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The Method of Orthogonal Decomposition for Differentials on Open Riemann Surfaces

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Introduction

In the investigation of harmonic differentials on open Riemann surfaces, L. Ahlfors introduced the method of orthogonal decomposition and proved its effectiveness. In reality, it is by this method that he established the existence and uniqueness of a harmonic differential which has preassigned singularities and periods and which is subject to a certain prescribed boundary behavior.

In the classical case of closed Riemann surfaces, one of the main problems is to construct harmonic differentials with given periods and singularities. When we try to generalize the classical results to open surfaces in a non-trivial manner, it becomes necessary to add some restrictive conditions. Our restrictions will not be imposed on the surfaces, but merely on the differentials that are brought under consideration. In fact, it seems natural to make restrictions on differentials so that they behave mildly near the ideal boundary.

L. Ahlfors introduced the following mode of boundary behavior:

"A harmonic differential ω whose only singularities are harmonic poles is said to be *distinguished* if

(1) there exist differentials $\omega_{hm} \in \Gamma_{hm}$, $\omega_{e0} \in \Gamma_{e0} \cap \Gamma^1$ such that $\omega = \omega_{hm} + \omega_{e0}$ outside of a compact set,

(2) ω^* has vanishing periods along all dividing cycles which lie outside of a sufficiently large compact set."

On the other hand, in order to describe the boundary behavior of harmonic functions, L. Sario introduced the linear operators $(P)L_1$ and L_0, which he called *principal operators*. He established the existence and uniqueness of a harmonic function which has preassigned singularities and the boundary behavior described by one of principal operators.

In L. Ahlfors and L. Sario [4], the above two methods, namely *the method of orthogonal decomposition* and *the method of linear operators* are described quite separately, and the relation between them is not touched. In this paper we shall show that the former method yields also the result obtained by the latter as stated above.

In order to prescribe boundary behavior of harmonic functions and differentials, we choose *an arbitrary closed linear subspace* Γ_χ *of* Γ_{he}. The proofs of our existence theorems depend solely on the following orthogonal decomposition:

$$\Gamma=\Gamma_{\chi}+\Gamma_{\chi}^{\perp}+\Gamma_{e0}+\Gamma_{e0}^{*}$$

where $\Gamma_{\chi}^{\perp}$ is the orthogonal complement of Γ_{χ} in Γ_{h}.

As for the notation and terminology, we follow L. Ahlfors and L. Sario [4]. In §0 some basic notions on differentials are briefly reviewed. In §1 we introduce the notion of *Γ_{χ}-behavior* and establish the existence and uniqueness of a harmonic function with preassigned singularities and Γ_{χ}-behavior. Defining Γ_{χ}-functions after Sario's principal functions, we express the reproducing kernels for periods or derivatives in some subspaces of Γ_{h} in terms of Γ_{χ}-functions and state the extremal properties of these kernels. For these investigations we are indebted to B. Rodin [10]. §2 is devoted to investigations of harmonic differentials having Γ_{χ}-behavior. In §3 we establish *a correspondence between the subspaces of* Γ_{he} *and the canonical operators* due to H. Yamaguchi [12], and we show, in particular, that Sario's principal operator method is included in our orthogonal decomposition method.

Finally in §4 we give *generalizations of the Riemann-Roch theorem and Abel's theorem of Kusunoki type* [5; 6; 7; 8]; cf. [10] too. We require that only the real parts of meromorphic functions and differentials have Γ_{χ}-behavior. We could restrict both real and imaginary parts to have Γ_{χ}-behavior and generalize the Riemann-Roch theorem as in H. Royden [11] and B. Rodin [10] and Abel's theorem as in L. Ahlfors [2]. However, since this condition seems to limit too strongly the class of surfaces on which the theory is meaningful, we shall not be concerned with such generalizations. See R. Accola [1] in this connection.

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§0. Preliminaries

0.1 The space $\Gamma(W)$

Let W be a Riemann surface, compact or not. Suppose that a differential ω of the first order on W has a local representation $\omega=adx+bdy$. Then the conjugate ω^* of ω is defined by $\omega^* = -b\,dx + a\,dy$. Note that $\omega^{**} = -\omega$.

To say that ω is square integrable means that the local coefficients a and b are Lebesgue measurable and that

$$\int_W \omega \wedge \bar{\omega}^* = \int_W (|a|^2 + |b|^2)\,dx\,dy < \infty.$$

The non-negative square root of this integral is called the Dirichlet norm of ω and is denoted by $\|\omega\|$.

For a pair of square integrable differentials $\omega_1 = a_1 dx + b_1 dy$, $\omega_2 = a_2 dx + b_2 dy$, the inner product (ω_1, ω_2) is defined by

$$(\omega_1, \omega_2) = \int_W \omega_1 \wedge \bar{\omega}_2^* = \int_W (a_1 \bar{a}_2 + b_1 \bar{b}_2)\,dx\,dy.$$

Note that $(\omega_1^*, \omega_2^*) = (\omega_1, \omega_2)$.

Two differentials are identified if their coefficients differ only on a set of measure zero in each local coordinate. With this convention, the space of all *real* (resp. *complex*) differentials with finite norm becomes a separable Hilbert space, which we denote by $\Gamma(W)$ (resp. $\Lambda(W)$).

If A is a subset of Γ, then A^* indicates the set of differentials whose conjugates are in A.

0.2 Weyl's lemma

First we list below some important subspaces of Γ.

Γ^∞: C^∞-differentials.

Γ_e^∞ (resp. Γ_c^∞): exact (resp. closed) C^∞-differentials.

Γ_e (resp. Γ_c): the closure in Γ of Γ_e^∞ (resp. Γ_c^∞).

The relations $\Gamma_e \cap \Gamma^\infty = \Gamma_e^\infty$ and $\Gamma_c \cap \Gamma^\infty = \Gamma_c^\infty$ are valid, but require non-trivial proofs.

Γ_{e0}^∞: $\{df\}$, where f is a C^∞-function with compact support.

Γ_{e0}: the closure in Γ of Γ_{e0}^∞.

Γ_h: harmonic differentials.

Now, Weyl's lemma: $\Gamma_c \cap \Gamma_c^* = \Gamma_h$, together with the well-known orthogonal decomposition: $\Gamma = \Gamma_c + \Gamma_{e0}^* = \Gamma_c^* + \Gamma_{e0}$, implies the following important orthogonal decompositions:

$$\Gamma = \Gamma_h + \Gamma_{e0} + \Gamma_{e0}^*, \quad \Gamma_c = \Gamma_h + \Gamma_{e0}.$$

0.3 Some important subspaces of Γ_h

Γ_{hse}: semi-exact harmonic differentials; these are differentials whose periods along dividing cycles are all zero.

Γ_{hm}: the orthogonal complement in Γ_h of Γ^*_{hse}.

More generally, we define for an arbitrary regular partition P of the ideal boundary of W. (L. Ahlfors and L. Sario [4], Ch. V, 15G)

$(P)\Gamma_{hse}$: harmonic differentials whose periods along P-dividing cycles are all zero.

$(P)\Gamma_{hm}$: the orthogonal complement in Γ_h of $(P)\Gamma^*_{hse}$.

Among regular partitions, the identical partition I and the canonical partition Q are the most important. Note that $\Gamma_{hse}=(Q)\Gamma_{hse}$ and $\Gamma_{hm}=(Q)\Gamma_{hm}$.

Γ_{he}: exact harmonic differentials.

On account of the decomposition $\Gamma_c=\Gamma_h+\Gamma_{e0}$, we have

$$\Gamma_e = \Gamma_{he}+\Gamma_{e0}.$$

Γ_{h0}: the orthogonal complement in Γ_h of Γ^*_{he}.

By definition we have the following orthogonal decompositions:

$$\Gamma_h = \Gamma_{he}+\Gamma^*_{h0} = \Gamma^*_{he}+\Gamma_{h0},$$

$$\Gamma_h = \Gamma_{hm}+\Gamma^*_{hse} = \Gamma^*_{hm}+\Gamma_{hse}.$$

$\Gamma_{he}\subset\Gamma_{hse}$ implies $\Gamma_{h0}\supset\Gamma_{hm}$. On the other hand, the inclusion relation $\Gamma_{hm}\subset\Gamma_{he}(\Leftrightarrow\Gamma_{hse}\supset\Gamma_{h0})$ is well-known. Hence.

$$\Gamma_{hm}\subset\Gamma_{he}\cap\Gamma_{h0}.$$

0.4 Extension lemma for closed differentials

The following lemma, which is substantially due to H. Yamaguchi, plays an important role later.

LEMMA 1. *Let W be a Riemann surface and Ω be a regularly imbedded connected subregion of W such that the relative boundary $\partial\Omega$ is compact. Set $V=W-\overline{\Omega}$. Let σ be a closed C^∞-differential on a neighborhood of $\overline{V}$. Then, in order that $\sigma|V$ can be extended to a closed C^∞-differential $\hat{\sigma}$ on W such that* (Supp $\hat{\sigma}$)$\cap\overline{\Omega}$ *is compact, it is necessary and sufficient that*

$$\int_{\partial\Omega}\sigma = 0.$$

PROOF. The necessity is obvious. To show the sufficiency we proceed by induction on the number of the contours of $\overline{\Omega}$. Set $\partial\Omega=\sum_{k=1}^{n}c_k$, where c_k are mutually disjoint analytic Jordan curves.

In case $n=1$, take a C^∞-function u such that $du=\sigma$ in a neighborhood of

c_1. Extend u to Ω so that $u \in C^\infty(\overline{\Omega})$ and $(\operatorname{Supp} u) \cap \overline{\Omega}$ is compact. Then define $\hat{\sigma}$ as follows:

$$\hat{\sigma} = \sigma \quad \text{on} \quad \overline{V}, \qquad \hat{\sigma} = du \quad \text{on} \quad \Omega.$$

In case $n \geqq 2$, take a quadrilateral subregion R of Ω such that one pair of opposite sides consists of subarcs of c_{n-1} and c_n, and that the other pair of opposite sides consists of arcs in Ω. In a neighborhood of $\bar{R}$, take a C^∞-function u such that $du = \sigma$ in a neighborhood of $\partial R \cap \partial \Omega$. Set $\Omega_1 = \Omega - \bar{R}$, $V_1 = W - \overline{\Omega}_1$ and define σ_1 on $\overline{V}_1$ as follows:

$$\sigma_1 = \sigma \quad \text{on} \quad \overline{V}, \qquad \sigma_1 = du \quad \text{on} \quad \bar{R}.$$

Then the number of contours of Ω_1 is $n-1$, and

$$\int_{\partial \Omega_1} \sigma_1 = \int_{\partial \Omega} \sigma - \int_{\partial R} du = 0.$$

We have thus completed the reduction process and consequently our proof.

0.5 Definition

By a neighborhood of the ideal boundary of W, we understand the complement of a compact subset of W. Consider a neighborhood V of the ideal boundary of W, which satisfies the following conditions:

(i) *V is regularly imbedded,*

(ii) *each component of V is not relatively compact,*

(iii) *$W - \overline{V}$ is non-empty and connected.*

We shall denote the set of all such V's by $\mathfrak{S}(W)$.

