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*On the Immersion Problem for Certain Manifolds**)

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§1. Introduction

In this note, let M^m denote a compact connected orientable C^∞-manifold (with boundary) of dimension m, and R^k the k-dimensional Euclidean space. We write $M^m \subseteq R^k$ (or $M^m \not\subseteq R^k$) to denote the existence (or the non-existence) of a C^∞-immersion of M^m into R^k.

The purpose of this note is to discuss the immersion problem for some manifolds M^m whose integral cohomology groups $H^i(M^m; Z)$ in positive dimensions are finite and have no 2-primary subgroups.

We obtain the following immersion theorems of such manifolds M^m into R^k, where k is near to $3m/2$.

Theorem 1. *Let M^m be a manifold of dimension $m=4n+r$ ($n>0$, $r=1$, 2, 3 or 4) which has the following properties* (i)-(iii):

(i) *$H^i(M^m; Z)$ is finite and has no 2-primary subgroup for any $i>2n-1$.*

(ii) *$\bar{p}_n=0$, where $\bar{p}_i \in H^{4i}(M^m; Z)$ is the i-th dual Pontrjagin class of M^m.*

(iii) *$H^{4n+3}(M^m; Z_3)=0$, if $r=4$.*

Then we have $M^{4n+r} \subseteq R^{6n+r-1}$.

Theorem 2. *Let M^m be a manifold of dimension $m=4n+r$ ($n>1$, $r=1$, 2; $n>2$, $r=3$) which has the following properties* (i)-(iii):

(i) *$H^i(M^m; Z)$ is finite and has no 2-primary subgroup for any $i>2n-3$.*

(ii) *$\bar{p}_n=0$ and $\bar{p}_{n-1}=0$.*

(iii) *$H^{4n-1}(M^m; Z_3)=0$.*

Then we have $M^{4n+r} \subseteq R^{6n+r-3}$.

As applications of these two theorems, we obtain the following two theorems about the lens space $L^n(p)=L^n(p; 1, \cdots, 1)$ of dimension $2n+1$.

Theorem 3. *Let p be an odd prime. If n is an even integer such that $\binom{n+1+n/2}{n/2} \equiv 0 \pmod{p}$, then $L^n(p) \subseteq R^{3n+2}$.*

Theorem 4. *Let p be a prime >3. If n is an odd integer such that $\binom{n+1+(n-1)/2}{(n-1)/2} \equiv 0 \pmod{p}$, then $L^n(p) \subseteq R^{3n+1}$.*

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The immersions of these two theorems are shown to be best possible for some kind of p and n (Theorems 7 and 8), by the non-immersion theorems obtained in the previous paper [5].

Finally, we remark that there is a lens space $L=L^n(p; q_1, \cdots, q_n)$, which has the homotopy type of $L^n(p)$ but that $L\subseteq R^{3n}$ and $L^n(p)\not\subseteq R^{3n+1}$ (Proposition 2).

The proofs of Theorems 1 and 2 are in §3, and are based on the following well-known theorem of M. W. Hirsch [3]:

(1.1) *If M^m is immersible in R^k with a transversal r-field, then M^m is immersible in R^{k-r}, where $m<k-r$.*

To apply this theorem, we use the obstruction theory for the existence of a cross-section of the r-frame bundle associated with the normal bundle of $M^m\subseteq R^k$. We recall in §2 some known facts about the cohomology and homotopy groups of the Stiefel manifolds $V_{n,r}$, which are used to determine these obstructions.

Theorems 3, 4, 7 and 8 are proved in §4. We notice that Theorems 3 and 4 are partial improvements of the following results of F. Uchida [15]:

$$L^n(p)\subseteq R^{2n+2[n/2]+4}, \qquad \textit{for an odd prime } p.$$

Also notice that D. Sjerve [10] has announced the following more general results:

$$L^n(p; q_1, \cdots, q_n)\subseteq R^{2n+2[n/2]+2}, \qquad \textit{for an odd prime } p.$$

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§2. Preliminaries

Let S^n be the n-sphere and let $V_{n,m}$ be the Stiefel manifold of orthonormal m-frames in R^n. In this section we list some known results about the integral cohomology groups and homotopy groups of $V_{n,m}$ which will be used in later sections (cf. [1], [9], [12] and [14]).

