Ω, that is, $p=0$, then the definition gives rise to that of the section of T for $x=0$.

When every $T_{I,K}$ has the section for $x=0$, then T has clearly the section $T(0, y)$. If this is the case, we shall call $T(0, y)$ the section of T in a narrow sense for $x=0$.

Let Ω, Ω' and $\chi = (\xi, \eta)$ be the same as before. Then the direct image $\chi T = \tilde{T}$ is represented by

$$\sum_{J,L} \tilde{T}'_{J,L}(x', y')\, dx'_J \wedge dy'_L, \quad \text{where} \quad \tilde{T}'_{J,L} = \sum_{I,K} T'_{I,K}(x', y') \frac{\partial(x_I, y_K)}{\partial(x'_J, y'_L)}.$$

$\tilde{T}$ is also the reciprocal image of T for the inverse map χ^{-1}. Let S be a current on Ω_0 and let $y' = \eta_0(y) = \eta(0, y)$. In a similar way the direct image

$\eta_0 S=\tilde{S}$ is represented by

$$\sum_L \tilde{S}'_L(y')dy'_L, \quad \text{where} \quad \tilde{S}'_L(y')=\sum_K S'_K(y')\frac{\partial \eta_{0K}^{-1}}{\partial y'_L}.$$

THEOREM 1. *If a current* $\overset{p}{T}$ *on* $\Omega\subset R^n_x\times R^m_y$ *has the section* $\overset{p}{S}$ *for* $x=0$, *then the direct image* $\chi\overset{p}{T}=\overset{p}{\tilde{T}}$ *also has the section* $\eta_0\overset{p}{S}=\overset{p}{\tilde{S}}$ *for* $x'=0$.

PROOF. Let $S=\sum_K S_K(y)dy_K$, $S_K(y)=\lim_{\lambda\to+0} T_{I,K}(\lambda x, y)$ for $|I|=0$. By Lemma 1 $\lim_{\lambda\to+0} T'_{I,K}(\lambda x', y')$ exists for $|I|=0$ and equals S'_K and $\lim_{\lambda\to+0}\lambda^{|I|}\times T'_{I,K}(\lambda x', y')=0$ for $|I|>0$. Put $a_{I,K,J,L}(x',y')=\dfrac{\partial(x_I, y_K)}{\partial(x'_J, y'_L)}$. Since $\xi(0, y)\equiv 0$, it follows that

$$|a_{I,K,J,L}(\lambda x', y')|=\begin{cases} O(\lambda^{|I|-|J|}) & \text{for} \quad |I|>|J| \\ O(1) & \text{for} \quad |I|\leqq|J| \end{cases}$$

as $\lambda\to+0$. Thus we have

$$\lim_{\lambda\to+0}\lambda^{|J|}\tilde{T}'_{J,L}(\lambda x', y')=\begin{cases} \sum_{|I|=0,K} T'_{I,K}(0, y')\dfrac{\partial y_K}{\partial y'_L}\Big|_{x'=0} & \text{for} \quad |J|=0 \\ 0 & \text{for} \quad |J|>0 \end{cases}$$

and again by Lemma 1 we have

$$\begin{aligned}\lim_{\lambda\to+0}\tilde{T}(\lambda x', y') &= \sum_L\Big(\sum_{|I|=0,K} T'_{I,K}(0, y')\frac{\partial y_K}{\partial y'_L}\Big|_{x'=0}\Big)dy'_L \\ &= \sum_L\sum_K S'_K(y')\frac{\partial \eta_{0K}^{-1}}{\partial y'_L}\,dy'_L \\ &= \sum_L \tilde{S}'_L(y')dy'_L=\tilde{S}(y'),\end{aligned}$$

which completes the proof.

For a current T on Ω, we shall define the section $T(0, y)$ to be the sum of the sections of the homogeneous components of T whenever they exist.

2. The section of a current on a submanifold

Let M be a manifold of dimension N. In what follows we always understand a manifold to be a differentiable manifold denumerable at infinity [4]. Let $\mathcal{D}(M)$ stand for the space of even C^∞ forms on M with compact support, equipped with the usual topology, and $\overset{p}{\mathcal{D}}(M)$ the subspace of p-forms $\in\mathcal{D}(M)$. $\underline{\mathcal{D}}(M)$ is the space of odd C^∞ forms with compact support. The spaces $\mathcal{D}'(M)$, $\overset{p}{\mathcal{D}}{}'(M)$, $\underline{\mathcal{D}}'(M)$ and $\overset{p}{\underline{\mathcal{D}}}{}'(M)$ are defined as the strong duals of $\underline{\mathcal{D}}(M)$, $\overset{N-p}{\underline{\mathcal{D}}}(M)$,

$\mathcal{D}(M)$ and $\overset{N-p}{\mathcal{D}}(M)$ respectively. We shall denote by $\mathcal{E}(M)$ the space of even C^∞ forms with the usual topology and by $\underline{\mathcal{E}}'(M)$ the strong dual of $\mathcal{E}(M)$, which consists of the odd currents $\in \underline{\mathcal{D}}'(M)$ with compact support. The same is true of $\underline{\mathcal{E}}(M)$ and $\mathcal{E}'(M)$.