Finally we introduce a standard notation. Let ω be a C^1-differential of the first order defined in a neighborhood of the ideal boundary of W. Let Ω denote a generic, relatively compact, regularly imbedded subregion of W. In the case that

$$\lim_{\Omega \uparrow W} \int_{\partial \Omega} \sigma$$

exists, we denote this limit by $\int_\beta \omega$. Here β stands for "the ideal boundary" of W.

0.6 The space Γ_χ

We choose an arbitrary closed linear subspace of $\Gamma_{he}(W)$ once for all and denote it by Γ_χ throughout this paper. We denote by $\Gamma_\chi^\perp$ the orthogonal complement in Γ_h of Γ_χ. Note the implications:

$$\Gamma_X \subset \Gamma_{he} \Leftrightarrow \Gamma_X^\perp \supset \Gamma_{h0}^* \Leftrightarrow \Gamma_X^{\perp *} \supset \Gamma_{h0}$$

and orthogonal decompositions:

$$\Gamma_h = \Gamma_X + \Gamma_X^\perp, \quad \Gamma = \Gamma_X + \Gamma_X^\perp + \Gamma_{e0} + \Gamma_{e0}^*.$$

It will be convenient to introduce three more spaces as follows:

$$H_X(W) = \{u \in HD(W) \colon du \in \Gamma_X\}, \quad D^\infty(W) = \{u \in C^\infty(W);\ du \in \Gamma\},$$

$$D_0^\infty(W) = \{u \in C^\infty(W) \colon du \in \Gamma_{e0}\}.$$

Naturally $H_{X_1}(W) \subset H_{X_2}(W)$ if $\Gamma_{X_1} \subset \Gamma_{X_2}$.

It should be noted that the orthogonal decomposition $\Gamma_e^\infty = \Gamma_{he} + (\Gamma_{e0} \cap \Gamma^\infty)$, or equivalently $D^\infty(W) = HD(W) + D_0^\infty(W)$ is a special case of the so-called *Royden decomposition*.

§1. Harmonic functions

1.1 Γ_X-behavior

DEFINITION. *Let u be a single-valued real harmonic function defined in a neighborhood of the ideal boundary of W. Suppose that u and $(du)^*$ admit the following representations in a neighborhood of the ideal boundary of W:*

$$u = u_X + u_{e0}, \qquad \text{where} \quad u_X \in H_X(W) \quad \text{and} \quad u_{e0} \in D_0^\infty(W),$$

$$(du)^* = \omega_{X^\perp}^* + \omega_{e0}, \qquad \text{where} \quad \omega_{X^\perp} \in \Gamma_X^\perp \quad \text{and} \quad \omega_{e0} \in \Gamma_{e0}.$$

Then we say that u has Γ_X-behavior.

REMARK. In the above representations, the component u_X is uniquely determined up to an additive constant. On the contrary, the component $\omega_{X^\perp}^*$ is determined only modulo a subspace (not necessarily closed) of Γ_{h0}.

PROPOSITION 1. *Let $V \in \mathfrak{S}(W)$ and let u be a C^∞-function in a neighborhood of $\bar{V}$. Then the following two statements are equivalent.*

(1) *u is Dirichlet finite in V and*

$$\int_\beta u\omega^* = 0 \qquad \left(\Leftrightarrow\ (du, \omega)_V = \int_{\partial V} u\omega^*\right)$$

for all $\omega \in \Gamma_X^\perp$.

(2) *There exist $u_X \in H_X(W)$ and $u_{e0} \in D_0^\infty(W)$ such that u is represented on V as follows:*

$$u = u_X + u_{e0}.$$

PROOF. (1)$\Rightarrow$(2). Extend $u|V$ to W to be a C^∞-function. We shall denote the extension by $\hat{u}$. Let $\hat{u} = u_1 + u_{e0}$ be the Royden decomposition of $\hat{u}$. Then by assumption, we have

$$0=\int_{\beta} u\omega^{*}=(du_1+du_{e0},\ \omega)=(du_1,\ \omega)$$

for all $\omega \in \Gamma_{\chi}^{\perp}$. Hence, $du_1 \in \Gamma_{\chi}$.

The converse part (2)$\Rightarrow$(1) is trivial.

Proposition 2. *Let $V \in \mathfrak{S}(W)$ and let ω be a C^{∞}-differential in a neighborhood of $\bar{V}$. Then the following two statements are equivalent.*

(1) *ω is closed and sguare integrable in V and*

$$\int_{\beta} v\omega=0 \qquad (\Leftrightarrow (dv,\ \omega^{*})_V=-\int_{\partial V} v\omega)$$

for all $v \in H_{\chi}$.

(2) *There exist $\omega_{\chi^{\perp}} \in \Gamma_{\chi}^{\perp}$ and $\omega_{e0} \in \Gamma_{e0}$ such that ω is represented on V as follows:*

$$\omega=\omega_{\chi^{\perp}}^{*}+\omega_{e0}.$$

Proof. (1)$\Rightarrow$(2). Since $1 \in H_{\chi}$, the assumption implies $0=\int_{\beta}\omega=-\int_{\partial V}\omega$. Hence, in virtue of Lemma 1, we can extend $\omega|V$ to W to be a closed C^{∞}-differential. We denote this extension by $\hat{\omega}$. Since ω is square integrable near the ideal boundary, $\hat{\omega} \in \Gamma_{c}^{\infty}(W)$. Here we use the orthogonal decomposition $\Gamma_c=\Gamma_h+\Gamma_{e0}$ to obtain

$$\hat{\omega}=\omega_1+\omega_{e0} \qquad \text{with} \quad \omega_1 \in \Gamma_h, \quad \omega_{e0} \in \Gamma_{e0}.$$

Then, by assumption,

$$0=\int_{\beta} v\hat{\omega}=\int_{\beta} v(\omega_1+\omega_{e0})=(dv,\ -\omega_1^{*}-\omega_{e0}^{*})=-(dv,\ \omega_1^{*})$$

for all $dv \in \Gamma_{\chi}$. Hence, $\omega_1^{*} \in \Gamma_{\chi}^{\perp}$.

The converse part (2)$\Rightarrow$(1) is trivial.

As an immediate consequence of Propositions 1 and 2, we obtain the following notable

Corollary. *Let $V \in \mathfrak{S}(W)$. Suppose u is harmonic on $\bar{V}$ and has Γ_{χ}-behavior. Then u and $(du)^{*}$ admit such representations on V as stated in the Definition.*

Proposition 3. *Constant functions have Γ_{χ}-behavior. If u has Γ_{χ}-behavior, then*

$$\int_{\beta}(du)^{*}=0.$$

More generally, if both u_1 and u_2 have Γ_{χ}-behavior, then

$$\int_{\beta} u_1(du_2)^* = 0.$$

PROOF. Since u_1 and $(du_2)^*$ are represented in the forms

$$u_1 = u_\chi + u_{e0}, \quad (du_2)^* = \omega^*_{\chi^\perp} + \omega_{e0}$$

near the ideal boundary, we have, for a sufficiently large regular subregion Ω of W,

$$\int_{\partial\Omega} u_1(du_2)^* = \int_{\partial\Omega} (u_\chi + u_{e0})(\omega^*_{\chi^\perp} + \omega_{e0})$$

$$= (du_\chi + du_{e0}, \omega_{\chi^\perp} - \omega^*_{e0})_\Omega.$$

The last term tends to $(du_\chi + du_{e0}, \omega_{\chi^\perp} - \omega^*_{e0})_W = 0$ as $\Omega \uparrow W$.

From this proposition we obtain

UNIQUENESS THEOREM. *If a harmonic function on W has Γ_χ-behavior, it is constant.*

1.2 Harmonic functions with preassigned singularities and Γ_χ-behavior

First we prove

LEMMA 2. *Let K be a compact subset of W and u be a C^∞-function in $W-K$ which vanishes identically near the ideal boundary of W. Suppose there is a closed C^∞-differential ω in $W-K$ such that $du+\omega^*$ vanishes identically near K and near the ideal boundary of W. Then there exists a harmonic function $\hat{u}$ in $W-K$ with Γ_χ-behavior such that $\hat{u}-u$ is Dirichlet finite.*

PROOF. Extend $du+\omega^*$ to W by 0 on K. Then $du+\omega^* \in \Gamma^\infty(W)$. We use the orthogonal decomposition

$$\Gamma = \Gamma_\chi + \Gamma^\perp_\chi + \Gamma_{e0} + \Gamma^*_{e0}$$

to obtain

$$du + \omega^* = \omega_\chi + \omega_{\chi^\perp} + \omega^{(1)}_{e0} + \omega^{(2)*}_{e0}.$$

On rewriting the equation in the form

$$du - \omega_\chi - \omega^{(1)}_{e0} = -\omega^* + \omega_{\chi^\perp} + \omega^{(2)*}_{e0},$$

we find that the differential on the left is closed and the differential on the right is coclosed in $W-K$. Hence $du-\omega_\chi-\omega^{(1)}_{e0}$ is harmonic in $W-K$. Set $\omega_\chi = du_\chi$, $\omega^{(1)}_{e0} = du_{e0}$ and $\hat{u} = u - u_\chi - u_{e0}$. Now it is obvious that $\hat{u}$ has the required properties.

Now we establish

THEOREM 1. *Suppose that at a finite number of points $p_j \in W$ there are* *given harmonic singularities of the form*

$$s_j = \operatorname{Re}\Big(\sum_{n=1}^{\infty} \frac{c_n^{(j)}}{z_j^n}\Big) + a_j \log|z_j|$$

where a_j is real and z_j is a local parameter near p_j such that $z_j(p_j)=0$. Then, in order that there exist a function u with Γ_χ-behavior which is harmonic on W except at $\{p_j\}$ and for which $u-s_j$ is harmonic at p_j for each j, it is necessary and sufficient that

$$\sum a_j = 0.$$

The function u is uniquely determined up to an additive constant.

PROOF. The necessity is obvious. To show the sufficiency, choose $r>0$ so small that

$$\sum_n \frac{c_n^{(j)}}{z_j^n}$$

converges in the punctured disk: $\{0<|z_j|<2r\}$ for every j and closed disks $\overline{\Delta}_j$: $\{|z_j|\leqq r\}$ are mutually disjoint. Set $V=\cup\Delta_j$ and define a singularity function s on $\overline{V}$ by setting $s=s_j$ on Δ_j. Extend s to W so that the extension is infinitely differentiable except at $\{p_j\}$ and vanishes in a connected neighborhood V of the ideal boundary. Denote this extension by $\hat{s}$.

On the other hand, by our assumption we have

$$\int_{\partial V} (ds)^* = 2\pi \sum a_j = 0.$$

Hence, in virtue of Lemma 1, we can extend $(ds)^*$ to W so that the extension is a closed C^∞-differential on $W-\{p_j\}$ and the closure in W of its support is compact. Denote this extension by σ.

Then $d\hat{s}+\sigma^*$ is identically zero on V and near the ideal boundary. Lemma 2 is now applied and the existence of u is shown. Since $u-s_j$ is harmonic except at p_j and square integrable, p_j is a removable singularity for $u-s_j$. By the uniqueness theorem in 1.1, u is unique up to an additive constant.