(2.1) *If n is odd, m is even and $k=n-m>0$, then*

$$H^i(V_{n,m}; Z)=\begin{cases} Z & \textit{for} \quad i=0, 2k+1, \\ Z_2 & \textit{for} \quad k<i<2k+1,\ i \textit{ even}, \\ 0 & \textit{for} \quad \textit{other } i<2k+1. \end{cases}$$

(2.2) $\pi_i(V_{n,m})=0 \qquad \textit{for} \quad i<n-m.$

(2.3) *If n is odd, m is even and $k=n-m>0$, then $\pi_i(V_{n,m})$ is a finite 2-*

primary group for $i<2k+1$ *and* $\pi_{2k+1}(V_{n,m})$ *is isomorphic to the direct sum of the infinite cyclic group* Z *and a finite 2-primary group.*

(2.4) *Let* p *be an odd prime. If* $n>1$ *is odd, then* $\pi_i(V_{n,2})$ *is finite for* $i>2n-3$ *and the* p*-primary component* $\pi_i(V_{n,2};p)$ *of* $\pi_i(V_{n,2})$ *is isomorphic to that of* $\pi_i(S^{2n-3})$, *and so*

$$\pi_{2n-3+i}(V_{n,2};p)=0 \quad \text{for} \quad n>2,\ i=1,2,4,5; \quad \text{or} \quad n>3,\ i=6,$$

$$\pi_{2n}(V_{n,2};3)=Z_3 \quad \text{for} \quad n>2,$$

$$\pi_{2n}(V_{n,2};p)=0 \quad \text{for} \quad n>2,\ p>3.$$

§3. Proofs of Theorems 1 and 2

To prove Theorems 1 and 2, we shall prove in the first the following

PROPOSITION 1. *Let* M^m *be a manifold of dimension* $m=4n+r$ $(n\geqq 0,$ $r=1, 2, 3$ *or* $4)$ *such that* $H^i(M^m;Z)$ *is finite and has no 2-primary subgroup for any* $i>2n+1$.

Then we have $M^{4n+r}\subseteq R^{6n+r+1}$.

PROOF. We remark that the manifold M^m is a manifold with boundary, because M^m is compact, connected, orientable and $H^m(M^m;Z)$ is finite. Then $H^m(M^m;G)=0$ for any abelian group G.

According to Whitney's theorem [16] we have

$$M^{4n+r}\subseteq R^{2(4n+r)-1}\subset R^{8n+r+3},$$

and let ν be its oriented $(4n+3)$-dimensional normal bundle over M^{4n+r} and $\nu^{(2n+2)}$ be the associated $(2n+2)$-frame bundle of ν. The obstructions for the existence of a cross-section of $\nu^{(2n+2)}$ are contained in $H^{i+1}(M^{4n+r};\pi_i(V_{4n+3,2n+2}))$. Here we notice that the local coefficients $\pi_i(V_{4n+3,2n+2})$ in these cohomology groups are trivial, because ν is orientable [8, p. 445].

According to (2.2) and (2.3),

$$\pi_i(V_{4n+3,2n+2})=\begin{cases}0, & \text{for} \quad i<2n+1,\\ \text{finite 2-primary group}, & \text{for} \quad 2n+1\leqq i\leqq 4n+2.\end{cases}$$

Therefore $H^{i+1}(M^{4n+r};\pi_i(V_{4n+3,2n+2}))=0$ for $i\leqq 4n+2$. By the above remark, $H^{4n+4}(M^{4n+4};\pi_{4n+3}(V_{4n+3,2n+2}))=0$. Hence $\nu^{(2n+2)}$ has a cross-section and M^{4n+r} $\subseteq R^{6n+r+1}$ by (1.1). q.e.d.

PROOF OF THEOREM 1. By Proposition 1 we have $M^{4n+r}\subseteq R^{6n+r+1}$. Let ν be its oriented $(2n+1)$-dimensional normal bundle and $\nu^{(2)}$ be the associated 2-frame bundle. According to (2.2)–(2.4), $\pi_i(V_{2n+1,2})$ is a finite group except for $i=4n-1$, and

$$\pi_i(V_{2n+1,2})=\begin{cases} 0, & \text{for } i<2n-1, \\ \text{finite 2-primary group, for } 2n-1\leqq i\leqq 4n+1,\ i\neq 4n-1, & \\ Z+\text{finite 2-primary group}, & \text{for } i=4n-1, \end{cases}$$

$$\pi_{4n+2}(V_{2n+1,2};3)=Z_3,$$

$$\pi_{4n+2}(V_{2n+1,2};p)=0, \qquad \text{if } p \text{ is a prime}>3.$$

Since M^m is a manifold with boundary, $H^m(M^m;G)=0$ for any abelian group G. Therefore, by the assumptions (i) and (iii), the primary (and the last) obstruction for the existence of a cross-section of $\nu^{(2)}$ lies in $H^{4n}(M^m;\pi_{4n-1}(V_{2n+1,2}))$ $(=H^{4n}(M^m;Z))$.