Let $\{\kappa\}$ be a complete family of coordinate systems in M, where κ is a homeomorphism of an open set $V_\kappa \subset M$ onto an open set $\tilde{V}_\kappa \subset R^N$, and the map

$$\kappa\kappa'^{-1}\colon \kappa'(V_\kappa \cap V_{\kappa'}) \to \kappa(V_\kappa \cap V_{\kappa'})$$

is a diffeomorphism for any κ, κ'. Let $T \in \mathcal{D}'(M)$. To every κ there is associated a current $T_{\tilde{V}_\kappa}$ on $\tilde{V}_\kappa$ such that $T_{\tilde{V}_\kappa} = \kappa\kappa'^{-1} T_{\tilde{V}_{\kappa'}}$ in $\kappa(V_\kappa \cap V_{\kappa'})$ and we can identify T with such a system as $\{T_{\tilde{V}_\kappa}\}$. Similar considerations hold true of an odd current $\underline{T}$. We consider a distribution on M as an even 0-current on M, or, what is the same, an element of $\overset{\circ}{\mathcal{D}}'(M)$.

Let M_0 be a submanifold of dimension $m < N$. Then to every $a \in M_0$ there is associated a coordinate system $\kappa = \{x_1, \cdots, x_n, y_1, \cdots, y_m\}$, $n+m=N$, which is valid on an open neighbourhood V_κ of a point a in M such that $x_1(a) = \cdots = x_n(a) = y_1(a) = \cdots = y_m(a) = 0$ and such that the restriction κ_0 of κ to

$$U_\kappa = V_\kappa \cap M_0 = \{b \in V_\kappa;\ x_1(b) = \cdots = x_n(b) = 0\}$$

forms a coordinate system in M_0. We have $\tilde{V}_\kappa = \{(x_1(b), \cdots, x_n(b), y_1(b), \cdots, y_m(b));\ b \in V_\kappa\}$, and $\tilde{U}_\kappa = \{(y_1(b), \cdots, y_m(b));\ b \in U_\kappa\}$.

If every $T_{\tilde{V}_\kappa}$ has the section $S_{\tilde{U}_\kappa}$ on $\tilde{U}_\kappa$, there exists a unique current $S \in \mathcal{D}'(M_0)$ determined by the system $\{S_{\tilde{U}_\kappa}\}$. This is an immediate consequence of Theorem 1. The consideration holds also true of the section in a narrow sense. If T is of degree p, then so is S. Then we can introduce

DEFINITION 2. Let $T \in \mathcal{D}'(M)$. If $T_{\tilde{V}_\kappa}$ has the section (resp. in a narrow sense) on $\tilde{U}_\kappa$ for every V_κ, the uniquely determined current $S \in \mathcal{D}'(M_0)$ is called the *section* of T (resp. in a narrow sense) on the submanifold M_0 and denoted by $T \mid M_0$.

As an application of the notion of the section of a current we can deal with an exterior product of two homogeneous currents $\overset{p}{S}$, $\overset{q}{T} \in \mathcal{D}'(M)$. Owing to the principle of localization, it suffices to define an exterior product in a coordinate neighbourhood V of every $a \in M$. Let $\overset{p}{S}_{\tilde{V}}$ and $\overset{q}{T}_{\tilde{V}}$ be written in the form

$$\overset{p}{S}_{\tilde{V}} = \textstyle\sum_I S_I(x)dx_I, \quad \overset{q}{T}_{\tilde{V}} = \sum_K T_K(x)dx_K, \quad S_I, T_K \in \overset{\circ}{\mathcal{D}}'(\tilde{V}).$$

We shall consider the current

$$\overset{p}{S}_{\tilde{V}} \otimes \overset{q}{T}_{\tilde{V}} = \textstyle\sum_{I,K} S_I(x) T_K(z) dx_I \wedge dz_K \qquad \text{in} \quad \tilde{V} \times \tilde{V},$$

where $S_I(x)T_K(z)$ denotes the multiplicative products [2, p. 78].

If $\overset{p}{S}_{\tilde V}\otimes\overset{q}{T}_{\tilde V}$ has the section to the diagonal $\varDelta_{\tilde V}$ of $\tilde V\times\tilde V$ for every V, then the system of the sections $\{(S_{\tilde V}\otimes T_{\tilde V})|\varDelta_{\tilde V}\}$ defines the current W on $\varDelta$, the diagonal of $M\times M$. The map $j\colon M\ni a\rightarrow(a, a)\in\varDelta$ is a diffeomorphism. The reciprocal image j^*W will be termed the exterior product of S and T with notation $S\wedge T$, a $(p+q)$-current.

From this definition it follows that

(1) If $S\wedge T$ exists, then so does $T\wedge S$ and we have $S\wedge T=(-1)^{pq}(T\wedge S)$.

(2) If $S\wedge T$ exists, then so do $(\alpha S)\wedge T$, $S\wedge(\alpha T)$ for every $\alpha\in C^\infty(M)$, and we have $\alpha(S\wedge T)=(\alpha S)\wedge T=S\wedge(\alpha T)$. If S and T are distributions on M, the definition is tantamount to that of the multiplicative product $S\cdot T$ given in [1, p. 165].

When S_IT_K exists for every I, K and V, it is clear from our definition that the exterior product $S\wedge T$ is well defined, and we can write

$$(S\wedge T)_{\tilde V}=\sum_{I,K}S_I(x)T_K(x)dx_I\wedge dx_K.$$

If this is the case, we shall say that the exterior product of S and T exists in a narrow sense.