DEFINITION. *We shall say that u has singularity s_j at p_j. A function which is harmonic on W except for a finite number of isolated singularities such as $\{s_j\}$ and has Γ_χ-behavior will be called a Γ_χ-function.*

1.3 The functions P_{χ,p_1,p_2} and $P_{\chi,p}^{(n)}$

Let p_j, $j=1, 2$, be two distinct points of W and let z_j be a local parameter near p_j such that $z_j(p_j)=0$. We denote by P_{χ,p_1,p_2} a Γ_χ-function which has singularity $(-1)^j \log|z_j|$ at p_j, $j=1, 2$. In case $\Gamma_\chi=\{0\}$, we write P_{0,p_1,p_2} for P_{χ,p_1,p_2}.

Let $p \in W$ and z be a local parameter near p such that $z(p)=0$. Denote by $P_{\chi,p}^{(n)}$ a Γ_{χ}-function which has $\mathrm{Re}(1/z^{n})$ as singularity at p, where n is a natural number. We note that $P_{\chi,p}^{(n)}$ depends on the particular choice of a local parameter at p. In case $\Gamma_{\chi}=\{0\}$, we write $P_{0,p}^{(n)}$ for $P_{\chi,p}^{(n)}$.

We shall write simply $P_0^{(0)}$, $P_{\chi}^{(0)}$ for P_{0,p_1,p_2}, P_{χ,p_1,p_2} respectively, and also $P_0^{(n)}$, $P_{\chi}^{(n)}$ for $P_{0,p}^{(n)}$, $P_{\chi,p}^{(n)}$ respectively.

THEOREM 2. *It holds that*

$$P_0^{(n)}-P_{\chi}^{(n)} \in H_{\chi} \qquad (n=0, 1, \cdots)$$

and

$$(du,\ d(P_0^{(n)}-P_{\chi}^{(n)}))=\begin{cases} 2\pi\{u(p_2)-u(p_1)\} & (n=0) \\ -\dfrac{2\pi}{(n-1)!}\dfrac{\partial^n u}{\partial x^n}(0) & (n=1, 2, \cdots) \end{cases}$$

for all $u \in H_{\chi}$.

The function $P=P_0^{(0)}-P_{\chi}^{(0)}$ (resp. $P=P_0^{(n)}-P_{\chi}^{(n)}$, $n\geqq 1$) *minimizes the functional*

$$\|du\|^2-4\pi\{u(p_2)-u(p_1)\} \qquad \left(\text{resp. } \|du\|^2-\frac{2\pi}{(n-1)!}\frac{\partial^n u}{\partial x^n}(0)\right)$$

on the space H_{χ}. *The minimum is* $-\|dP\|^2$, *and the deviation from the minimum is* $\|du-dP\|^2$.

PROOF. Let $\varepsilon>0$ be so small that parametric disks Δ_j: $\{|z_j|<\varepsilon\}$, $j=1, 2$, are disjoint. Let $u \in H_{\chi}$. Then, by Proposition 1, we have

$$\begin{aligned}(du,\ dP_0^{(0)})_{W-\Delta_1-\Delta_2} &= \int_{\partial\Delta_1}\log|z_1|(du)^*-\int_{\partial\Delta_2}\log|z_2|(du)^*+O(\varepsilon) \\ &= O(\varepsilon)\to 0 \qquad \text{as } \varepsilon\downarrow 0,\end{aligned}$$

and, by Proposition 2,

$$\begin{aligned}-(du,\ dP_{\chi}^{(0)})_{W-\Delta_1-\Delta_2} &= \int_{\partial\Delta_1}u(dP_{\chi}^{(0)})^*+\int_{\partial\Delta_2}u(dP_{\chi}^{(0)})^* \\ &= -\int_{\partial\Delta_1}u\,d\arg z_1+\int_{\partial\Delta_2}u\,d\arg z_2+O(\varepsilon)\to 2\pi\{u(p_2)-u(p_1)\} \qquad \text{as } \varepsilon\downarrow 0.\end{aligned}$$

By addition we obtain $(du,\ d(P_0^{(0)}-P_{\chi}^{(0)}))=2\pi\{u(p_2)-u(p_1)\}$.

To prove the second equality, let Δ be a parametric disk $\{|z|<\varepsilon\}$. Let $u \in H_{\chi}$. Then, for $n\geqq 1$,

$$(du,\ dP_0^{(n)})_{W-\Delta}=-\int_{\partial\Delta}P_0^{(n)}(du)^*=\int_{\partial\Delta}u^*dP_0^{(n)}=\int_{\partial\Delta}u^*d\left(\mathrm{Re}\frac{1}{z^n}\right)+O(\varepsilon),$$

and

$$-(du,\ dP_X^{(n)})_{W-\Delta}=\int_{\partial\Delta}u\Big(dP_X^{(n)}\Big)^*=\int_{\partial\Delta}u\,d\Big(\mathrm{Im}\frac{1}{z^n}\Big)+O(\varepsilon).$$

By addition

$$\begin{aligned}(du,\ d(P_0^{(n)}-P_X^{(n)}))_{W-\Delta}&=\int_{\partial\Delta}\Big\{u^*d\Big(\mathrm{Re}\frac{1}{z^n}\Big)+u\,d\Big(\mathrm{Im}\frac{1}{z^n}\Big)\Big\}+O(\varepsilon)\\ &=\mathrm{Im}\int_{\partial\Delta}(u+iu^*)d\Big(\frac{1}{z^n}\Big)+O(\varepsilon)\\ &=\mathrm{Im}\Big[(-n)\frac{2\pi i}{n!}\Big\{\frac{\partial^n u}{\partial x^n}(0)+i\frac{\partial^n u^*}{\partial x^n}(0)\Big\}\Big]+O(\varepsilon)\\ &\to-\frac{2\pi}{(n-1)!}\frac{\partial^n u}{\partial x^n}(0)\qquad\text{as}\quad\varepsilon\downarrow 0.\end{aligned}$$

In the following corollaries, $n=0,1,\cdots$.

COROLLARY 1. *Let p_1 be an arbitrary point of W. If p_2 runs through all points near p_1 (in case $n=0$) or if p runs through all points of a non-empty open subset of W (in case $n\geqq 1$), then*

$$d(P_0^{(n)}-P_X^{(n)})$$

span Γ_X.

PROOF. Assume

$$(du,\ d(P_0^{(n)}-P_X^{(n)}))=0.$$

This assumption does not depend on the choice of a parameter with respect to which a singularity is given. First consider the case $n=0$. By Theorem 2 u is constant near p_1 and hence in W. Therefore $du=0$, and it is shown that $d(P_0^{(0)}-P_X^{(0)})$ span Γ_X.

Next let $n\geqq 1$. Suppose p runs through all points of a disk Δ: $\{|z|<r\}$. By Theorem 2 $\partial^n u/\partial x^n=0$ in Δ. Let v be a conjugate harmonic function of u in Δ and set $f=u+iv$. Develop f into

$$f(z)=a_0+a_1z+\cdots\qquad\text{in}\quad\Delta.$$

Since $d^nf/dz^n=\partial^n u/\partial x^n+i\partial^n v/\partial x^n=i\partial^n v/\partial x^n$ in Δ and hence is constant, $f(z)$ has the form

$$f(z)=a_0+\cdots+a_mz^m,\qquad 0\leqq m\leqq n.$$

The same is true for any other local parameter. If $a_m\neq 0$ for $m\geqq 1$, consider the parameter ζ defined by $z=\zeta+\zeta^{n+1}$. For a sufficiently small positive ε, this defines a one-to-one conformal mapping of Δ_ζ: $\{|\zeta|<\varepsilon\}$ into Δ. We have

$$f(\zeta+\zeta^{n+1})=a_0+\cdots+a_m\zeta^{m(n+1)}\qquad\text{in}\quad\Delta_\zeta.$$

This is impossible because $\deg f = m(n+1) > n$. Therefore f and hence u is constant. It follows that $d(P_0^{(n)} - P_X^{(n)})$ span Γ_X.

COROLLARY 2. *Let Γ_{X_1} and Γ_{X_2} be closed linear subspaces of Γ_{he}. If $P_{X_1}^{(n)} = P_{X_2}^{(n)}$ for all such pairs (p_1, p_2) as described in Corollary 1 (in case $n=0$) or for all p in a non-empty open subset of W (in case $n \geqq 1$), then $\Gamma_{X_1} = \Gamma_{X_2}$.*

In particular, if any function with Γ_{X_1}-behavior has Γ_{X_2}-behavior, then $\Gamma_{X_1} = \Gamma_{X_2}$.

COROLLARY 3. *Let $\Gamma_{X_1} \subset \Gamma_{X_2}$ be two closed linear subspaces of Γ_{he}. Then*

$$P_{X_1}^{(n)} - P_{X_2}^{(n)} \in H_{X_1}^{\perp} \cap H_{X_2}$$

and

$$(du,\ d(P_{X_1}^{(n)} - P_{X_2}^{(n)})) = \begin{cases} 2\pi\{u(p_2) - u(p_1)\} & (n=0) \\ -\dfrac{2\pi}{(n-1)!}\dfrac{\partial^n u}{\partial x^n}(0) & (n=1, 2, \cdots) \end{cases}$$

for all $u \in H_{X_1}^{\perp} \cap H_{X_2}$.

The function $P_{X_1}^{(n)} - P_{X_2}^{(n)}$ has an extremal property similar to the one in the theorem.

1.4 The functions $Q_{X,c}$ and $Q_{X,p}^{(n)}$

Let c be a simple arc on W and put $\partial c = p_2 - p_1$. Take a parametric disk Δ: $\{|z| < 1\}$ which contains c. Set $\zeta_j = z(p_j)$, $j = 1, 2$. Consider the function

$$v(z) = \arg(z - \zeta_2) - \arg(z - \zeta_1)$$

in $\Delta - c$, and extend it to $W - c$ to obtain a C^∞-function which is identically zero outside a concentric compact disk. We denote the extension by $\hat{v}$. Then $d\hat{v}$ is a C^∞-differential on $W - \{p_1, p_2\}$.

Next consider the function

$$u(z) = \log|z - \zeta_2| - \log|z - \zeta_1|$$

in $\Delta - \{p_1, p_2\}$, and extend it to $W - \{p_1, p_2\}$ to obtain a C^∞-function which vanishes identically near the ideal boundary. We denote the extension by $\hat{u}$.

Then $d\hat{v} - (d\hat{u})^*$ vanishes identically in Δ and near the ideal boundary. By Lemma 2 we obtain a differential ω in $W - \{p_1, p_2\}$ which has the following properties:

(i) ω is harmonic on $W - \{p_1, p_2\}$ and $\omega - dv$ is square integrable in Δ,

(ii) ω is the differential of a harmonic function in $W - c$ with Γ_X-behavior.

We shall denote the harmonic function in (ii) by $Q_{X,c}$ and ω by $dQ_{X,c}$.

By (i) and (ii) $Q_{X,c}$ is uniquely determined up to an additive constant. $Q_{X,c}$ does not depend on the choice of a parameter z, and $dQ_{X,c}$ depends merely on the homotopy class of c with fixed end points p_1 and p_2.

Let $p \in W$ and let z be a local parameter near p such that $z(p)=0$. We denote by $Q_{X,p}^{(n)}$ a Γ_X-function which has $\operatorname{Im} z^{-n}$ as singularity. Here n is a natural number. It should be noted that $Q_{X,p}^{(n)}$ depends on the particular choice of a local parameter at p. In case $\Gamma_X=\{0\}$, we write $Q_{0,p}^{(n)}$ for $Q_{X,p}^{(n)}$.