Let E be the total space of the bundle $\nu^{(2)}$ and let $\pi\colon E\to M^m$ be the projection of $\nu^{(2)}$. Consider the following commutative diagram:

$$\begin{array}{ccccc} H^{4n-1}(V_{2n+1,2};Z) & \xrightarrow{\delta} & H^{4n}(E,V_{2n+1,2};Z) & \xleftarrow{\bar{\pi}^*} & H^{4n}(M^m;Z) \\ & & {\scriptstyle j^*}\searrow & & \swarrow{\scriptstyle \pi^*} \\ & & & H^{4n}(E;Z) & \end{array}$$

where $j\colon E\to(E,V_{2n+1,2})$ is the injection, $\bar{\pi}\colon(E,V_{2n+1,2})\to(M^m,*)$ is π, and δ is the coboundary homomorphism. Note that $\pi_1(M^m)$ acts trivially on $H_*(V_{2n+1,2};Z)$ [8, p. 445] and that $H_i(V_{2n+1,2};Z)$ is a finite 2-primary group for $i<4n-1$ (cf. §2). Thus, by Theorem 1.B of J. -P. Serre [9, p.268], we see that $\bar{\pi}^*$ is a monomorphism since $H^{4n}(M^m;Z)$ has no 2-primary subgroup.

Theorem 30.10 of A. Borel and F. Hirzebruch [2, p. 377] implies that, for a generator ι of $H^{4n-1}(V_{2n+1,2};Z)=Z$ (cf. (2.1)), 2ι is transgressive and that

$$p_n(\nu)=(-1)^{n+1}\bar{\pi}^{*-1}\delta(2\iota) \qquad \text{modulo a 2-primary group,}$$

where $p_n(\nu)$ denote the n-th Pontrjagin class of ν. Since $H^{4n}(M^m;Z)$ has no 2-primary subgroup, we have $p_n(\nu)=\bar{p}_n$ and $\bar{p}_n=(-1)^{n+1}\bar{\pi}^{*-1}\delta(2\iota)$.

Let $c\in H^{4n}(M^m;\pi_{4n-1}(V_{2n+1,2}))$ be the primary (and the last) obstruction to the construction of a cross-section of $\nu^{(2)}$. Then $\pi^*c=0$ [11, p. 188], and so $j^*\bar{\pi}^*c=0$. By the exactness of the cohomology sequence, there is an element $x\in H^{4n-1}(V_{2n+1,2};Z)$ such that $\delta x=\bar{\pi}^*c$. Since ι is a generator, $x=2q\iota$ or $(2q+1)\iota$ for some integer q.

If $x=2q\iota$, then $c=(-1)^{n+1}q\bar{p}_n$.

If $x=(2q+1)\iota$, then $\iota=x-2q\iota$, and hence ι is transgressive. Thus we may take $y=\bar{\pi}^{*-1}\delta\iota$. Therefore $2y=(-1)^{n+1}\bar{p}_n$. By the assumption (i), there is an odd integer $2s-1$ such that $(2s-1)y=0$. Then $y=2sy=(-1)^{n+1}s\bar{p}_n$. Hence $c=y+(-1)^{n+1}q\bar{p}_n=(-1)^{n+1}(s+q)\bar{p}_n$.

Therefore, in both cases we see that $c=0$ if $\bar{p}_n=0$. Hence, by the assumption (ii), $\nu^{(2)}$ has a cross-section and we have $M^{4n+r}\subseteq R^{6n+r-1}$ by (1.1).
q.e.d.

