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Overrings of Almost Multiplication Rings

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1. Introduction

In [3] Gilmer introduced the notion of almost Dedekind domain. In [1] Butts and Phillips studied a class of rings with the property that the integral domains in this class are the almost Dedekind domains; they called these rings almost multiplication rings. In this paper we shall obtain several results concerning almost multiplication rings which reduce to known results when stated for almost Dedekind domains. In particular, results are obtained concerning overrings of a certain type of almost multiplication ring.

All rings which we consider in this paper will be commutative rings with unity. Let R be such a ring. We shall always denote by K the total quotient ring of R. If S is a multiplicatively closed set in R, and if S contains no zero-divisors, we shall assume that the ring of quotients R_S is imbedded in K in the natural way. If P is a prime ideal of R we denote by R_P the ring of quotients of R with respect to P, by $S(P)$ the multiplicatively closed set of non-zero-divisors in R but not in P, and by $N(P)$ the set of all $x \in R$ such that $sx=0$ for some $s \in R$ with $s \notin P$. By an *overring* of R we mean a ring T such that $R \subseteq T \subseteq K$. We shall write $A \subset B$ when A is a proper subset of B; an ideal A of R will be referred to as proper if $A \subset R$.

A ring R is called an *almost multiplication ring* (AM-ring) if for every proper prime ideal P of R, each ideal of R_P can be written as a product of prime ideals. If this is the case, then for every proper prime ideal P of R, the ring R_P is either a discrete rank one valuation ring or a special primary ring. Several sets of equivalent conditions for a ring to be an AM-ring have been given by Butts and Phillips [1; thm. 2.0].

An element of a ring R will be called *regular* if it is not a zero-divisor. An ideal A will be called *regular* if it contains at least one regular element; otherwise A will be called a *Z-ideal*. *A* ring R is said to have *few zero-divisors* if there are only a finite number of maximal elements in the set of Z-ideals of R. This notion was introduced by Davis in [2]. We shall use several times the following consequence of the fact that a ring has few zero-divisors: if R has few zero-divisors then every regular ideal of R is generated by its regular elements, and every finitely generated regular ideal of R has a finite set of generators consisting of regular elements [2; lemma B]. *A* ring R is called a *quasi-valuation ring* if R has few zero-divisors and if the set of regular ideals of R is totally ordered by inclusion; this is equivalent to the definition of Davis [2; 203].

2. Overrings of almost multiplication rings

In this section we shall study the overrings of certain AM-rings, showing that they are AM-rings and determining their regular prime ideals. To begin with, we need a result concerning quasi-valuation rings. Let R be a quasi-valuation ring. Then, either R consists entirely of units and zero-divisors, or R has exactly one regular maximal ideal P. If, in the latter case, P is principal, then R is called a *discrete* quasi-valuation ring [2; 208].

LEMMA 1. *Let R be a quasi-valuation ring with regular maximal ideal P. Then R_P is a discrete rank one valuation ring if and only if R is a discrete quasi-valuation ring and $\bigcap_{n=1}^{\infty} P^n = N(P)$.*

PROOF. Assume that R_P is a discrete rank one valuation ring. Let A be a proper regular ideal of R. Then $A \subseteq P$ and $AR_P = P^k R_P$ for some positive integer k. If M is a maximal ideal of R which is not regular then both A and P^k meet $R \setminus M$, so $AR_M = R_M = P^k R_M$. Thus $A = P^k$. It follows that every ascending chain of regular ideals of R has only a finite number of distinct terms. Now, since R has few zero-divisors, every regular ideal of R is generated by its regular elements. Starting from this fact we can use the standard argument employed to show that the ACC implies the finite basis condition to prove that every regular ideal of R can be generated by a finite number of regular elements. It follows immediately that P is principal.