Writing simply $Q_0^{(0)}$, $Q_X^{(0)}$, $Q_0^{(n)}$, $Q_X^{(n)}$, $n\geqq 1$, for $Q_{0,c}$, $Q_{X,c}$, $Q_{0,p}^{(n)}$, $Q_{X,p}^{(n)}$ respectively, we derive

THEOREM 4. $\qquad dP_0^{(n)}+(dQ_X^{(n)})^* \in \Gamma_X^{\perp *} \qquad (n=0, 1, \cdots)$

and

$$(\omega,\ dP_0^{(n)}+(dQ_X^{(n)})^*)=\begin{cases} 2\pi\int_c \omega & (n=0) \\ -\dfrac{2\pi}{(n-1)!}\dfrac{\partial^n u}{\partial x^n}(0) & (n=1, 2, \cdots) \end{cases}$$

for all $\omega \in \Gamma_X^{\perp *}$, *where* $\omega=du$ *near* p.

PROOF. Draw two sufficiently small disks Δ_j: $\{|z-\zeta_j|<\varepsilon\}$, $j=1, 2$. Consider a component of $c\frown(\Delta-\Delta_1-\Delta_2)$ which connects Δ_1 and Δ_2, and denote it by c_ε. Then for any $\omega \in \Gamma_X^{\perp *}$, we have by Proposition 1

$$\begin{aligned}(\omega,\ (dQ_X^{(0)})^*)_{W-\Delta_1-\Delta_2} &= -(dQ_X^{(0)}, \omega^*)_{W-\Delta_1-\Delta_2} \\ &= \int_{c_\varepsilon^+ + c_\varepsilon^- - \partial\Delta_1 - \partial\Delta_2} Q_X^{(0)}\omega \to 2\pi\int_c \omega \qquad \text{as } \varepsilon\downarrow 0.\end{aligned}$$

It was already shown in the proof of Theorem 1 that

$$(\omega,\ dP_0^{(0)})_{W-\Delta_1-\Delta_2} \to 0 \qquad \text{as} \quad \varepsilon\downarrow 0$$

for any $\omega \in \Gamma_h$. The first equality follows immediately.

To prove the second equality, let Δ denote the parametric disk $\{|z|<\varepsilon\}$. Then

$$\begin{aligned}(\omega,\ dP_0^{(n)}+(dQ_X^{(n)})^*)_{W-\Delta} &= (\omega,\ dP_0^{(n)})_{W-\Delta}+(\omega^*,\ -dQ_X^{(n)})_{W-\Delta} \\ &= \int_\beta P_0^{(n)}\omega^* - \int_{\partial\Delta} P_0^{(n)}\omega^* + \int_\beta Q_X^{(n)}\omega - \int_{\partial\Delta} Q_X^{(n)}\omega \\ &= \int_{\partial\Delta} u^* dP_0^{(n)} + \int_{\partial\Delta} u\, dQ_X^{(n)} \\ &= \int_{\partial\Delta} u^* d\left(\operatorname{Re}\frac{1}{z^n}\right) + \int_{\partial\Delta} u\, d\left(\operatorname{Im}\frac{1}{z^n}\right) + O(\varepsilon) \\ &= \operatorname{Im}\int_{\partial\Delta}(u+iu^*)d\left(\frac{1}{z^n}\right)+O(\varepsilon)\end{aligned}$$

$$= \mathrm{Im}\left[(-n)\frac{2\pi i}{n!}\left\{\frac{\partial^n u}{\partial x^n}(0)+i\frac{\partial^n u^*}{\partial x^n}(0)\right\}\right]+O(\varepsilon)$$

$$\to -\frac{2\pi}{(n-1)!}\frac{\partial^n u}{\partial x^n}(0) \qquad \text{as} \quad \varepsilon \downarrow 0.$$

COROLLARY 1. *If c runs through all arcs in Δ which have p_1 as the initial point (in case $n=0$) or if p runs through all points of a non-empty open subset of W (in case $n \geqq 1$), then*

$$dP_0^{(n)}+(dQ_X^{(n)})^*$$

span $\Gamma_X^{\perp *}$.

COROLLARY 2. *Let Γ_{X_1} and Γ_{X_2} be closed linear subspaces of Γ_{he}. If $Q_{X_1}^{(n)}=Q_{X_2}^{(n)}$ for all arcs c as described in Corollary 1 (in case $n=0$) or all points p in an non-empty open subset of W (in case $n \geqq 1$), then $\Gamma_{X_1}=\Gamma_{X_2}$.*

COROLLARY 3. *Let Γ_{X_1} and Γ_{X_2} be closed linear subspaces of Γ_{he} such that $\Gamma_{X_1} \subset \Gamma_{X_2}^{\perp *}$. Then*

$$dP_{X_1}^{(n)}+(dQ_{X_2}^{(n)})^* \in \Gamma_{X_1}^{\perp} \cap \Gamma_{X_2}^{\perp *},$$

and

$$(\omega,\ dP_{X_1}^{(n)}+(dQ_{X_2}^{(n)})^*)=\begin{cases} 2\pi\int_c \omega & (n=0) \\ -\dfrac{2\pi}{(n-1)!}\dfrac{\partial^n u}{\partial x^n}(0) & (n=1,2,\cdots) \end{cases}$$

for all $\omega \in \Gamma_{X_1}^{\perp} \cap \Gamma_{X_2}^{\perp *}$, where $\omega = du(z)$ near p.

REMARK. We have considered only "Γ_X-functions" with simple singularities. However, it should be noted that [4], Ch. III, Theorem 9E (Sario's main theorem for principal functions) can be similarly generalized.

§2. Harmonic differentials

2.1 Γ_X-behavior

A differential will be said to have Γ_X-behavior if it coincides with du in a neighborhood of the ideal boundary, where u is a real harmonic function with Γ_X-behavior.

UNIQUENESS THEOREM. *In case $\Gamma_X \supset \Gamma_{hm}$ ($\Leftrightarrow \Gamma_X^{\perp *} \subset \Gamma_{hse}$), if $\omega \in \Gamma_{he}(W)$ has Γ_X-behavior, then ω is identically zero.*

PROOF. Let $\omega = \omega_X + \omega_{e0}^{(1)}$ and $\omega^* = \omega_{X^\perp}^* + \omega_{e0}^{(2)}$ with $\omega_X \in \Gamma_X$, $\omega_X^{\perp} \in \Gamma_X^{\perp}$ and $\omega_{e0}^{(1)}, \omega_{e0}^{(2)} \in \Gamma_{e0}$. Since ω, ω_X and $\omega_{e0}^{(1)}$ are exact, we can set $\omega = du$, $\omega_X = du_X$ and

$\omega_{e0}^{(1)} = du_{e0}$. Set $v = u - u_X - u_{e0}$. Take a sufficiently large regular subregin Ω of W. Then v is constant in each component of $W - \Omega$.

If $\Gamma_X \supset \Gamma_{hm}$, ω^* is semi-exact. Hence

$$\int_{\partial\Omega} v\omega^* = 0, \quad \text{i.e.,} \quad \int_{\partial\Omega} u\omega^* = \int_{\partial\Omega} (u_X + u_{e0})\omega^*.$$

Thus

$$(\omega, \omega)_\Omega = \int_{\partial\Omega} u\omega^* = (\omega_X + \omega_{e0}^{(1)}, \omega_{X^\perp} - \omega_{e0}^{(2)*})_\Omega \to 0 \qquad \text{as} \quad \Omega \uparrow W.$$

In case $\Gamma_X \not\supset \Gamma_{hm}$ the uniqueness theorem does not hold. This will be shown in 2.2.

2.2 Period reproducing differentials

THEOREM 4. *Let c be a cycle on W. Then, there exists a unique differential $\sigma_X(c) \in \Gamma_X^\perp$ such that*

$$(\omega, \sigma_X(c)^*) = \int_c \omega$$

for all $\omega \in \Gamma_X^{\perp *}$. It has the following properties:

(i) *$\sigma_X(c)$ has Γ_X-behavior,*

(ii) *$\int_d \sigma_X(c) = c \times d$ for any cycle d, where $c \times d$ indicates the intersection number of c and d, i.e., the number of times that d crosses c from left to right. In particular, in case c is a dividing cycle, $\sigma_X(c)$ is exact,*

(iii) *if c runs through all cycles, then $\sigma_X(c)$ span $\Gamma_X^\perp \cap Cl(\Gamma_X + \Gamma_{ho})$.*

PROOE. Since $\omega \to \int_c \omega$ $(\omega \in \Gamma_X^{\perp *})$ is a continuous linear functional on $\Gamma_X^{\perp *}$, the existence and uniqueness of $\sigma_X(c)$ follow from an elementary theorem of Hilbert space theory. However, the properties (i) and (ii) of $\sigma_X(c)$ do not seem to be immediate consequences of the defining property of $\sigma_X(c)$. Therefore, in order to show that $\sigma_X(c)$ has the properties (i) and (ii), we shall have to construct $\sigma_X(c)$ in a direct way.

In 1.4 we have constructed the function $Q_{X,c}$ for an arc c. We can extend $dQ_{X,c}$ to any 1-chain c, in particular to any cycle c, by the linearity in c. In case c is a cycle, Theorem 3 shows

$$\sigma_X(c) = dQ_{X,c}.$$

Hence, (i) and (ii) follow immediately from the method of construction for $Q_{X,c}$.

However, in case c is a cycle, we can construct $dQ_{X,c}$ more easily. In fact, it suffices to treat the case where c is an oriented analytic Jordan curve.

Take a relatively compact ring domain R which contains c. Define the function v on $R-c$ as follows:

$$v=\begin{cases}1 & \text{on the left side of } c,\\ 0 & \text{on the right side of } c.\end{cases}$$

Extend v to $W-c$ so that it becomes a C^∞-function with support relatively compact in W. Denote the extension by $\hat{v}$. Then $d\hat{v}\in\Gamma_c^\infty(W)$. Now we use the orthogonal decomposition

$$\Gamma_c=\Gamma_h+\Gamma_{e0}=\Gamma_\chi+\Gamma_\chi^\perp+\Gamma_{e0}$$

to obtain

$$d\hat{v}=\omega_\chi+\omega_{\chi^\perp}+\omega_{e0}.$$

Then it is obvious that

(i) $\omega_{\chi^\perp}$ has Γ_χ-behavior,

(ii) $\int_d\omega_{\chi^\perp}=c\times d$ for all cycle d.

Moreover we have, for any $\omega\in\Gamma_\chi^{\perp *}$,

$$(\omega,\omega_{\chi^\perp}^*)=(\omega,(d\hat{v})^*-\omega_\chi^*-\omega_{e0}^*)$$
$$=(\omega,(d\hat{v})^*)=\int_{c^++c^-}v\omega=\int_c\omega.$$

Thus $\omega_{\chi^\perp}$ has the reproducing property and coincides with $\sigma_\chi(c)$.

As an application of the above theorem we shall show the existence of a non-zero $\omega\in\Gamma_{he}(W)$ with Γ_χ-behavior in the case $\Gamma_\chi\not\supset\Gamma_{hm}$. Since $\Gamma_\chi^{\perp *}\not\subset\Gamma_{hse}$, there exist $\omega_0\in\Gamma_\chi^{\perp *}$ and a dividing cycle c such that $\int_c\omega_0\neq 0$. Then $\sigma_\chi(c)$ has Γ_χ-behavior and $(\omega_0,\sigma_\chi(c))=\int_c\omega_0\neq 0$. This implies $\sigma_\chi(c)\neq 0$.