We know that on an oriented manifold every odd current is associated with an even current in a natural way. On the other hand, every coordinate neighbourhood V_κ is supposed to be oriented according to the natural ordering of coordinates in κ. To every odd current $\underline{S}$ there is associated a system of currents $\underline{S}_{\tilde V_\kappa}$ such that

$$\underline{S}_{\tilde V_\kappa}=\sum_I S_I(x)dx_I,$$

but with the rules of transformations:

$$\underline{S}_{\tilde V_{\kappa'}}(x')=\frac{J_{\kappa\kappa'^{-1}}}{|J_{\kappa\kappa'^{-1}}|}\sum_{I,J}\underline{S}'_I(x')\frac{\partial x_I}{\partial x'_J}dx'_J\qquad\text{in}\quad\kappa'(V_\kappa\cap V_{\kappa'}).$$

This observation leads us to the definition of the exterior products between currents of any kind. For example, let us consider two currents $\underline{S}$ and T on M. If $W_\kappa=\underline{S}_{\tilde V_\kappa}\wedge T_{\tilde V_\kappa}$ exists for every κ, we can see that $\{W_\kappa\}$ uniquely determines an odd current $\underline{W}$, a fact which is verified straight forward. Then we call $\underline{W}$ the exterior product $\underline{S}\wedge T$ of $\underline{S}$ and T. The parity of the exterior product obeys to the usual law for the exterior multiplication when one of the factors is a C^∞ form.

Now we turn to the consideration about the section of an odd current $\underline{T}\in\mathcal{D}'(M)$ on a submanifold M_0, where the injection $j\colon M_0\rightarrow M$ is supposed to be oriented. We shall continue to use the notations as before. The map j assigns to the canonical orientation of U_κ a fixed orientation of V_κ in each point of U_κ, which may or may not coincide with the canonical orientation of V_κ and accordingly we define $\varepsilon(p)$, $p\in U_\kappa$, to be 1 or -1. Taking this into account, if the section $S_{\tilde U_\kappa}$ of $T_{\tilde V_\kappa}$ for $x=0$ exists for every κ, we can conclude that $\{\varepsilon S_{\tilde{U}_\kappa}\}$ uniquely determines an odd current $\underline{S}$ on M_0, which we shall call the section of $\underline{T}$ on M_0 and denote it by $\underline{T}|M_0$.

The same is true of the section in a narrow sense.

3. Sections and reciprocal images

Consider a C^∞ map ξ of a manifold M' of N'-dimension into a manifold M of dimension N. The reciprocal image $\xi^*\alpha$, $\alpha \in \overset{p}{\mathcal{D}}(M)$, belongs to $\overset{p}{\mathcal{E}}(M')$. Then the integral

$$\int \underline{\beta} \wedge \xi^*\alpha, \quad \text{where} \quad \underline{\beta} \in \overset{N'-p}{\underline{\mathcal{D}}}(M'),$$

defines a continuous linear form on $\overset{p}{\mathcal{D}}(M)$, and in turn an odd current $\xi\underline{\beta}$ of degree $N-p$ which is called the direct image of $\xi\underline{\beta}$.

Now consider a current $T \in \overset{p}{\mathcal{D}}{}'(M)$. If $\xi\underline{\beta} \wedge T$ exists for every $\underline{\beta} \in \overset{N'-p}{\underline{\mathcal{D}}}(M')$, the linear map

$$\underline{\beta} \to \int \xi\underline{\beta} \wedge T$$

will be continuous. Indeed, it is enough to show the assertion when M', M are open subsets Ω', Ω of Euclidean spaces of dimension N' and of dimension N. In this case we may write $\xi\underline{\beta}$ and T in the following forms:

$$\xi\underline{\beta} = \sum_I S_I(x)dx_I, \qquad S_I \in \overset{\circ}{\mathcal{E}}{}'(\Omega),$$

$$T = \sum_K T_K(x)dx_K, \quad T_K \in \overset{\circ}{\mathcal{D}}{}'(\Omega),$$

and therefore

$$\xi\underline{\beta} \wedge T = \left(\sum_I (-1)^{\rho(I,CI)} S_I(x) T_{CI}(x)\right) dx,$$

where $(-1)^{\rho(I,K)}$ denotes the signature of the permutation $\{I, K\}$ of $\{1, 2, \cdots, N\}$, and we used the notation $\sum_I (-1)^{\rho(I,CI)} S_I(x) T_{CI}(x)$ for the abbreviation of $\lim_{\lambda \to +0} \sum_I (-1)^{\rho(I,CI)} S_I(x) T_{CI}(x+\lambda u)$. By making use of a restricted δ-sequence $\{\rho_k\}$, we obtain

$$\xi\underline{\beta} \wedge T = \lim_{k\to\infty} \sum_I (-1)^{\rho(I,CI)} S_I(T_{CI} * \rho_k) dx,$$

so we can conclude the assertion in virtue of the Banach-Steinhaus theorem.

DEFINITION 3. Given $T \in \overset{p}{\mathcal{D}}{}'(M)$, if $\xi\underline{\beta} \wedge T$ exists for every $\underline{\beta} \in \overset{N'-p}{\underline{\mathcal{D}}}(M')$, the current $\xi^* T$ determined by the equation

$$<\underline{\beta}, \xi^* T> = \int \xi\underline{\beta} \wedge T$$

is called the *reciprocal image* of T under the map ξ.

We note that if ξ^*T exists for every $T \in \overset{p}{\mathcal{D}}'(M')$, then $\xi\beta$ is an odd $(N-p)$-form. This follows from the fact that a distribution on Ω which admits the multiplicative product with every distribution on Ω must belong to $\mathcal{E}(\Omega)$ [1, p. 166].