We have $\bigcap_{n=1}^{\infty} P^n R_P = 0$. Under the canonical homomorphism from R into R_P every element of $\bigcap_{n=1}^{\infty} P^n$ is mapped into $\bigcap_{n=1}^{\infty} P^n R_P$, so $\bigcap_{n=1}^{\infty} P^n \subseteq N(P)$. If $x \in N(P)$ then $sx = 0$ for some $s \in R \setminus P$. Since P^n is P-primary this implies that $x \in P^n$. Thus $N(P) \subseteq \bigcap_{n=1}^{\infty} P^n$.

Conversely, suppose that R is a discrete quasi-valuation ring with $\bigcap_{n=1}^{\infty} P^n = N(P)$. Then $N(P)$ is a prime ideal of R: the proof of this is very much like the proof of [4; lemma 2.10]. Hence R_P is an integral domain. Since $(A' \cap R)R_P = A'$ for every ideal A' of R_P, we see that R_P is a valuation ring. The maximal ideal of R_P is PR_P, which is principal, so R_P is discrete and rank one.

Let R be a quasi-valuation ring with regular maximal ideal P. We shall call R *strongly discrete* if R_P is a discrete rank one valuation ring.

To obtain significant results concerning the overrings of an AM-ring R we shall have to assume that R has few zero-divisors. Since Noetherian rings and integral domains have few zero-divisors our results will apply to a wide class of AM-rings. Note that R has few zero-divisors if and only if K has few zero-divisors. Hence, if R has few zero-divisors the same is true of each of its overrings. Our first result characterizes AM-rings (with few zero-divisors) in terms of their overrings.

THEOREM 1. *Let R be a ring with few zero-divisors. Then R is an AM-ring if and only if*

(i) *K is an AM-ring, and*

(ii) *for every regular maximal ideal P of R the ring $R_{S(P)}$ is a strongly discrete quasi-valuation ring.*

PROOF. Assume that R is an AM-ring. Let R' be a ring of quotients of R and let A' be an ideal of R' such that $\mathrm{rad}(A')=P'$, a prime ideal of R'. Then $\mathrm{rad}(A')\cap R=P'\cap R=P$ is a prime ideal of R, and since $\mathrm{rad}(A'\cap R)=\mathrm{rad}(A')\cap R$, we have $A'\cap R=P^n$ for some positive integer n. Then $A'=P^nR'=P'^n$; thus R' is an AM-ring [1; thm. 2.0]. Therefore, K is an AM-ring, as is $R_{S(P)}$ for every regular maximal ideal P of R. By [1; thm. 2.11] and [7; thm. 5], $R_{S(P)}$ is a P-ring (the concept of P-ring is due to Davis [2; 203]). Hence $R_{S(P)}$ is a quasi-valuation ring (see [2; thm. 3]). It follows from [1; thm. 2.12] and Lemma 1 that $R_{S(P)}$ is strongly discrete.

Conversely, suppose that (i) and (ii) hold. Let P be a prime ideal of R. If P is a Z-ideal then $R_{S(P)}=K$ and $R_P\cong K_{PK}$ [8; 231]. Hence, each ideal of R_P can be written as a product of prime ideals by (i).

Suppose that P is regular. Let M be a maximal ideal of R with $P\subseteq M$; then M is regular and $R_{S(M)}\subseteq R_{S(P)}$. Since $R_{S(P)}\neq K$ (for, the regular elements of P are regular nonunits of $R_{S(P)}$) we have $R_{S(P)}=(R_{S(M)})_{S(PR_{S(P)}\cap R_{S(M)})}$ [2; prop. 5]. Hence $PR_{S(P)}\cap R_{S(M)}$ is a regular prime ideal of $R_{S(M)}$. Therefore, since $R_{S(M)}$ is a discrete quasi-valuation ring, $PR_{S(P)}\cap R_{S(M)}=MR_{S(M)}$, and consequently $R_{S(P)}=R_{S(M)}$ and $PR_{S(P)}=MR_{S(M)}$. Then, by the strong discreteness of $R_{S(M)}$, R_P is a discrete rank one valuation ring. Therefore, R is an AM-ring.