From the defining property of $\sigma_\chi(c)$:

$$(\omega,\sigma_\chi(c)^*)=\int_c\omega \qquad \text{for any} \quad \omega\in\Gamma_\chi^{\perp *}$$

it follows that

$$\sigma_\chi(c)^*\perp(\Gamma_\chi^{\perp *}\cap\Gamma_{he}),\quad \text{i.e.,}\quad \sigma_\chi(c)\perp(\Gamma_\chi^\perp\cap\Gamma_{he}^*).$$

Hence, by making use of the orthogonal decomposition:

$$\Gamma_\chi^\perp=(\Gamma_\chi^\perp\cap\Gamma_{he}^*)+\{\Gamma_\chi^\perp\cap Cl(\Gamma_\chi+\Gamma_{h0})\}$$

we find that $\sigma_\chi(c)$ span $\Gamma_\chi^\perp\cap Cl(\Gamma_\chi+\Gamma_{h0})$.

REMARK. If $\Gamma_\chi=\{0\}$, $\Gamma_\chi^{\perp *}=\Gamma_h$ and $\Gamma_\chi^\perp\cap Cl(\Gamma_\chi+\Gamma_{h0})=\Gamma_{h0}$. If $\Gamma_\chi=\Gamma_{hm}$, $\Gamma_\chi^{\perp *}=\Gamma_{hse}$ and $\Gamma_\chi^\perp\cap Cl(\Gamma_\chi+\Gamma_{h0})=\Gamma_{hse}^*\cap\Gamma_{h0}$. If $\Gamma_\chi=\Gamma_{he}$, $\Gamma_\chi^{\perp *}=\Gamma_{h0}$ and $\Gamma_\chi^\perp\cap Cl(\Gamma_\chi+\Gamma_{h0})=\Gamma_{h0}^*\cap Cl(\Gamma_{he}+\Gamma_{h0})$.

COROLLARY. *If* $\Gamma_\chi^\perp\cap Cl(\Gamma_\chi+\Gamma_{h0})\subset\Gamma_\chi^{\perp *}$, *then*

$$(\sigma_\chi(c_1),\ \sigma_\chi(c_2)^*)=c_1\times c_2$$

for any cycles c_1 *and* c_2.

REMARK. Note the implications:

$$\Gamma_\chi\subset\Gamma_{hm}\Rightarrow\Gamma_\chi\perp\Gamma_\chi^*\Rightarrow(\Gamma_\chi+\Gamma_{h0})\subset\Gamma_\chi^{\perp *}\Rightarrow\Gamma_\chi^\perp\cap Cl(\Gamma_\chi+\Gamma_{h0})\subset\Gamma_\chi^{\perp *}.$$

2.3 A special Riemann's bilinear relation

Let $\{A_j,\ B_j\}$ be a canonical homology basis of W modulo dividing cycles. It has the following intersection property:

$$A_i\times A_j=B_i\times B_j=0,\quad A_i\times B_j=\delta_{ij}\qquad\text{for all}\quad i,\ j.$$

Let $\omega\in\Gamma_h(W)$ have Γ_χ-behavior and denote A_j, B_j-periods of ω by x_j, y_j. Then except for a finite number of j, x_j and y_j are zero. The differential

$$\sum_j\{-x_j\sigma_\chi(B_j)+y_j\sigma_\chi(A_j)\}\qquad\text{(a finite sum)}$$

has the same A_j, B_j-periods as ω.

Now assume $\Gamma_\chi\supset\Gamma_{hm}$. Then by the uniqueness theorem in 2.1 we have

$$\omega=\sum_j\{-x_j\sigma_\chi(B_j)+y_j\sigma_\chi(A_j)\}.$$

Hence, for any $\omega_1\in\Gamma_\chi^{\perp *}$, we obtain by the reproducing property of σ_χ

$$(\omega,\ \omega_1^*)=\sum_j\Big(\int_{A_j}\omega\int_{B_j}\omega_1-\int_{B_j}\omega\int_{A_j}\omega_1\Big).$$

2.4 Differentials with preassigned singularities, periods and Γ_χ-behavior

We shall return to the general case where Γ_χ is an arbitrary closed linear subspace of $\Gamma_{he}(W)$.

THEOREM 5. *Suppose that at a finite number of points* $p_j\in W$ *there are given singularities of the form*

$$\sigma_j=d\Big[\mathrm{Re}\Big(\sum_{n=1}^\infty\frac{c_n^{(j)}}{z_j^n}\Big)+a_j\log|z_j|+b_j\arg z_j\Big]$$

where z_j *is a local parameter near* p_j *such that* $z_j(p_j)=0$. *Choose a canonical homology basis* $\{A_k,\ B_k\}$ *of* W *modulo dividing cycles such that none of* A_k, B_k *passes through any* p_j. *Give a sequence of pairs* $\{x_k,\ y_k\}$ *of real numbers such* *that, except for a finite number of k, x_k and y_k are zero.*

Then, in order that there exist a harmonic differential ω on W, which has preassigned singularities σ_j, has x_k, y_k as A_k, B_k-periods and has Γ_X-behavior, it is necessary and sufficient that

$$\sum a_j = \sum b_j = 0.$$

In case $\Gamma_X \supset \Gamma_{hm}$, the differential ω is uniquely determined.

PROOF. The necessity is obvious. To prove the sufficiency, assume $\sum a_j = \sum b_j = 0$. Let u be a function obtained in Theorem 1 corresponding to the singularities

$$s_j = \mathrm{Re}\left(\sum_{n=1}^{\infty} \frac{c_n^{(j)}}{z_j^n}\right) + a_j \log |z_j|.$$

Connect p_1 and p_j ($j \geqq 2$) by a simple arc c_j, and set

$$\omega_1 = \sum_{j \leqq 2} b_j dQ_{X, c_j}.$$

Let x_k', y_k' be A_k, B_k-periods of ω_1. Since ω_1 has Γ_X-behavior, only a finite number of x_k' and y_k' are different from zero. Then the differential

$$\omega = du + \omega_1 + \sum_k \{-(x_k - x_k')\sigma_X(B_k) + (y_k - y_k')\sigma_X(A_k)\} \qquad \text{(a finite sum)}$$

has the required properties. The uniqueness of ω in case $\Gamma_X \supset \Gamma_{hm}$ follows from the uniqueness theorem in 2.1.

§3. Canonical operators

In this section we shall bring forth the notion of canonical operators introduced by H. Yamaguchi [12].

3.1 Normal operators

It seems instructive to compare Yamaguchi's canonical operators with Sario's normal operators. So we begin with reviewing normal operators.

Let W be an open Riemann surface and let V be an element of $\mathfrak{S}(W)$ (defined in 0.5). By $C^{\omega}(\partial V)$ we denote the linear space which consists of all real analytic functions on ∂V, and by $H(\bar{V})$ we denote the linear space which consists of all real harmonic functions on $\bar{V}$.

Consider a linear mapping

$$L: C^{\omega}(\partial V) \to H(\bar{V}).$$

L is called normal if it satisfies the following conditions:

(1) $Lf = f$ on ∂V,

(2) $L1=1$,
(3) $Lf \geqq 0$ if $f \geqq 0$,
(4) $\int_{\partial V}(dLf)^{*}=0$.

Conditions (2) and (3) are equivalent to the validity of the maximum-minimum principle:

$$m \leqq f \leqq M \quad \text{implies} \quad m \leqq Lf \leqq M.$$

3.2 Canonical operators

Consider a mapping

$$L: C^{\omega}(\partial V) \to H(\bar{V})$$

which satisfies the following conditions:

(1) $Lf=f$ on ∂V,
(2) $L1=1$,
(3′) $\|dLf\|_{V}<\infty$,
(4′) $(dLf, dLg)_{V}=\int_{\partial V} f(dLg)^{*}$ for all $f, g \in C^{\omega}(\partial V)$.

The linearity of L follows from these conditions. Note that (4) follows from (2) and (4′). In this connection, we remark that it is an open question whether normal operators have the property (3′) or not.

H. Yamaguchi called such an operator L canonical. Sario's principal operators $(P)L_{1}$ and L_{0} are canonical as well as normal.

3.3 Correspondence between canonical operators and closed linear subspaces of Γ_{he}

Denote by $\boldsymbol{L}_{V}$ the set of all canonical operators defined with respect to V and by $\boldsymbol{\Gamma}_{he}$ the set of all closed linear subspaces of $\Gamma_{he}(W)$. We shall establish

THEOREM 6. *There exists a one-to-one correspondence between* $\boldsymbol{L}_{V}$ *and* $\boldsymbol{\Gamma}_{he}$ *such that, for any* $u \in H(\bar{V})$, *the following conditions* (i) *and* (ii) *are equivalent to each other*:

(i) $u=Lu$ *on* V, *where* L *belongs to* $\boldsymbol{L}_{V}$,
(ii) u *has* Γ_{L}*-behavior, where* $\Gamma_{L} \in \boldsymbol{\Gamma}_{he}$ *corresponds to* L.

PROOF. We shall first establish the mapping $L \to \Gamma_{L}$. To this end, let $f \in C^{\omega}(\partial V)$ and extend f to be a complex-valued analytic function in a neighborhood of ∂V. Its real part is a harmonic extension of f, which will be also denoted by f. Since $Lf-f$ is harmonic on the left side of ∂V and vanishes on ∂V, it can be and hence Lf can be extended harmonically across ∂V. Therefore we can extend Lf to W to be of class $C^{\infty}(W)$, and use the orthogonal decomposition

$$D^{\infty}(W)=HD(W)+D_0^{\infty}(W)$$

to obtain

$$Lf=u_f+v_f \qquad \text{on } V \text{ with} \quad u_f \in HD(W) \quad \text{and} \quad v_f \in D_0^{\infty}(W).$$

Here, the component u_f is uniquely determined by f, up to an additive constant.

On the other hand, L satisfies $\int_{\partial V}(dLf)^*=0$. Hence, in virtue of lemma 1 we can extend $(dLf)^*$ to W so that the extension is of class $\Gamma_c^{\infty}(W)$. We use the orthogonal decomposition

$$\Gamma_c=\Gamma_h+\Gamma_{e0}$$

to obtain

$$(dLf)^*=\sigma_f+\omega_f \qquad \text{on } V \text{ with} \quad \sigma_f \in \Gamma_h(W) \quad \text{and} \quad \omega_f \in \Gamma_{e0}(W).$$

The component σ_f is uniquely determined by f modulo a certain subspace of $\Gamma_{h0}(W)$.

With this notation we assert that

$$du_f \perp \sigma_g{}^*$$

for all f, $g \in C^{\omega}(\partial V)$. This fact follows from condition (4′) as follows:

$$0=\int_{\beta}(Lf)(dLg)^*=\int_{\beta}(u_f+v_f)(\sigma_g+\omega_g)$$

$$=(du_f+dv_f,\ -\sigma_g^*-\omega_g^*)_W=-(du_f,\ \sigma_g^*)_W.$$

Therefore, if we set

$$\Gamma_L=Cl\{du_f: f \in C^{\omega}(\partial V)\},$$

we have

$$\sigma_f \in \Gamma_L^{\perp *}.$$

Thus for such Γ_L, (i) implies (ii).