PROOF OF THEOREM 2. By Theorem 1 we have $M^{4n+r}\subseteq R^{6n+r-1}$. Let ν be its oriented $(2n-1)$-dimensional normal bundle and $\nu^{(2)}$ be the associated 2-frame bundle. According to (2.2)-(2.4), $\pi_i(V_{2n-1,2})$ is a finite group except for $i=4n-5$, and

$$\pi_i(V_{2n-1,2})=\begin{cases} 0, & \text{for } i<2n-3,\\ \text{finite 2-primary group, for } 2n-3\leqq i<4n+2, & \\ \qquad\qquad i\neq 4n-5,\ 4n-2, & \\ Z+\text{finite 2-primary group, for } i=4n-5, & \end{cases}$$

$$\pi_{4n-2}(V_{2n-1,2};\, 3)=Z_3,$$

$$\pi_{4n-2}(V_{2n-1,2};\, p)=0, \text{ if } p \text{ is a prime}>3.$$

Since M^m is a manifold with boundary, $H^m(M^m;G)=0$ for any abelian group G. Therefore, the assumptions (i) and (iii) imply that the primary (and the last) obstruction for the existence of a cross-section of $\nu^{(2)}$ lies in $H^{4n-4}(M^m;\pi_{4n-5}(V_{2n-1,2}))$ $(=H^{4n-4}(M^m;Z))$. In the similar way to the proof of Theorem 1, we can see that the obstruction vanishes if $\bar{p}_{n-1}=0$. Thus $\nu^{(2)}$ has a cross-section by (ii) and we have $M^{4n+r}\subseteq R^{6n+r-3}$ by (1.1). q.e.d.

§4. Applications for lens spaces

Let $p>2$ be an integer and let Γ be the cyclic group of order p with generator t. Let $S^{2n+1}\subset C^{n+1}$ be the unit $(2n+1)$-sphere in the complex $(n+1)$-space. Given $n+1$ primitive p-th roots $\alpha_0\ (=e^{2\pi i/p})$, $\alpha_1, \cdots, \alpha_n (\in C)$ of unity, define an action of Γ on S^{2n+1} by the formula:

$$t(z_0, z_1, \cdots, z_n)=(\alpha_0 z_0, \alpha_1 z_1, \cdots, \alpha_n z_n),$$

where z_j $(j=0, 1, \cdots, n)$ are complex numbers with $\sum_{j=0}^{n}|z_j|^2=1$. The quotient manifold S^{2n+1}/Γ is called a *lens space*. Set

$$\alpha_j=\alpha_0{}^{q_j}=e^{2\pi i q_j/p}.$$

The lens space S^{2n+1}/Γ is written by $L^n(p;\, q_1, q_2, \cdots, q_n)$ (or briefly L^{2n+1}). The notation $L^n(p)$ will be used for the lens space $L^n(p;\, 1, 1, \cdots, 1)$.

The lens space L^{2n+1} has a structure of a CW-complex with one cell in each dimension. The cohomology groups of L^{2n+1} are given as follows:

$$H^j(L^{2n+1}; Z) = \begin{cases} Z & \text{for } j=0, 2n+1, \\ Z_p & \text{for } j=2, 4, \cdots, 2n, \\ 0 & \text{for other } j, \end{cases}$$

$$H^j(L^{2n+1}; Z_p) = Z_p \qquad \text{for } 0 \leqq j \leqq 2n+1.$$

Let $x \in H^2(L^{2n+1}; Z)$ be a generator. The total Pontrjagin class of L^{2n+1} $(=L^n(p; q_1, q_2, \cdots, q_n))$ is given by the formula ([13], Corollary 3.2):

$$(4.1) \qquad p(L^{2n+1}) = (1+x^2)(1+q_1^2x^2)(1+q_2^2x^2)\cdots(1+q_n^2x^2).$$

Let L_0^{2n+1} denote the set $L^{2n+1} - \mathrm{Int}\, D$, where D is a $(2n+1)$-dimensional disk contained in the interior of the highest dimensional cell of the given CW-decomposition. Then L_0^{2n+1} is the compact connected orientable manifold (with boundary $S^{2n} = \dot{D}$) of dimension $2n+1$. Let j: $L_0^{2n+1} \to L^{2n+1}$ be the inclusion map. It is easily seen that the induced homomorphism j^*: $H^i(L^{2n+1}; Z) \to H^i(L_0^{2n+1}; Z)$ is an isomorphism for $i < 2n+1$ and that $H^{2n+1}(L_0^{2n+1}; G) = 0$, where G is any abelian group. Thus we may identify the Pontrjagin class of L_0^{2n+1} and that of L^{2n+1}.