THEOREM 2. *Let R be a ring with few zero-divisors. If R is an AM-ring then every overring of R is an AM-ring.*

PROOF. Since K is the total quotient ring of an overring R' of R, (i) of Theorem 1 holds for R'. We may now assume that R' is not K. Let P' be a regular maximal ideal of R'. By [2; prop. 6] $R'_{S(P')}$ is a quasi-valuation ring. Let $P=P'\cap R$. We wish to conclude that $R_{S(P)}\subseteq R'_{S(P')}$, and in order to draw this conclusion we note that $S(P)\subseteq S(P')$. For, it follows from the fact that R is a P-ring that R' is a flat overring of R [7; thm. 5]. Hence, every regular element of R is also a regular element of R' by a well-known property of flat modules (see N. Bourbaki, Algèbre Commutative, Chap. 1, Hermann, Paris, 1961, p. 41). Thus, we do have $R_{S(P)}\subseteq R'_{S(P')}$ and it follows that $R_{S(P)}\neq K$.

Thus P is a regular prime ideal of R and it follows from the proof of Theorem 1 that $R_{S(P)}$ is a strongly discrete quasi-valuation ring. Since $R'_{S(P')} \neq K$ it follows as in the proof of Theorem 1 that $R_{S(P)} = R'_{S(P')}$. Hence $R'_{S(P')}$ is strongly discrete. Thus (ii) of Theorem 1 holds and we conclude that R' is an AM-ring.

Note that, in the notation of the preceding proof, we have $PR_{S(P)} = P'R'_{S(P')}$, and

$$R_P \cong (R_{S(P)})_{PS_{S(P)}} = (R'_{S(P')})_{P'R'_{S(P')}} \cong R'_{P'}.$$

A ring R is called a *multiplication ring* if whenever A and B are ideals of R with $A \subseteq B$ there is an ideal C of R such that $A = BC$. Since multiplication rings are AM-rings [1; lemma 2.4] we have the

Corollary. *Let R be a multiplication ring having few zero-divisors. Then every overring of R is an AM-ring.*

Butts and Phillips have shown that the cancellation law for regular ideals holds in an AM-ring [1; thm. 2.11]. Our next result gives at least a partial answer to the question of the extent to which the cancellation law for regular ideals characterizes AM-rings.

Theorem 3. *Let R be a ring with few zero-divisors. Then R is an AM-ring if and only if*

(i) *K is an AM-ring,*

(ii) *$AB = AC$ and A regular imply $B = C$, and*

(iii) *for every regular maximal ideal P of R we have*

$$\bigcap_{n=1}^{\infty} P^n = N(P).$$

Proof. If R is an AM-ring, then (i) and (ii) hold. Furthermore, if P is a regular maximal ideal of R then (2) of [1; thm. 2.12] applies and so (iii) holds.

Conversely, suppose (i)-(iii) hold. Let P be a regular maximal ideal of R; we shall show that $R_{S(P)}$ is a strongly discrete quasi-valuation ring. By (ii), R is a P-ring [7; thm. 5], so that $R_{S(P)}$ is a quasi-valuation ring. Also by (ii), $P^2 \neq P$, and so $P^2R_{S(P)} \neq PR_{S(P)}$.

Let $a/s \in \bigcap_{n=1}^{\infty} P^n R_{S(P)}$ and let n be a positive integer. Then $a/s = b/t$ where $b \in P^n$; hence ta $\in P^n$, and since $t \notin P$, we have $a \in P^n$. Then, by (iii), $a \in N(P)$ and so there is an element $u \in R \setminus P$ such that $ua = 0$. But $u \in R_{S(P)} \setminus PR_{S(P)}$ and $ua/s = 0$. Thus $a/s \in N(PR_{S(P)})$. Therefore, $\bigcap_{n=1}^{\infty} P^n R_{S(P)} = N(PR_{S(P)})$. Thus $R_{S(P)}$ is strongly discrete if it is discrete. This fact is a consequence of the following