In order to see that conversely (ii) implies (i), suppose (ii) holds. Then by the Corollary stated after Proposition 2, we have

$$u=u_L+v_{e0},$$

$$(du)^*=\omega_{L^{\perp}}^*+\omega_{e0} \qquad \text{on} \quad V.$$

Then by the definition of Γ_L, Lu and $(dLu)^*$ admit similar representation as u and $(du)^*$. Since $u=Lu$ on ∂V,

$$\|d(u-Lu)\|_V^2=\int_{\beta}(u-Lu)\{d(u-Lu)\}^*=0$$

proving that $u=Lu$ on V.

We shall next establish a mapping of Γ_{he} into $\boldsymbol{L}_V$. For all $\Gamma_X \in \boldsymbol{\Gamma}_{he}$ and $f \in C^\infty(\partial V)$, we denote by $H(\Gamma_X, f)$ the set of functions u defined on $\bar{V}$ which satisfy the following conditions:

⟨1⟩ u is harmonic in V and $D_V(u)<\infty$,

⟨2⟩ $u=f$ on ∂V,

⟨3⟩ there exist $u_X \in H_X(W)$ and $v_{e0} \in D_0^\infty(W)$ such that $u=u_X+v_{e0}$ on V,

⟨4⟩ $\int_{\partial V}(du)^*=0$.

In order to see that $H(\Gamma_X, f)$ is non-empty, let v be the Dirichlet solution with respect to V with boundary values f on ∂V and 0 on the ideal boundary β of W. It is evident that v satisfies conditions ⟨1⟩, ⟨2⟩ and ⟨3⟩. If $\int_{\partial V}(dv)^* \neq 0$, then $W \notin O_G$. In this case, let v_1 be the harmonic measure of β with respect to V. Then $\int_{\partial V}(dv_1)^* \neq 0$. It is easily seen that

$$v-\left\{\int_{\partial V}(dv)^* \Big/ \int_{\partial V}(dv_1)^*\right\}v_1 \in H(\{0\}, f) \subset H(\Gamma_X, f).$$

Finally we shall show that $H(\Gamma_X, f)$ is complete with respect to the Dirichlet norm. To see this, consider a finite number of ring regions $\{D_j\}$ on W such that $\cup D_j \supset \partial V$, f is extended harmonically to $\cup D_j$ and each D_j can be mapped conformally onto $1/r<|z|<r$ so that $|z|=1$ corresponds to $\partial V \cap D_j$. Set $G=\cup D_j \cup V$. Then every function of $H(\Gamma_X, f)$ has a harmonic extension to G. Let $\{u_n\}$ be a Cauchy sequence in $H(\Gamma_X, f)$, i.e.,

$$\|d(u_m-u_n)\|_V \to 0 \quad \text{as} \quad m, n \to \infty.$$

Since

$$\|d(u_m-u_n)\|_G \leqq 2\|d(u_m-u_n)\|_V,$$

u_n converges in G to a function u in norm and locally uniformly. It is obvious that u satisfies conditions ⟨1⟩, ⟨2⟩ and ⟨4⟩. To see that u satisfies ⟨3⟩, we have only to recall Proposition 1 in 1.1.

Since $H(\Gamma_X, f)$ is convex, there exists a unique function in $H(\Gamma_X, f)$ whose differential has the smallest norm in V. We denote this function by $L_X f$. Then

$$dL_X f \perp dH(\Gamma_X, 0) \quad \text{in} \quad \Gamma_h(V).$$

We want to establish

⟨5⟩ $\int_\beta w(dL_X f)^* = 0 \qquad$ for any $\quad w \in H_X(W)$.

To see this, take any $w_1 \in H(\{0\}, w)$. On account of ⟨4⟩ one can apply Lemma

1 and extend $(dL_X f)^*$ to a differential $\omega \in \Gamma_c^\infty$. Since w_1 is equal to a member v_{e0} of $D_0^\infty(W)$ on V and $\Gamma_{e0} \perp \Gamma_c^*$,

$$\int_\beta w_1(dL_X f)^* = -(dv_{e0}, \omega^*) = 0.$$

Obviously $w - w_1 \in H(\Gamma_X, 0)$ and hence

$$0 = \big(d(w-w_1),\ dL_X f\big)_V = \int_\beta (w-w_1)(dL_X f)^* = \int_\beta w(dL_X f)^*.$$

Now it is clear that L_X is canonical. Let us see that, for any $u \in H(\overline{V})$, the following conditions (i′) and (ii′) are equivalent to each other:

(i′) $u = L_X u$ on V,
(ii′) u has Γ_X-behavior.

The implication (i′)⇒(ii′) follows from ⟨3⟩, ⟨5⟩ and Proposition 2 in 1.1. Conversely, assume (ii′) and take any $v \in H(\Gamma_X, u)$. Since $u \in H(\Gamma_X, u)$, $u - v \in H(\Gamma_X, 0)$. Express $u - v = w_X + w_{e0}$ with $w_X \in H_X(W)$ and $w_{e0} \in D_0^\infty(W)$, and $(du)^* = \omega_{X^\perp}^* + \omega_{e0}$ with $\omega_{X^\perp} \in \Gamma_X^\perp$ and $\omega_{e0} \in \Gamma_{e0}$. We have

$$(du - dv,\ du)_V = \int_\beta (u-v)(du)^* = (dw_X + dw_{e0},\ \omega_{X^\perp} - \omega_{e0}^*) = 0.$$

Therefore, $0 \leqq \|du - dv\|_V^2 = \|dv\|_V^2 - \|du\|_V^2$ and hence $u = L_X u$.

Finally we shall show that $\Gamma_{L_1} = \Gamma_{L_2}$ means $L_1 = L_2$ and that $L_{X_1} = L_{X_2}$ implies $\Gamma_{X_1} = \Gamma_{X_2}$. Suppose first that $\Gamma_{L_1} = \Gamma_{L_2}$. For any $f \in C^\omega(\partial V)$ it holds that $L_1(L_1 f) = L_1 f$ on V. Since (i) implies (ii), $L_1 f$ has Γ_{L_1}-behavior which is equal to Γ_{L_2}-behavior. We apply (ii)⇒(i) and conclude $L_1 f = L_2(L_1 f) = L_2 f$ on V. This shows $L_1 = L_2$.

Next suppose that $L_{X_1} = L_{X_2}$. Let $P_{X_j,p}^{(1)}$ $(j=1, 2)$ be a function obtained in 1.3 for $p \notin \overline{V}$. On account of the equivalence (i′)⇔(ii′), we have $P_{X_1,p}^{(1)} = P_{X_2,p}^{(1)}$. From Corollary 2 of Theorem 2 it follows that $\Gamma_{X_1} = \Gamma_{X_2}$.

3.4 Sario's principal operators

We shall show that the canonical operator which corresponds to the subspace $(P)\Gamma_{hm}$ (resp. Γ_{he}) in the sense of Theorem 6 is Sario's principal operator $(P)L_1$ (resp. L_0).

For the sake of simplicity, we will take up only Γ_{hm}. Let $V \in \mathfrak{S}(W)$ and denote by L_{hm} the operator which is defined with respect to V and corresponds to Γ_{hm}. For $f \in C^\omega(\partial V)$, put $u = L_{hm} f$. Since u has Γ_{hm}-behavior,

(a) $u = u_{hm} + u_{e0}$ on V, where $du_{hm} \in \Gamma_{hm}$ and $du_{e0} \in \Gamma_{e0}$,

(b) $(du)^* = \omega_{hse} + \omega_{e0}$ on V, where $\omega_{hse} \in \Gamma_{hse}$ and $\omega_{e0} \in \Gamma_{e0}$.

On account of Proposition 1 in 1.1, (a) is equivalent to (a′) $\int_{\beta} u\omega = 0$, or equivalently $(du, \omega)_V = \int_{\partial V} u\omega$ for any $\omega \in \Gamma_{hse}$.

Let us show that (b) is equivalent to

(b′) $\int_{\gamma} (du)^* = 0$ for any dividing cycle $\gamma \subset V$.

It is evident that (b) implies (b′). Assume (b′). Since $\int_{\partial V} (du)^* = 0$, $(du)^*$ can be extended to a closed C^∞-differential σ on W by Lemma 1. We apply the decomposition $\Gamma_c = \Gamma_h + \Gamma_{e0}$ and write $\sigma = \omega_h + \omega_{e0}$. Condition (b′) implies that $\int_c \omega_h = 0$ for all dividing cycles c. Hence ω_h is semi-exact. Thus (b) follows.

We know that $u = L_{hm} f$ is characterized by (a), (b) and the boundary condition $u = f$ on ∂V. On the other hand, it is known (cf. [9]) that $L_1 f$ satisfies (a′) and (b′), where L_1 is a Sario's principal operator. Hence, $L_{hm} = L_1$.

In this connection, we remark that a differential harmonic on W save for a finite number of isolated singularities is *distinguished in the Ahlfors' sense* ([4], Ch, V, 21D) if and only if it has Γ_{hm}-behavior in our sense in 2.1. It is now seen that *Kusunoki's semi-exact canonical differentials* in [6] are identical with meromorphic differentials with distinguished real parts.

We next consider the operator which is defined with respect to V and corresponds to $\Gamma_{he}(W)$. We denote it by L_{he}. For $f \in C^\infty(\partial V)$, set $u = L_{he} f$. Then

(α) $u = u_{he} + u_{e0}$ on V, where $u_{he} \in HD(W)$ and $u_{e0} \in D_0^\infty(W)$,

(β) $(du)^* = \omega_{h0} + \omega_{e0}$ on V, where $\omega_{h0} \in \Gamma_{h0}$ and $\omega_{e0} \in \Gamma_{e0}$.

However, (α) is superfluous, because any square integrable harmonic function on $\bar{V}$ admits such a representation. On account of Proposition 2 in 1.1, (β) is equivalent to

(β') $\int_{\beta} v(du)^* = 0$, or equivalently $(dv, du) = \int_{\partial V} v(du)^*$ for any $v \in HD(W)$.

Hence $u = L_{he} f$ is characterized by (β') and the boundary condition $u = f$ on ∂V. It is known (cf. [9]) that $L_0 f$ satisfies (β'), where L_0 is a Sario's principal operator. Therefore L_0 coincides with our L_{he}.

Now we see that our Theorems 2, 3, 4 are generalizations of B. Rodin [10], Theorems 1, 2 and L. Ahlfors and L. Sario [4], Ch. III, Theorem 10F.

§4. Riemann-Roch theorem

Throughout this section we assume $\Gamma_\chi \supset \Gamma_{hm}$, or equivalently $\Gamma_\chi^{\perp *} \subset \Gamma_{hse}$. The regular analytic differentials on W whose real parts have Γ_χ-behavior along the ideal boundary of W form a linear space over the real number field. We denote this space by $\Lambda_\chi(W)$.

We remark that, for the interior W of a compact bordered Riemann surface, there exist infinitely many closed linear subspaces between $\Gamma_{hm}(W)$ and $\Gamma_{he}(W)$. This follows from the fact that the dimension of $\Gamma_{hm}(W)$ is finite while the dimension of $\Gamma_{he}(W)$ is infinite.

4.1 Periods and singularities

The validity of the following existence and uniqueness assertions will be readily seen by virtue of the results obtained in §§1, 2.