Hereafter, we assume that p is an odd prime. We shall apply the previous results to the problem of finding the least integer $k > 0$ such that $L^n(p) = L^n(p; 1, \cdots, 1)$ can be immersed in R^{2n+1+k}. According to (1.1), such an integer $k > 0$ is equal to the geometrical dimension[1)] of $-\tau_0(L^n(p))$ (written by $g\text{-}\dim(-\tau_0(L^n(p)))$), where $\tau_0(L^n(p))$ is the stable class of the tangent bundle $\tau(L^n(p))$ of $L^n(p)$. Some results about the non-immersibility and the non-embeddability of $L^n(p)$ were obtained in [4], [5] and [6].

Let x be a generator of $H^2(L^n(p)_0; Z)$ $(=H^2(L^n(p); Z))$. (4.1) shows that the total Pontrjagin class of $L^n(p)_0$ is given by the formula:

$$p(L^n(p)_0) = (1+x^2)^{n+1},$$

and so the dual Pontrjagin class is given by the formula:

$$\bar{p}(L^n(p)_0) = (1+x^2)^{-n-1} = \sum_{i=0}^{[n/2]} (-1)^i \binom{n+i}{i} x^{2i}.$$

Since $L^n(p)$ is naturally embedded in $L^{n+1}(p)_0$, Theorems 3 and 4 are immediate consequences of the following two theorems.

THEOREM 5. *Let p be an odd prime. If n is an even integer such that* $\binom{n+1+n/2}{n/2} \equiv 0 \pmod p$, *then* $L^{n+1}(p)_0 \subseteq R^{3n+2}$.

1) The *geometrical dimension* of $\alpha \in \widetilde{KO}(X)$ is the least integer k such that $\alpha + k = \theta(\beta)$ for some $\beta \in \varepsilon(X)$, where θ: $\varepsilon(X) \to KO(X)$ is the natural map of the set of equivalence classes $\varepsilon(X)$ of real vector bundles over a CW-complex X into the associated Grothendieck group $KO(X)$.

PROOF. By the assumption, we have

$$\bar{p}_{n/2}(L^{n+1}(p)_0)=(-1)^{n/2}\binom{n+1+n/2}{n/2}x^n=0,$$

and so we get $L^{n+1}(p)_0\subseteq R^{3n+2}$ by Theorem 1 (for $r=3$). q.e.d.

THEOREM 6. *Let p be a prime >3. If n is an odd integer such that $\binom{n+1+(n-1)/2}{(n-1)/2}\equiv 0 \pmod p$, then $L^{n+1}(p)_0\subseteq R^{3n+1}$.*

PROOF. By the assumption, we have

$$\binom{n+1+(n+1)/2}{(n+1)/2}=\frac{n+1+(n+1)/2}{(n+1)/2}\binom{n+1+(n-1)/2}{(n-1)/2}$$

$$=3\binom{n+1+(n-1)/2}{(n-1)/2}\equiv 0 \qquad (\mathrm{mod}\ p).$$

Thus we have $\bar{p}_{(n-1)/2}(L^{n+1}(p)_0)=0$ and $\bar{p}_{(n+1)/2}(L^{n+1}(p)_0)=0$, and hence we get $L^{n+1}(p)_0\subseteq R^{3n+1}$ by Theorem 2 (for $r=1$). q.e.d.

If we combine these two results with the non-immersion theorems which we have obtained in the previous paper ([5], Theorems 4 and 5), we obtain the following results.

THEROEM 7. *Assume that either of the conditions* I) *and* II) *below is satisfied.*

I) *$p=6k+1$ $(k>0)$ is a prime, α and β are even integers such that $0<\alpha\leqq(2p-2)/3$ and $\beta=(2p-2)/3$, and $l>1$ is an integer.*

II) *$p=6k-1$ $(k>0)$ is a prime, α and β are odd integers such that $0<\alpha\leqq(2p-1)/3$ and $\beta=(p-2)/3$, and l is an integer such that $l>1$ if $\alpha>1$ and $l>2$ if $\alpha=1$.*

Then, for $n=\alpha p^l+\beta$, we have

$$L^n(p)\subseteq R^{3n+2},\quad L^n(p)\not\subseteq R^{3n+1}.$$

PROOF. $L^n(p)\not\subseteq R^{3n+1}$ is a consequence of Theorem 4 in [5].