LEMMA 2. *Let R be a quasi-valuation ring with regular maximal ideal P. If $\bigcap_{n=1}^{\infty} P^n = N(P)$ then R is a discrete quasi-valuation ring.*

PROOF. Since P is regular and $N(P)$ is not regular we must have $P^2 \neq P$. Since R has few zero-divisors there is a regular element $a \in P$ with $a \notin P^2$. Let b be any regular element in P. Either $a/b \in R$ or $b/a \in R$. If $b/a \in R$ then $b \in (a)$. Suppose $b/a \notin R$. Then $a/b \in R$ and in fact, since this element is a regular non-unit in R, it belongs to P. Thus $a \in bP \subseteq P^2$, a contradiction. Thus we have $P = (a)$.

Since AM-rings with few zero-divisors are P-rings, we have the following corollaries to the results we have proved. For the first we use [2; cor. 2 to thm. 3].

COROLLARY 1. *Let R be an AM-ring with few zero-divisors. If R' is an overring of R other than K then*

$$R' = \bigcap_{P \in \Delta} R_{S(P)}$$

where Δ is some set of regular maximal ideals.

Noetherian P-rings are called D-rings [2]. If P is a prime ideal of a D-ring then $\bigcap_{n=1}^{\infty} P^n = N(P)$. Hence we have

COROLLARY 2. *A Noetherian ring R is an AM-ring if and only if K is an AM-ring and R is a D-ring.*

Our next result gives information concerning the prime ideals of an overring of an AM-ring with few zero-divisors. It includes a strengthening of Corollary 1, and is a generalization of [3; thm. 4].

THEOREM 4. *Let R be an AM-ring with few zero-divisors. Let R' be an overring of R. Let Δ be the set of regular prime ideals P of R such that $PR' \neq R'$.*

(i) *If P' is a regular maximal ideal of R' and $P = R \cap P'$ then $R'_{S(P')} = R_{S(P)}$ and $P' = PR_{S(P)} \cap R'$.*

(ii) *If P is a regular prime ideal of R then $P \in \Delta$ if and only if $R' \subseteq R_{S(P)}$, and $R' = \bigcap_{P \in \Delta} R_{S(P)}$.*

(iii) *If A' is an ideal of R' and $R \cap A' = A$ then $A' = AR'$.*

(iv) *$\{PR'\}_{P \in \Delta}$ is the set of proper regular prime ideals of R'.*

PROOF. (i) follows from the proof of Theorem 2. The first part of (ii) is a consequence of [7; thms. 1 and 5]; the other part follows from [7; thm. 3] and (i). Let A' and A be as in the statement of (iii). To show that $A' = AR'$ it is sufficient to show that $A'R'_{S(P')} = (AR')R'_{S(P')}$ for an arbitrary regular maximal ideal P' of R' [7; lemma 1]. Let $P = R \cap P'$. By (i), $R' \subseteq R'_{S(P')} = R_{S(P)}$. Let

$a'/s' \in A'R'_{S(P')}$, $a' \in A'$, $s' \in S(P')$. Then $a'=a/u$ and $s'=s/v$ where $a, s \in R$ and $u, v \in S(P)$. Now, $a=a'u \in R \cap A'=A$ and $s=s'v \in S(P') \cap R \subseteq S(P)$. Hence $a'/s'=av/su \in AR_{S(P)}=AR'_{S(P')}=(AR')R'_{S(P')}$. Thus $A'R'_{S(P')} \subseteq (AR')R'_{S(P')}$; the other inclusion certainly holds. Now let Δ' be the set of proper regular prime ideals of R'. It follows from (iii) that $\Delta' \subseteq \{PR'\}_{P \in \Delta}$. If $P \in \Delta$ then $R' \subseteq R_{S(P)}$ and by (iii), $PR_{S(P)} \cap R'=(PR_{S(P)} \cap R' \cap R)R'=(PR_{S(P)} \cap R)R'=PR'$ is a regular prime ideal of R'.

3. Miscellaneous results

In this section we shall give several additional results concerning *AM*-rings.