[I] Let $p \in W$, and give an analytic singularity at p:

$$s=\sum_{n=1}^{\infty}\frac{c_n}{z^n}$$

where z is a local parameter near p such that $z(p)=0$. Then there exists an analytic function (multi-valued in general) which has s as its singularity and whose real part is single-valued and has Γ_χ-behavior. This function is uniquely determined up to an additive constant. We denote by Ψ_s one of them.

[II] For any cycle c there exists a unique differential $\varphi(c)\in\Lambda_\chi$ such that

$$\operatorname{Re}\int_d \varphi(c)=c\times d \qquad \text{for any cycle } d.$$

[III] Let c be an arbitrary arc on W, and put $\partial c=p_2-p_1$. Let λ be a complex number. Then there exists an analytic differential which has simple poles at p_j with residues $(-1)^j\lambda$, $j=1, 2$ and whose real part is exact in $W-c$ and has Γ_χ-behavior. This differential is uniquely determined by λ and c (more precisely the homotopy class of c with fixed end points p_1 and p_2). We denote this differential by $\phi_\lambda(c)$. By the linearity in c we extend the definition of $\phi_\lambda(c)$ to the case where c is an arbitrary 1-chain.

4.2 Riemann-Roch theorem

We shall now establish Riemann-Roch theorem of Kusunoki type. Our proof will be similar to that in Y. Kusunoki [5; 6]. First we give a lemma which plays a fundamental role in our proof.

Let φ and ϕ be analytic differentials on W which have only isolated singularities and whose real parts have Γ_χ-behavior. Assume further that φ has no non-zero residues. Take a canonical homology basis $\{A_j, B_j\}$ of W modulo dividing cycles so that the following conditions are satisfied:

(i) Any one of A_j, B_j does not pass through any singularity of φ and ϕ,

(ii) A_j and B_j are oriented analytic Jordan curves such that $A_j\cap A_k=B_j\cap B_k=\phi$ for any j, k, $A_j\cap B_k=\phi$ for $j\neq k$ and $A_j\cap B_j$ consists of one point.

Cut W along A_j, B_j and denote by W_0 the resulting planar surface. Since the real part of φ has Γ_X-behavior, $\mathrm{Im}\,\varphi$ is expressed by $\omega^*_{X^\perp}+\omega_{e0}$ in some neighborhood V of the ideal boundary. By our assumption $\Gamma_{hse}\supset\Gamma_X^{\perp *}$, $\int_\gamma \mathrm{Im}\,\varphi=0$ for any dividing cycle γ in V and hence for any dividing cycle γ in W. This, together with the fact that W_0 is planar and φ has no singularities with non-zero residues, implies that φ is exact in W_0. Hence, we can set $\varphi=df$ on W_0.

Now we state

LEMMA 3. (Kusunoki) *For differentials φ, ψ and a basis $\{A_j, B_j\}$ just explained, we have*

$$\mathrm{Re}(\sum \mathrm{Res}\, f\psi)=-\frac{1}{2\pi}\sum \mathrm{Im}\left\{\int_{A_j}\varphi\int_{B_j}\psi-\int_{B_j}\varphi\int_{A_j}\psi\right\} \qquad \textit{(a finite sum).}$$

PROOF. Let Ω be a relatively compact subregion of W such that each component of $W-\Omega$ is not compact and has only one analytic contour in common with $\bar{\Omega}$. Suppose that Ω contains all singularities of φ and those of ψ. Suppose moreover that $\partial\Omega$ intersects none of A_j, B_j. Then, integrating $f\psi$ along contours of $\Omega\cap W_0$, we have

$$2\pi i\sum \mathrm{Res}\, f\psi=-\sum_{A_j, B_j\subset\Omega}\left\{\int_{A_j}\varphi\int_{B_j}\psi-\int_{B_j}\varphi\int_{A_j}\psi\right\}+\int_{\partial\Omega}f\psi.$$

On the other hand we have

$$\begin{cases}\mathrm{Re}\,\varphi=\omega_X+\omega_{e0}^{(1)}\\ \mathrm{Im}\,\varphi=\omega^*_{X^\perp}+\omega_{e0}^{(2)}\end{cases} \quad\text{and}\quad \begin{cases}\mathrm{Re}\,\psi=\omega'_X+\omega_{e0}^{(3)}\\ \mathrm{Im}\,\psi=\omega'^*_{X^\perp}+\omega_{e0}^{(4)}\end{cases}$$

outside of a compact subset of W. Hence, for a sufficiently large Ω, we have

$$\mathrm{Im}\int_{\partial\Omega}f\psi=(\omega_X+\omega_{e0}^{(1)},\ \omega'_{X^\perp}-\omega_{e0}^{(4)*})_\Omega+(\omega^*_{X^\perp}+\omega_{e0}^{(2)},\ -\omega'^*_X-\omega_{e0}^{(3)*})_\Omega.$$

Since Γ_X, $\Gamma_X^\perp$, Γ_{e0} and Γ^*_{e0} are pairwise orthogonal, it follows from the above equality that

$$\mathrm{Im}\left(\int_{\partial\Omega}f\psi\right)\to 0 \qquad \text{as}\quad \Omega\uparrow W.$$

We need also the following well-known algebraic fact.

LEMMA 4. *Let X and Y be two linear spaces over a field K, and consider a bilinear form (x, y) defined over $X\times Y$. Denote the left kernel $\{x\in X: (x, y)=0$ for all $y\in Y\}$ by X_0 and the right kernel $\{y\in Y: (x, y)=0$ for all $x\in X\}$ by Y_0. If the quotient space X/X_0 is finite dimensional, then there is an isomorphism $X/X_0\cong Y/Y_0$.*

PROOF. Let (ξ, η) be the bilinear form induced by (x, y) on $(X/X_0)\times(Y/Y_0)$, i.e., $(\xi, \eta)=(x, y)$ where $x\in\xi$ and $y\in\eta$. Each $\eta\in Y/Y_0$ determines a linear form on X/X_0: $\xi\rightarrow(\xi, \eta)$, which we denote by $l(\eta)$. Since the form (ξ, η) is non-degenerate, l is an isomorphism from Y/Y_0 into the dual space $(X/X_0)^*$ of X/X_0. There is a similar isomorphism from X/X_0 into the dual space $(Y/Y_0)^*$. Hence, $\dim(Y/Y_0)\leqq\dim(X/X_0)^*$ and $\dim(X/X_0)\leqq\dim(Y/Y_0)^*$. Since $\dim(X/X_0)$ is finite by assumption, $(X/X_0)^*$ has the same dimension as X/X_0. It follows from the inequalities obtained above that four spaces X/X_0, $(X/X_0)^*$, Y/Y_0, and $(Y/Y_0)^*$ have the same dimension.

Next let D be a finite divisor on W. We introduce two linear spaces $M_X[D]$ and $\Lambda_X[D]$ over the real number field as follows:

$M_X[D]=\{f: f$ is a single-valued meromorphic function on W such that the real part of f has Γ_X-behavior and that $(f)>D\}$,

$\Lambda_X[D]=\{\alpha$: α is a meromorphic differential on W such that the real part of α has Γ_X-behavior and that $(\alpha)>D\}$.

Here (f) and (α) denote the divisors of f and α respectively, and $(f)>D$ means that $(f)-D$ is non-negative.

With this notation we state

THEOREM 7. *Let D be a divisor on W, and set $D=B-A$ where A and B are disjoint non-negative divisors. Then we have*

(1) $\dim M_X[-D]=2\{\deg B+1-\min(1, \deg A)\}-\dim(\Lambda_X[-A]/\Lambda_X[D])$.

In case W has finite genus g, $\dim\Lambda_X[-A]=2\{g+\deg A-\min(1, \deg A)\}$ and the above equality (1) is simplified into the following form:

$$\dim M_X[-D]=2(\deg D-g+1)+\dim\Lambda_X[D].$$

PROOF. Choose a canonical homology basis $\{A_j, B_j\}$ of W modulo dividing cycles so that none of A_j, B_j intersects D. Consider a meromorphic function on W whose real part is single-valued and has Γ_X-behavior. In case $A\neq 0$ we assume furthermore that a branch of the function vanishes at $a_1\in A$. Cut W along A_j, B_j and denote the resulting surface by W_0. In case $A\neq 0$ consider the branch which vanishes at a_1. As shown before Lemma 3 any branch is single-valued on W_0.

The meromorphic functions f on W_0, which are obtained in the manner described just above and have the property: $(f)>-B$, form a real linear space, which will be denoted by $M(-B)$.

Set $A=\sum_{j=1}^{\nu}m_ja_j$ and $B=\sum_{k=1}^{\mu}n_kb_k$. We consider the bilinear form

$$(f, \alpha)=\mathrm{Re}\Big(\sum_{k=1}^{\mu}\mathrm{Res}_{b_k}f\alpha\Big)$$

defined on the product space $M(-B)\times\Lambda_X[-A]$. Let us show that the left kernel

$$\{f \in M(-B) : (f, \alpha) = 0 \quad \text{for all} \quad \alpha \in \Lambda_X[-A]\}$$

is equal to $M_X[-D]$ and the right kernel

$$\{\alpha \in \Lambda_X[-A] : (f, \alpha) = 0 \quad \text{for all} \quad f \in M(-B)\}$$

is equal to $\Lambda_X[D]$. Since the real part of f is single-valued on W by assumption, the formula in Lemma 3 is written in the following form:

$$(2) \quad \mathrm{Re}\Big(\sum_{k=1}^{\mu} \mathrm{Res}_{b_k} f\alpha\Big) = -\frac{1}{2\pi} \sum \Big\{\Big(\mathrm{Im}\int_{A_j} df\Big)\Big(\mathrm{Re}\int_{B_j} \alpha\Big) - \Big(\mathrm{Im}\int_{B_j} df\Big)\Big(\mathrm{Re}\int_{A_j} \alpha\Big)\Big\} - \mathrm{Re}\Big(\sum_{j=1}^{\nu} \mathrm{Res}_{a_j} f\alpha\Big).$$

Suppose $f \in M(-B)$ satisfies $(f, \alpha) = 0$ for all $\alpha \in \Lambda_X[-A]$. We replace α by $\varphi(A_j)$ (resp. $\varphi(B_j)$) in (2) and find that $\mathrm{Im}\int_{A_j} df = 0$ (resp. $\mathrm{Im}\int_{B_j} df = 0$). Hence, f is single-valued on W. In case $A \neq 0$, denote the degree of f at a_j by m_j'. Since $f(a_1) = 0$, $m_1' \geqq 1$. Take a local parameter z near a_1 such that $z(a_1) = 0$. Let

$$f = \lambda z^{m_1'} + \cdots, \qquad \lambda \neq 0.$$

Suppose $m_1' < m_1$ and replace α in (2) by $d\Psi_s$, where $s = \bar{\lambda}/z^{m_1'}$. Then $(f, \alpha) = -m_1'\lambda\bar{\lambda} \neq 0$, which is a contradiction. We have thus shown $m_1' \geqq m_1$. Next we shall show $m_j' \geqq m_j$ $(j \geqq 2)$. Suppose $m_j' < m_j$. In case $m_j' \geqq 1$, the same reasoning as above leads to a contradiction. In case $m_j' = 0$, draw an arc c in W_0 such that $\partial c = a_j - a_1$. Replace α in (2) by $\phi_\lambda(c)$ where $\lambda = f(a_j) \neq 0$. Then $0 = \mathrm{Re}(\sum \mathrm{Res}_{a_j} f\alpha) = f(a_j)\overline{f(a_j)} \neq 0$. This is impossible. Hence $f \in M_X[-D]$. Conversely $f \in M_X[-D]$ implies $(f, \alpha) = 0$ for all $\alpha \in \Lambda_X[-A]$.