Under the condition I),

$$\binom{n+1+n/2}{n/2}=\binom{\frac{3\alpha}{2}p^l+p}{\frac{\alpha}{2}p^l+\frac{p-1}{3}}\equiv 0 \qquad (\mathrm{mod}\ p)^{2)},$$

and under the condition II),

2) If $a=\sum_i a_ip^i$ and $b=\sum_i b_ip^i$ are p-adic expansions, then $\binom{a}{b}\equiv\prod_i\binom{a_i}{b_i} \pmod p$.

$$\binom{n+1+n/2}{n/2}=\binom{\frac{3\alpha-1}{2}p^l+\frac{p-1}{2}p^{l-1}+\cdots+\frac{p-1}{2}p^2+\frac{p+1}{2}p}{\frac{\alpha-1}{2}p^l+\frac{p-1}{2}p^{l-1}+\cdots+\frac{p-1}{2}p^2+\frac{p-1}{2}p+\frac{2p-1}{3}}$$

$$\equiv 0 \pmod p.$$

Therefore, by Theorem 3, we have $L^n(p)\subseteq R^{3n+2}$. q.e.d.

THEOREM 8. *Assume that either of the conditions* III) *and* IV) *below is satisfied.*

III) $p=6k-1$ $(k>0)$ *is a prime,* α *is an even integer such that* $0<\alpha\leqq(2p-2)/3$, $\beta=(2p-1)/3$, *and* $l>1$ *is an integer.*

IV) $p=6k+1$ $(k>0)$ *is a prime,* α *is an odd integer such that* $0<\alpha\leq(2p-1)/3$, $\beta=(p-1)/3$, *and* l *is an integer such that* $l>1$ *if* $\alpha>1$ *and* $l>2$ *if* $\alpha=1$.

Then, for $n=\alpha p^l+\beta$, *we have*

$$L^n(p)\subseteq R^{3n+1},\quad L^n(p)\not\subseteq R^{3n}.$$

PROOF. $L^n(p)\not\subseteq R^{3n}$ is a consequence of Theorem 5 in [5]. Under the condition III),

$$\binom{n+1+(n-1)/2}{(n-1)/2}=\binom{\frac{3\alpha}{2}p^l+p}{\frac{\alpha}{2}p^l+\frac{p-2}{3}}\equiv 0 \pmod p,$$

and under the condition IV),

$$\binom{n+1+(n-1)/2}{(n-1)/2}=\binom{\frac{3\alpha-1}{2}p^l+\frac{p-1}{2}p^{l-1}+\cdots+\frac{p-1}{2}p^2+\frac{p+1}{2}p}{\frac{\alpha-1}{2}p^l+\frac{p-1}{2}p^{l-1}+\cdots+\frac{p-1}{2}p^2+\frac{p-1}{2}p+\frac{2p-2}{3}}$$

$$\equiv 0 \pmod p.$$

Thus, by Theorem 4, we have $L^n(p)\subseteq R^{3n+1}$. q.e.d.

If the number of the non-zero terms of the p-adic expansions of n is larger than 2, we have many types of results corresponding to Theorems 7 and 8. For examples, we have the following (cf. [5], Theorems 4′ and 5′).

THEOREM 7′. *Assume either of the conditions* I′) *and* II′) *below is satisfied.*

I′) $p=6k+1$ $(k>0)$ *is a prime;* $m>2$ *is an integer;* α_i $(i=1, 2, \cdots, m)$ *are even integers such that* $0<\alpha_i\leqq(2p-2)/3$ *for* $i\geqq 2$ *and* $\alpha_1=(2p-2)/3$; *and* $l_i(i=1, 2, \cdots, m)$ *are integers with* $l_m>l_{m-1}>\cdots>l_2>l_1=0$.

II′) $p=6k-1$ $(k>0)$ *is a prime;* $m>2$ *is an even integer;* α_i $(i=1, 2, \cdots, m)$ *are odd integers such that* $0<\alpha_i\leqq(2p-1)/3$ *if* i *is even,* $0<\alpha_i\leqq(p-2)/3$ *if* i

is odd >1, *and* $\alpha_1=(p-2)/3$; *and* $l_i (i=1, 2, \cdots, m)$ *are integers with* $l_m>l_{m-1}>\cdots>l_2>l_1=0$.