Now, let us consider a special case in which M' is a submanifold M_0 of M as in the preceding section. Let j: $M_0 \to M$ be the injection, which is a C^∞ map. Then we can show

THEOREM 2. *Given* $T \in \overset{p}{\mathcal{D}}'(M)$, $0 \leqq p \leqq m$, *the reciprocal image* j^*T *exists if and only if the section* $T|M_0$ *exists. And if this is the case, we have* $j^*T = T|M_0$.

PROOF. We shall continue to use the notations as before. For any $\alpha \in \overset{p}{\mathcal{D}}(M)$ and $\beta \in \overset{m-p}{\mathcal{D}}(M_0)$ with support $\subset\subset U_\kappa$, it is easy to verify the relation:

$$\int \beta \wedge j^*\alpha = \int_{\tilde{U}_\kappa} \beta_{\tilde{U}_\kappa} \wedge (j^*\alpha)_{\tilde{U}_\kappa} = \int_{\tilde{V}_\kappa} (\delta(x)dx \wedge \beta_{\tilde{V}_\kappa}) \wedge \alpha_{\tilde{V}_\kappa},$$

which implies that

$$(j\beta)_{\tilde{V}_\kappa} = \delta(x)dx \wedge \beta_{\tilde{U}_\kappa}.$$

Suppose j^*T exist, then, since the exterior product $(j\beta)_{\tilde{V}_\kappa} \wedge T_{\tilde{V}_\kappa}$ exists for any β, it follows that $(\delta(x)dx \wedge dy_J) \wedge T_{\tilde{V}_\kappa}$ must exist for any J with $|J| = m-p$. Putting $T_{\tilde{V}_\kappa} = \sum_{I,K} T_{I,K}(x, y)dx_I \wedge dy_K$, $|I| + |K| = p$, we can write

$$\begin{aligned}
&(\delta(x)dx \wedge dy_J) \wedge T_{\tilde{V}_\kappa} \\
&\quad = \lim_{\lambda \to +0} \textstyle\sum_{I,K} \delta(x+\lambda u) T_{I,K}(x, y) d(x+\lambda u) \wedge d(y+\lambda v)_J \wedge dx_I \wedge dy_K \\
&\quad = \lim_{\lambda \to +0} \textstyle\sum (-1)^{\rho(CL,L)} \lambda^{|L|} \delta(x+\lambda u) T_{I,K}(x, y) dx_{CL} \wedge du_L \wedge d(y+\lambda v)_J \wedge dx_I \wedge dy_K.
\end{aligned}$$

We can conclude from these equalities that

$$\lim_{\lambda \to +0} \lambda^{|I|} \delta(x+\lambda u) T_{I,K}(x, y)$$

exists for every $T_{I,K}$, and in addition if $|I| > 0$, the limit is 0. Indeed, choose $J = CK$ for any K with $|K| = p$, then it is easy to see that the assertion is true of $|K| = p$, and

$$\lim_{\lambda \to +0} \sum_{|K| \leqq p-1} (-1)^{\rho(CL,L)} \lambda^{|L|} \delta(x+\lambda u) T_{I,K}(x, y) dx_{CL} \wedge du_L \wedge d(y+\lambda v)_J \wedge dx_I \wedge dy_K$$

exists. Then a similar argument can be applied to obtain the results for the case $|K| = p-1$ when $p \geqq 1$. The repeated use of this procedure will lead us to the conclusion. It then follows from Lemma 2 that the section $T_{\tilde{V}_\kappa}|\tilde{U}_\kappa$ exists.

Conversely, let us assume that the section $T|M_0$ exists. This implies that if we write $T_{\tilde{V}_\kappa}=\sum_{I,K}T_{I,K}(x, y)dx_I\wedge dy_K$, then $\lim_{\lambda\to+0}\lambda^{|I|}T_{I,K}(\lambda x, y)$ exists for every $T_{I,K}$ and equals 0 for $|I|>0$. Putting $\lim_{\lambda\to+0}T_{I,K}(\lambda x, y)=S_K(y)$ for $|I|=0$, we obtain $(T|M_0)_{\tilde{U}_\kappa}=\sum_K S_K(y)dy_K$. From these facts together with Lemma 2 it will be easily verified that we obtain with $\underline{\beta}_{\tilde{U}_\kappa}=\sum_J \underline{\beta}_J(y)dy_J$

$$\begin{aligned}(j\underline{\beta})_{\tilde{V}_\kappa}\wedge T_{\tilde{V}_\kappa}\\ &=\lim_{\lambda\to+0}\sum(-1)^{\rho(CL,L)}\lambda^{|L|}\delta(x+\lambda u)\beta_J(y+\lambda v)\wedge\\ &\qquad\wedge T_{I,K}(x, y)\wedge dx_{CL}\wedge du_L\wedge d(y+\lambda v)_J\wedge dx_I\wedge dy_K\\ &=\sum_J\delta(x)dx\wedge\beta_J(y)S_{CJ}(y)dy_J\wedge dy_{CJ}\end{aligned}$$

and

$$\int(j\underline{\beta})_{\tilde{V}_\kappa}\wedge T_{\tilde{V}_\kappa}=\int\underline{\beta}_{\tilde{U}_\kappa}\wedge(T|M_0)_{\tilde{U}_\kappa},$$

which implies that $j^*T=T|M_0$. Thus the proof is complete.