Suppose there is α in the right kernel such that $\alpha = (a_n z^n + a_{n+1} z^{n+1} + \cdots)dz$ with $n < n_k$ and $a_n \neq 0$, where z is a local parameter near b_k such that $z(b_k) = 0$. Choose an $f \in M(-(n+1)b_k) \subset M(-B)$ such that

$$f = \bar{a}_n / z^{n+1} + \cdots.$$

Then $0 = (f, \alpha) = a_n \bar{a}_n \neq 0$. This is impossible. Hence $\alpha \in \Lambda_X[D]$. Conversely, if $\alpha \in \Lambda_X[D]$, then $(f, \alpha) = 0$ for all $f \in M(-B)$. Thus the right kernel is equal to $\Lambda_X[D]$.

Now to find the dimension of $M(-B)$ we take a local parameter z_k near b_k such that $z_k(b_k) = 0$. Then a basis of $M(-B)$ is given by

$$\Psi_{s_{k,\rho}}, \Psi_{\tilde{s}_{k,\rho}} \quad (1 \leqq \rho \leqq n_k,\ 1 \leqq k \leqq \mu) \quad \text{where} \quad s_{k,\rho} = 1/z_k^\rho,\ \tilde{s}_{k,\rho} = i/z_k^\rho,$$

and constant functions 1, i in case $A = 0$; and by

$$\Psi_{s_{k,\rho}} \text{ and } \Psi_{\tilde{s}_{k,\rho}} \text{ normalized so that } \Psi(a_1) = 0 \text{ in case } A \neq 0.$$

Thus we find

$$\dim M(-B)=\begin{cases}2(\deg B+1) & \text{if} \quad A=0,\\ 2\deg B & \text{if} \quad A\neq 0.\end{cases}$$

Since $M(-B)$ is finite dimensional, we apply Lemma 4 and obtain an isomorphism

$$M(-B)/M_X[-D]\cong \Lambda_X[-A]/\Lambda_X[D].$$

Equating the dimensions of both sides, we obtain (1).

Finally suppose W has finite genus g. Let $\{A_l, B_l\}_{l=1}^{g}$ be a canonical homology basis of W modulo dividing cycles. Let z_j be a local parameter near a_j such that $z_j(a_j)=0$. Then a basis of $\Lambda_X[-A]$ is given by the following differentials:

$$\varphi(A_l),\ \varphi(B_l) \qquad (l=1, 2, \cdots, g),$$

$$d\Psi_{s_{j,\rho}} \quad \text{with} \quad s_{j,\rho}=1/z_j^{\rho},$$

$$d\Psi_{\tilde{s}_{j,\rho}} \quad \text{with} \quad \tilde{s}_{j,\rho}=i/z_j^{\rho} \qquad (1\leqq \rho\leqq m_j-1,\ \ 1\leqq j\leqq \nu) \quad \text{and}$$

$$\phi_\lambda(c_j)\ (\lambda=1, i;\ 2\leqq j\leqq \nu) \quad \text{where } c_j \text{ is an arc such that } \partial c_j=a_j-a_1.$$

If $A=0$, a basis consists only of $\{\varphi(A_l)\}$ and $\{\varphi(B_l)\}$ and $\dim \Lambda_X[-A]=2g$. If $A\neq 0$,

$$\dim \Lambda_X[-A]=2\{g+\sum_{j=1}^{\nu}(m_j-1)+(\nu-1)\}=2(g+\deg A-1).$$

Thus $\quad \dim \Lambda_X[-A]=2\{g+\deg A-\min(1, \deg A)\}.$

4.3 Abel's theorem

The following theorem is a generalization of Y. Kusunoki [6], Theorem 10.

THEOREM 8. *Let D be a divisor on W such that* $\deg D=0$. *Then, in order that there exist a meromorphic function f on W such that* $\mathrm{Re}(\log f)$ *has Γ_X-behavior and $(f)=D$, it is necessary and sufficient that*

$$\mathrm{Re}\int_c \varphi(A_j), \quad \mathrm{Re}\int_c \varphi(B_j)$$

are all integers, where c is an arbitrary chain such that $\partial c=D$.

Such a function f is determined up to a non-zero constant factor.

PROOF. To prove the necessity, suppose that there exists a function f such that $\mathrm{Re}(\log f)$ has Γ_X-behavior and $(f)=D$ and choose a canonical homology basis $\{A_j, B_j\}$ of W modulo dividing cycles so that none of A_j, B_j intersects D. Cut W along A_j, B_j and denote the resulting surface by W_0. Since $\deg D=0$, we can take a chain c_0 in W_0 such that $\partial c_0=D$. We may consider

c_0 instead of c because, $c-c_0$ being a cycle, both $\operatorname{Re}\int_{c-c_0}\varphi(A_j)$ and $\operatorname{Re}\int_{c-c_0}\varphi(B_j)$ are integers.

Let $\varphi\in\Lambda_\chi$ and let Φ be an integral of φ in W_0, i.e., $d\Phi=\varphi$. We infer that Φ is single-valued for the same reason as before. By Lemma 3 we have

$$2\pi\operatorname{Re}(\sum\operatorname{Res}\Phi\, d\log f)=-\sum\Big\{\Big(\operatorname{Re}\int_{A_j}\varphi\Big)\Big(\int_{B_j}d\arg f\Big)-\Big(\operatorname{Re}\int_{B_j}\varphi\Big)\Big(\int_{A_j}d\arg f\Big)\Big\}.$$

Substitute $\varphi(A_k)$ for φ. The right hand side of the equality is then equal to $\int_{A_k}d\arg f$ which is a multiple of 2π. If D is expressed as $\sum m_p b_p-\sum n_q a_q$ with m_p, $n_q>0$, then

$$\sum\operatorname{Res}\Phi\, d\log f=\sum_p m_p\int^{b_p}\varphi(A_k)-\sum_q n_q\int^{a_q}\varphi(A_k)=\int_{c_0}\varphi(A_k).$$

Hence $\operatorname{Re}\int_{c_0}\varphi(A_k)$ is an integer. If φ is replaced by $\varphi(B_k)$, then it is concluded that $\operatorname{Re}\int_{c_0}\varphi(B_k)$ is an integer.

Conversely suppose $\operatorname{Re}\int_{c_0}\varphi(A_k)$ and $\operatorname{Re}\int_{c_0}\varphi(B_k)$ are all integers. We consider the differential $\psi=\phi_1(c_0)$. For $\varphi\in\Lambda_\chi$ we have

$$2\pi\operatorname{Re}(\sum\operatorname{Res}\Phi\psi)=-\sum\Big\{\Big(\operatorname{Re}\int_{A_j}\varphi\Big)\Big(\operatorname{Im}\int_{B_j}\psi\Big)-\Big(\operatorname{Re}\int_{B_j}\varphi\Big)\Big(\operatorname{Im}\int_{A_j}\psi\Big)\Big\}$$

where Φ is an integral of φ in W_0. We replace φ by $\varphi(A_k)$ and observe that $\sum\operatorname{Res}\Phi\psi=\int_{c_0}\varphi(A_k)$. The right hand side of the equality is equal to $\operatorname{Im}\int_{A_k}\psi$. Therefore, this is equal to a multiple of 2π. We see that the same is true for $\operatorname{Im}\int_{B_k}\psi$. Hence $\exp\Big(\int\psi\Big)$ is single-valued and has the required properties.

Finally suppose there are two functions f_1 and f_2 with the properties stated in the theorem. Then $\operatorname{Re}(\log(f_1/f_2))$ is harmonic on W and has Γ_χ-behavior. Hence it is a constant on account of the uniqueness theorem in 1.1. It follows that f_1/f_2 is a constant on W. Therefore a function f in the theorem has the form

$$f=ae^{\int\phi_1(c)},\quad \text{where } a \text{ is a non-zero constant.}$$

Remark. We assume that W has finite genus g in this remark. The following two statements are equivalent to each other.

1) There exists a chain c such that $\partial c=D$ and $\operatorname{Re}\int_c\varphi(A_j)$, $\operatorname{Re}\int_c\varphi(B_j)$ are all integers.

2) There exists a chain c_1 such that $\partial c_1=D$ and that

$$\mathrm{Re}\int_{c_1}\varphi=0$$

for all $\varphi\in\Lambda_\chi$.

To see that 1) implies 2), we only have to set $c_1=c+\sum(-m_jB_j+n_jA_j)$ where $m_j=\mathrm{Re}\int_c\varphi(A_j)$, $n_j=\mathrm{Re}\int_c\varphi(B_j)$. Conversely 2) implies 1) trivially.

If $W\in O_{AD}$, i.e., if $\Gamma_{hm}=\Gamma_{he}$, then there exists only one Γ_χ such that $\Gamma_{hm}\subset\Gamma_\chi\subset\Gamma_{he}$. In this case, Λ_χ coincides with the set of all semi-exact square integrable analytic differentials. Therefore 2) is equivalent to the following classical statement.

3) D can be written in the form ∂c_1 where c_1 has the property that

$$\int_{c_1}\alpha=0$$

for all semi-exact square integrable analytic differentials.

References

[1] R. D. M. Accola: *Some classical theorems on open Riemann surfaces*, Bull. Amer. Math. Soc., **73** (1967), 13-26.

[2] L. V. Ahlfors: *Abel's theorem for open Riemann surfaces*, Seminars on Analytic Functions, II, 7-19, Institute for Advanced Study, Princeton, 1958.

[3] L. V. Ahlfors: *The method of orthogonal decomposition for differentials on open Riemann surfaces*, Ann. Acad. Sci. Fenn. Ser. A. I, no. 249/7 (1958), 15 pp.

[4] L. V. Ahlfors and L. Sario: *Riemann surfaces*, Princeton Univ. Press (1960).

[5] Y. Kusunoki: *Contributions to Riemann-Roch's theorem*, Mem. Coll. Sci. Univ. Kyoto Ser. A, **31** (1958), 161-180.

[6] Y. Kusunoki: *Theory of Abelian integrals and its applications to conformal mapping*, Ibid., **32** (1959), 235-258.

[7] Y. Kusunoki: *Supplements and corrections to my former papers*, Ibid., **33** (1961), 429-433.

[8] Y. Kusunoki: *Abelian differentials on open Riemann surfaces*, Symposium at Shugakuin (in Kyoto), 1961, 24 pp. (in Japanese).

[9] K. Oikawa: *Minimal slit regions and linear operator method*, Kōdai Math. Sem. Rep., **17** (1965), 187-190.

[10] B. Rodin: *Reproducing kernels and principal functions*, Proc. Amer. Math. Soc., **13** (1962), 982-992.

[11] H. L. Royden: *The Riemann-Roch theorem*, Comm. Math. Helv., **34** (1960), 37-51.

[12] H. Yamaguchi: *Regular operators and spaces of harmonic functions with finite Dirichlet integral on open Riemann surfaces*, to appear in J. Math. Kyoto Univ., 8 (1968).

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