Then, for $n=\sum_{i=1}^{m}\alpha_i p^{l_i}$, *we have*

$$L^n(p)\subseteq R^{3n+2}, \quad L^n(p)\not\subseteq R^{3n+1}.$$

THEROEM 8′. *Assume that either of the conditions* III′) *and* IV′) *below is satisfied.*

III′) $p=6k-1$ $(k>0)$ *is a prime*; $m>2$ *is an integer*; $\alpha_i (i=2, 3, \cdots, m)$ *are even integers such that* $0<\alpha_i\leqq(2p-2)/3$, *and* $\alpha_1=(2p-1)/3$; *and* l_i $(i=1, 2, \cdots, m)$ *are integers with* $l_m>l_{m-1}>\cdots>l_2>l_1=0$.

IV′) $p=6k+1$ $(k>0)$ *is a prime*; $m>2$ *is an integer*; α_i $(i=3, 4, \cdots, m)$ *are even integers such that* $0<\alpha_i\leqq(2p-2)/3$, α_2 *is an odd integer such that* $0<\alpha_2\leqq(2p-1)/3$, *and* $\alpha_1=(p-1)/3$; *and* l_i $(i=1, 2, \cdots, m)$ *are integers with* $l_m>l_{m-1}>\cdots>l_2>l_1=0$.

Then, for $n=\sum_{i=1}^{m}\alpha_i p^{l_i}$, *we have*

$$L^n(p)\subseteq R^{3n+1}, \quad L^n(p)\not\subseteq R^{3n}.$$

The proof of Theorem 7′ (or Theorem 8′) is similar to that of Theorem 7 (or Theorem 8), and so we omit the details.

§5. Remarks

In this section we shall give an example of the lens space $L=L^n(p; q_1, q_2, \cdots, q_n)$ which has the homotopy type of $L^n(p)$ but has the geometrical dimension of the stable normal bundle different from that of $L^n(p)$.

First, we recall Theorem VI of P. Olum [7, p. 468] about the homotopy types of lens spaces:

(5.1) *Two lens spaces* L^n $(p; q_1, q_2, \cdots, q_n)$ *and* L^n $(p; q'_1, q'_2, \cdots, q'_n)$ *have the same homotopy type if and only if*

$$q_1q_2\cdots q_n=\pm k^{n+1}q'_1q'_2\cdots q'_n \qquad (\text{mod } p)$$

for some integer k *relatively prime to* p.

PROPOSITION 2. *Let* $n=3\cdot5^l+1=2m$ $(l>1)$, *and let*

$$L=L^n(5; \overbrace{1, \cdots, 1}^{m-1}, \overbrace{2, \cdots, 2}^{m+1}).$$

Then we have

1) L *and* $L^n(5)$ *have the same homotopy type.*
2) $L^n(5)\subseteq R^{3n+2}$ *and* $L^n(5)\not\subseteq R^{3n+1}$, *that is,* $g\cdot dim(-\tau_0(L^n(5)))=n+1$.
3) $L\subseteq R^{3n}$, *that is,* $g\cdot dim(-\tau_0(L))\leqq n-1$.

PROOF. 1) $2^{m+1}=2^{5^l+2\cdot5^{l-1}+\cdots+2\cdot5+4}$

$$=2^{5^l}\cdot4^{5^{l-1}+\cdots+5}\cdot16\equiv\pm2^{5^l}\equiv\pm2 \qquad (\bmod 5),$$

$$2^{n+1}=2^{3\cdot5^l+2}\equiv-8^{5^l}\equiv2^{5^l}\equiv2 \qquad (\bmod 5).$$

Thus, by (5.1) we see that L and $L^n(5)$ have the same homotopy type.

2) This fact is a consequence of Theorem 7 in §4.

3) Consider the $(2n+3)$-dimensional manifold (with boundary)

$$L_0'=L^{n+1}(5;\overbrace{1,\cdots,1}^{m},\overbrace{2,\cdots,2}^{m+1})_0.$$

According to (4.1), we have

$$p(L_0')=(1+x^2)^{m+1}(1-x^2)^{m+1}=(1-x^4)^{m+1},$$

$$\bar{p}(L_0')=(1-x^4)^{-m-1}=\sum_i\binom{m+i}{i}x^{4i}.$$

Thus,

$$\bar{p}_m=\binom{m+m/2}{m/2}x^n=\binom{2\cdot5^l+5^{l-1}+\cdots+5+2}{3\cdot5^{l-1}+\cdots+3\cdot5+4}x^n=0$$

and, clearly, $\bar{p}_{m-1}=0$. Therefore, by Theorem 2 (for $r=3$), we have $L_0'\subseteq R^{3n}$. Since L is naturally embedded in L_0', we obtain $L\subseteq R^{3n}$. q.e.d.

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