If ξ is an oriented C^∞ map of M' into M, we can define in a similar way the reciprocal image $\xi^*\underline{T}\in\underline{\mathcal{D}}'^{p}(M')$ for an odd current $\underline{T}\in\underline{\mathcal{D}}'^{p}(M)$ under the map ξ. In particular, when ξ is an oriented injection j of a submanifold M_0 into M, Theorem 2, as we see easily, also remains true of the oriented injection j and the odd current $\underline{T}$.

As an application we can show Stokes' formula for a current of any kind. Before going to a general discussion, we consider the integral $\int_a^b S'(x)dx$, where S is a distribution on the real line. If the values $S(a)$, $S(b)$ exist, the integral is defined to be $S(b)-S(a)$. Now we shall consider it in more detail: Let h be the characteristic function of the interval $[a, b]$. Then $h'=\delta_a-\delta_b$. It is known [1, p. 162] that the following conditions for a distribution S are equivalent:

(1) The values $S(a)$, $S(b)$ exist.
(2) The multiplicative product $h'S$ exists.
(3) The multiplicative product hS' exists.
(4) The multiplicative products hS and hS' exist.

Let us assume that any one of these equivalent conditions is satisfied for S. Then $(hS)'=h'S+hS'$. Consequently we have

$$\int hS'dx=-\int h'Sdx=\int(S(b)\delta_b-S(a)\delta_a)dx=S(b)-S(a).$$

Therefore if we understand in general the integral $\int_a^b T(x)dx$ of a distribution T to be $\int hT dx$ when the multiplicative product hT exists, we obtain

$$\int_a^b S'(x)dx = S(b) - S(a)$$

under the assumption made above.

Let Ω be a domain in the manifold M. We assume that Ω is a domain with regular boundary, that is, the boundary $b\Omega$ is a closed $(N-1)$-dimensional manifold and we can find for each point $a \in b\Omega$ its coordinate neighbourhood V with coordinates $x, y_1, \cdots, y_{N-1}$ such that $V \cap \bar{\Omega}$ is the set of all points $b \in V$ with $x(b) \leqq 0$. We can assign to each point a of $b\Omega$ a tangent vector at a in M entering into $C\Omega$, so that $b\Omega$ is transversally oriented in a familiar way. Thus the injection $b\Omega \to M$ is oriented. We note that if M is orientable, then so is $b\Omega$.

Let $\underline{T}$ be an odd $(N-1)$-current defined on M. Let I_Ω denote the characteristic function of Ω. If $I_\Omega \wedge \underline{T}$ exists with compact support, we define

$$\int_\Omega \underline{T} = \int I_\Omega \wedge \underline{T},$$

where the right side has a meaning since $I_\Omega \wedge \underline{T} \in \mathcal{E}'(M)$. Before going to the statement of Stokes' formula for an odd current, we show a proposition needed later on.

Proposition 1. *If $\underline{T} \mid b\Omega$ exists in a narrow sense, then the exterior products $I_\Omega \wedge \underline{T}$, $I_\Omega \wedge d\underline{T}$ and $dI_\Omega \wedge \underline{T}$ in a narrow sense exist and we have*

$$d(I_\Omega \wedge \underline{T}) = dI_\Omega \wedge \underline{T} + I_\Omega \wedge d\underline{T}.$$

Proof. It is enough to show the assertions in a neighbourhood of each point $a \in b\Omega$. Let V be taken as before and put $U = \{b \in V;\ x(b) = 0\}$. We can write $\underline{T}_{\tilde{V}}$ in the form:

$$\underline{T}_{\tilde{V}} = T_0(x,y)dy + \textstyle\sum_j T_j(x,\ y)dx \wedge dy_1 \wedge \cdots \wedge \widehat{dy_j} \wedge \cdots \wedge dy_{N-1},$$

where the circumflex indicates omission. The assumption that $\underline{T} \mid b\Omega$ exists in a narrow sense means that the section $T_k(0,\ y)$, $0 \leqq k \leqq N-1$, exists. Consequently the multiplicative product $\delta(x)T_k(x,\ y)$ exists and equals $\delta(x)T_k(0,\ y)$. Let $Y(x)$ be the Heaviside function. Then we have $(I_\Omega)_{\tilde{V}} = Y(-x) \otimes 1_y$ in $\tilde{V}$. Since

$$\frac{\partial}{\partial x}(Y(-x) \otimes 1_y) = -\delta(x) \otimes 1_y,$$

$$\frac{\partial}{\partial y_j}(Y(-x) \otimes 1_y) = 0, \qquad j = 1, 2, \cdots, N-1,$$

we can conclude that the multiplicative products $(I_\Omega)_{\tilde V} T_k(x, y)$, $(I_\Omega)_{\tilde V} \frac{\partial T_k}{\partial x}$ and $(I_\Omega)_{\tilde V} \frac{\partial T_k}{\partial y_j}$ exist for $k=0, 1, \cdots, N-1$, $j=1, 2, \cdots, N-1$ [1, p. 168]. This implies that $(I_\Omega)_{\tilde V} \wedge \underline{T}_{\tilde V}$ and $(I_\Omega)_{\tilde V} \wedge d\underline{T}_{\tilde V}$ exist in a narrow sense and we have

$$d((I_\Omega)_{\tilde V} \wedge \underline{T}_{\tilde V}) = d(I_\Omega)_{\tilde V} \wedge \underline{T}_{\tilde V} + (I_\Omega)_{\tilde V} \wedge d\underline{T}_{\tilde V},$$

which completes the proof.