Theorem 5. *A Noetherian ring is an AM-ring if and only if it is a multiplication ring.*

Proof. Let R be a Noetherian *AM*-ring. The zero ideal of R has no imbedded prime divisors. For, let P_1 and P_2 be prime divisors of the zero ideal and let Q_1 and Q_2 be the corresponding primary ideals in a reduced primary decomposition of the zero ideal. Suppose $P_1 \subset P_2$. By [1; thm. 2.12] $P_1=N(P_2)$ and $Q_1=P_1$. Also $Q_2=P_2^n$ for some positive integer n [1; thm. 2.3]. Hence $Q_1 \subset Q_2$ and Q_2 is redundant, a contraction. Since P is the only P-primary ideal of R when P is a non-maximal prime ideal [1; thm. 2.12], we have $0=P_1 \cap \cdots \cap P_s \cap Q_1 \cap \cdots \cap Q_t$, where the P_i are non-maximal prime ideals and the Q_i are primary ideals with maximal radicals. Since there are no inclusion relations between the radicals of these ideals, they are comaximal [1; thm. 2.12]. Hence $R=R/P_1 \oplus \cdots \oplus R/P_s \oplus R/Q_1 \oplus \cdots \oplus R/Q_t$. Since a homomorphic image of a Noetherian *AM*-ring is a Noetherian *AM*-ring, each R/P_i is a Dedekind domain and each R/Q_i is a special primary ring. Thus R is a multiplication ring (see [5; 51]). We have already noted that a multiplication ring is an *AM*-ring.

This result has been obtained independently by J. L. Mott (see Notices Amer. Math. Soc. 13 (1966) p. 825, thm. 4).

The next theorem is an extension of the result of [1; thm. 2.8]. It does not require that R have few zero-divisors, nor does the result following.

Theorem 6. *A ring R is an AM-ring if and only if for every proper primary ideal Q of R and for every maximal ideal M of R with $Q \subseteq M$, either Q is a power of M or $Q=N(M)$.*

Proof. Suppose R is an *AM*-ring, Q a proper primary ideal of R, rad $(Q)=P$, and $Q \subseteq M$ where M is a maximal ideal. Then $P \subseteq M$. If $N(P)$ is not a prime then $P=M$ and Q is a power of P by [1; thm. 2.2]. If $N(P)$ is a prime but $P \neq N(P)$ then $P=M$ and Q is a power of P by [1; thm 2.3]. If $P=N(P)$

then $Q=P$ by [1; thm. 2.4]; furthermore, $N(P)=N(M)$ by [1; thm. 2.12]. Conversely, if the condition of the theorem holds then R is an AM-ring by [1; thm. 2.8].

THEOREM 7. *Let R be an AM-ring in which every regular ideal is contained in only a finite number of maximal ideals. Then every regular ideal of R can be written as a product of maximal ideals.*

PROOF. Let A be a regular ideal of R and let P be a minimal prime divisor of A. By [1; thm. 2.12] P is a maximal ideal of R. Conversely, every maximal ideal of R containing A is a minimal prime divisor of A. Let $P_1, \cdots, P_n$ be the distinct maximal ideals of R which contain A; they are finite in number by hypothesis. Then rad $(A)=P_1 \cap \cdots \cap P_n = P_1 \cdots P_n$. Therefore, A is a product of powers of $P_1, \cdots, P_n$ by [1; thm. 2.10].

This result is an extension to AM-rings of the first conclusion of [3; thm. 3]; for AM-rings with few zero-divisors the second conclusion of that result of Gilmer has the following analogue.

THEOREM 8. *Let R be an AM-ring with few zero-divisors and with only a finite number of maximal ideals. Then every finitely generated regular ideal of R is principal.*

Since every finitely generated regular ideal of R has a finite set of regular generators it is sufficient to show that if a and b are regular elements of R then the ideal (a, b) is principal. If we consider $R_{S(M)}$ instead of R_M, for the maximal ideals M of R, the proof of [6; thm. 5] can be used.

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