THEOREM 3 (Stokes' formula). *Let $\Omega \subset M$ be a domain with regular boundary and let $\underline{T}$ be an odd $(N-1)$-current on M such that* $\operatorname{supp} \underline{T} \cap \bar{\Omega}$ *is compact. If $\underline{T}$ has the section $\underline{T} \mid b\Omega$ in a narrow sense, then*

$$\int_\Omega d\underline{T} = \int_{b\Omega} j^* \underline{T},$$

where j is the oriented injection of $b\Omega$ into M.

PROOF. From Proposition 1 we have

$$d(I_\Omega \wedge \underline{T}) = dI_\Omega \wedge \underline{T} + I_\Omega \wedge d\underline{T}.$$

Consequently we have

$$\int_\Omega d\underline{T} = \int I_\Omega \wedge d\underline{T} = -\int dI_\Omega \wedge \underline{T}.$$

Hence it remains to show that $-\int dI_\Omega \wedge \underline{T} = \int_{b\Omega} j^* \underline{T}$. To do so, it is enough to show that

$$-\int \phi \cdot (dI_\Omega \wedge \underline{T}) = \int_{b\Omega} (j^*\phi)(j^*\underline{T}), \qquad \phi \in \mathring{\mathcal{D}}(V)$$

in a neighbourhood V of each point $a \in b\Omega$. Let V be taken as before. Then we can see from the proof of Theorem 2 that

$$-\phi \cdot (dI_\Omega \wedge \underline{T})_{\tilde V} = \phi(0, y)\delta(x) dx \wedge \underline{T}_{\tilde V} = \phi(0, y)\delta(x) dx \wedge (j^* \underline{T})_{\tilde U}$$

and then

$$-\int \phi \cdot (dI_\Omega \wedge \underline{T}) = \int_{\tilde V} \phi(0, y)\delta(x) dx \wedge (j^* \underline{T})_{\tilde U}$$

$$= \int_{\tilde U} \phi(0, y)(j^* T)_{\tilde U} = \int_{b\Omega} (j^*\phi)(j^* \underline{T}),$$

which completes the proof.

REMARK. When M is oriented, the boundary $b\Omega$ can be oriented as indicated before. We can prove in a like manner that Stokes' formula is also valid for an even current T.

It may happen that $\underline{T}|b\Omega$ exists in a wider sense but not $I_\Omega \wedge d\underline{T}$. Indeed, put $\Omega=\{(x,y)\in R^2;\ x<0\}$. Let $\alpha, \beta \in \overset{\circ}{\mathcal{D}}(R)$ be equal to 1 in a 0-neighbourhood and $\underline{T}=\alpha(x)\beta(y)y\dfrac{d}{dx}(\log|\log|x||)dx$. $\log|\log|x||$ has no value at 0 and $\dfrac{d}{dx}(\log|\log|x||)$ no mass at 0 [3, p. 23] and therefore $Y(-x)\cdot\dfrac{d}{dx}(\log|\log|x||)$ does not exist. Then it is easy to verify that $\underline{T}|b\Omega=0$ but $I_\Omega\wedge d\underline{T}$ does not exist. Similarly the existence of $I_\Omega\wedge d\underline{T}$ does not imply the existence of $\underline{T}|b\Omega$. Let Ω be the same as above. If we put $\underline{T}=d(f(x)g(y))$ with $f(x)=g(x)=\log(\min\{1, |x|\})$, then $d\underline{T}=0$. Since $\underline{T}=\dfrac{1}{x}\log|y|\,dx+\dfrac{1}{y}\log|x|\,dy$ in a 0-neighbourhood it follows that $\underline{T}|b\Omega$ does not exist even in a wider sense.

4. Fixations and trace maps

Let M be a manifold of dimension N and M_0 a submanifold of dimension m of M. Let j be the injection $M_0\to M$. We shall first define the trace map. To do so, let $\mathcal{H}(M)\subset\overset{p}{\mathcal{D}}'(M)$ be a locally convex space with topology finer than that of $\mathcal{D}'(M)$ and assume that $\mathcal{H}(M)\cap\overset{p}{\mathcal{E}}(M)$ is dense in $\mathcal{H}(M)$. If the map of $\mathcal{H}(M)\cap\overset{p}{\mathcal{E}}(M)$ into $\overset{p}{\mathcal{D}}'(M_0)$ which transforms $\alpha\in\mathcal{H}(M)\cap\overset{p}{\mathcal{E}}(M)$ into the restriction of α to M_0 can be continuously extended from $\mathcal{H}(M)$ into $\overset{p}{\mathcal{D}}'(M_0)$, then the extended map is called a trace map on M_0, and the current $\in\overset{p}{\mathcal{D}}'(M_0)$ which corresponds to $T\in\mathcal{H}(M)$ will be called the trace of T and denoted by $T|[M_0]$.

Proposition 2. *Let $\mathcal{H}(M)$ be a barrelled space. If the section $T|M_0$ on M_0 exists for every $T\in\mathcal{H}(M)$, then the trace $T|[M_0]$ exists for every $T\in\mathcal{H}(M)$ and $T|[M_0]=T|M_0$.*

Proof. We shall continue to employ the same notations as used in the preceding sections. For each point $a\in M_0$ we may assume that there exists a neighbourhood V of a such that

$$\tilde{V}=\{(x,\ y);\ |x|<\delta,\ |y|<\delta\},$$

$$\tilde{U}=\{y;\ |y|<\delta\},\qquad U=V\cap M_0$$

for some constant $\delta>0$. Put $T_{\tilde{V}}=\sum_{I,K}T_{I,K}(x,y)dx_I\wedge dy_K$ and let $\{\rho_k\}$ be a restricted δ-sequence with $\operatorname{supp}\rho_k\subset B_\delta\subset R^n$. Since $T|M_0$ exists, the limit

$$\lim_{k\to\infty}< T_{I,K}(x,y),\rho_k(x)> = S_K(y)\in\overset{\circ}{\mathcal{D}}'(\tilde{U}),\quad |K|=p,$$

exists for $|I|=0$. The linear map

$$\mathcal{H}(M) \ni T \to < T_{I,K}(x, y), \quad \rho_k(x)> \in \mathcal{D}'(\tilde{U}), \qquad |K| = p,$$

is clearly continuous. Since $\mathcal{H}(M)$ is barrelled, the map $\mathcal{H}(M) \ni T \to S_K(y) \in \overset{\circ}{\mathcal{D}}'(\tilde{U})$ will be continuous by the Banach-Steinhaus theorem. Thus the map

$$\mathcal{H}(M) \ni T \to T | M_0 = \sum_K S_K(y) dy_K \in \mathcal{D}'(\tilde{U})$$

is continuous. Especially if $T = \alpha \in \mathcal{H}(M) \cap \mathcal{E}(M)$ then $\alpha(x, y) | \tilde{U} = \alpha(x, y) | [\tilde{U}]$. Consequently the trace $T | [M_0]$ exists and equals $T | M_0$, which completes the proof.

Owing to Theorem 2, we can also restate that if $j^* T$ exists for every $T \in \mathcal{H}(M)$, the map $T \to j^* T \in \overset{p}{\mathcal{D}}'(M_0)$ is continuous.

In a similar way we can show

PROPOSITION 3. *Let S be a q-current on M. If $S \wedge T$ exists for every p-current T of a barrelled space $\mathcal{H}(M)$, then the map $\mathcal{H}(M) \ni T \to S \wedge T \in \overset{p+q}{\mathcal{D}'}(M)$ is continuous.*

Propositions 2 and 3 hold also true of odd currents with necessary modifications.

Now, we assume that $M = R^{n+m}$.

PROPOSITION 4. *Let T be a distribution on R^{n+m}. If $(T * \rho_k) | M_0$ converges in $\overset{\circ}{\mathcal{D}}'(M_0)$ for any δ-sequence $\{\rho_k\}$, then the section $T | M_0$ exists and $T | M_0 = \lim_{k \to \infty} (T * \rho_k) | M_0$.*

PROOF. It is sufficient to show the assertion near any point $a \in M_0$. By a linear coordinate transformation, we may assume that a is the origin and that M_0 is defined in a neighbourhood of 0 by a system of equations:

$$\begin{cases} x_i = f_i(v_1, \cdots, v_m), & i = 1, 2, \cdots, n, \\ y_j = v_j, & j = 1, 2, \cdots, m, \end{cases}$$

in a neighbourhood of $v = 0$, where f_i is a C^∞ function with $f_i(0) = 0$. Consider the coordinate transformation:

$$\begin{cases} x_i = f_i(v_1, \cdots, v_m) + u_i, & i = 1, 2, \cdots, n, \\ y_j = v_j, & j = 1, 2, \cdots, m, \end{cases}$$

where (u, v) remains in a neighbourhood of $(0, 0)$. Let $\sigma_k(u)$ and $\tau_l(v)$ be any δ-sequences. Then $\rho_{k,l}(x, y) = \sigma_k(x) \tau_l(y)$ is also a δ-sequence and we have

$$\begin{aligned} (T * \rho_{k,l}) | M_0 &= < T(x', y'), \rho_{k,l}(x - x', y - y') >_{x', y'} | M_0 \\ &= < T(x', y'), \sigma_k(f(v) - x') \tau_l(v - y') >_{x', y'} \\ &= < T'(u', v'), \sigma_k(f(v) - f(v') - u') \tau_l(v - v') >_{u', v'}. \end{aligned}$$

Then, for any $\phi(v)dv \in \overset{m}{\mathcal{D}}(R^m)$ with support in a 0-neighbourhood, we can write

$$<(T*\rho_{k,l})|M_0, \phi(v)>_v$$
$$= < T'(u', v'), \int\sigma_k(f(v)-f(v')-u')\tau_l(v-v')\phi(v)dv >_{u',v'}$$
$$= < T'(u', v'), \int\sigma_k(f(v+v')-f(v')-u')\tau_l(v)\phi(v+v')dv >_{u',v'}.$$

Consequently we obtain

$$\lim_{k,l\to\infty} <(T*\rho_{k,l})|M_0, \phi(v)>_v = \lim_{k\to\infty} < T'(u', v), \sigma_k(-u')\phi(v)>_{u',v},$$

which implies that $\lim_{k\to\infty} < T'(u', v), \sigma_k(-u')>_{u'}$ exists for every (restricted) δ-sequence σ_k, and that $T|M_0$ exists near the origin and

$$\lim_{k,l\to\infty} <(T*\rho_{k,l})|M_0, \phi(v)>_v = < T|M_0, \phi(v)>_v,$$

which completes the proof.

Corollary. *Let $\mathcal{H}(M)\subset\overset{\circ}{\mathcal{D}}'(R^{n+m})$ have the approximation property by reguralization. If the trace exists for every $T\in\mathcal{H}(M)$, then the section exists also for every $T\in\mathcal{H}(M)$ and both coincide.*

5. Admissible maps

Let M and M_1 be manifolds with dimensions N and N_1 respectively. Let ξ be a C^∞ map of M into M_1.

Definition 4. ξ is called *admissible* if ξ^*T exists for every $T\in\overset{\circ}{\mathcal{D}}'(M_1)$.

As remarked in Section 3, the definition is equivalent to asserting that the direct image $\xi\underline{\phi}$ is a C^∞ form for every $\underline{\phi}\in\overset{N}{\underline{\mathcal{D}}}(M)$, or that the map $\overset{\circ}{\mathcal{D}}(M_1)\ni\alpha \to\xi^*\alpha\in\overset{\circ}{\mathcal{E}}(M)$ can be continuously extended from $\overset{\circ}{\mathcal{D}}'(M_1)$ into $\overset{\circ}{\mathcal{D}}'(M)$.

First we remark that if ξ is admissible, then we can conclude that the reciprocal image ξ^*T of any $T\in\overset{p}{\mathcal{D}}'(M_1)$ exists, or, what is the same, the direct image $\xi\underline{\phi}$ of any $\underline{\phi}\in\overset{N-p}{\underline{\mathcal{D}}}(M)$ is a C^∞ form. Indeed, it is sufficient to show the assertion when M and M_1 are open subsets Ω and Ω' in Euclidean spaces respectively. Put $T=\sum_K T_K dx'_K$, $|K|=p$, where T_K is a distribution on Ω'. By assumption, ξ^*T_K exists for every K. Now we have

$$\begin{aligned}<\underline{\phi}, \textstyle\sum_K(\xi^*T_K)\xi^*(dx'_K)> &= \textstyle\sum_K<\underline{\phi}\wedge\xi^*(dx'_K), \xi^*T_K>\\ &= \textstyle\sum_K<\xi\underline{\phi}\wedge dx'_K, T_K>\\ &= <\xi\underline{\phi}, \textstyle\sum_K T_K dx'_K>,\end{aligned}$$

which shows that $\xi^* T$ exists and equals $\sum_K (\xi^* T_K)\xi^*(dx'_K)$.

From these considerations we see that ξ is admissible if and only if the following condition (C) [5, p. 377] is satisfied:

(C) *The image of every odd current with compact support which is defined by a C^∞ form is also a C^∞ form.*

If ξ is an admissible map of M into M_1, then we must have $N \geqq N_1$. Many of the results established in [2, p. 67–p. 85] can be generalized for currents. We shall state here some of them without proofs, because we can show them by the same procedure as therein made.

PROPOSITION 5. *Let ξ be an admissible map of M into M_1 and η an admissible map of M into M_2 of dimension N_2. Suppose that $N = N_1 + N_2$. Then the multiplicative product $(\xi^* S)(\eta^* T)$ exists for every $S \in \overset{\circ}{\mathcal{D}}'(M_1)$ and $T \in \overset{\circ}{\mathcal{D}}'(M_2)$ if and only if the map $\chi = (\xi, \eta)$ of M into $M_1 \times M_2$ is locally diffeomorphic.*

PROPOSITION 6. *If ξ is a C^∞ map of M onto M_1 with no critical point, then the reciprocal map ξ^* of $\overset{p}{\mathcal{D}}'(M_1)$ into $\overset{p}{\mathcal{D}}'(M)$ is a monomorphism for every p with $0 \leqq p \leqq N_1$.*

PROPOSITION 7. *Let ξ be an admissible map of M into M_1, where we assume M_1 to be connected. Then the following conditions are equivalent to each other:*

(1) $\xi^*(\overset{p}{\mathcal{D}}'(M_1)) = \overset{p}{\mathcal{D}}'(M)$ *for some p with $0 \leqq p \leqq N_1$.*

(2) $\xi^*(\overset{p}{\mathcal{D}}'(M_1)) = \overset{p}{\mathcal{D}}'(M)$ *for every p with $0 \leqq p \leqq N_1$.*

(3) $\xi^*(\overset{p}{\mathcal{E}}'(M_1)) = \overset{p}{\mathcal{E}}'(M)$ *for some p with $0 \leqq p \leqq N_1$.*

(4) $\xi^*(\overset{p}{\mathcal{E}}'(M_1)) = \overset{p}{\mathcal{E}}'(M)$ *for every p with $0 \leqq p \leqq N_1$.*

(5) *The map ξ is a diffeomorphism of M onto M_1.*

The analogues of Propositions 6 and 7 remain valid for an oriented map and for odd currents.

References

[1] M. Itano, *On the theory of the multiplicative products of distributions*, this Journal, **30** (1966), 151-181.

[2] ———and A. Jôichi, *On C^∞ maps which admit transposed image of every distribution*, this Journal, **31** (1967), 75-88.

[3] S. Łojasiewicz, *Sur la fixation des variables dans une distribution*, Studia Math., **17** (1958), 1-64.

[4] G. de Rham, *Variétés différentiables*, Paris, Hermann (1955).

[5] L. Schwartz, *Théorie des distributions*, Paris, Hermann (1966).

[6] R. Shiraishi, *On the value of distributions at a point and the multiplicative products*, this Journal, **31** (1967), 89-104.

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