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Roots of Scalar Operator-valued Analytic Functions and their Functional Calculus

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Introduction

Let X be a Banach space, T a linear bounded operator acting in X and f an analytic complex function defined in a neighborhood of $\sigma(T)$. Let us suppose also that f is non-constant in each connected component of its domain of definition which intersects $\sigma(T)$.

In this paper we study the spectral properties of T if $f(T)$ is a spectral operator of scalar type. The example of Stampfli (see [18]) shows that in general T is not a scalar operator.

We shall prove that T is a Φ-scalar operator in the sense of [15], where Φ is a suitable basic algebra.

1. Preliminaries

Throughout the paper we shall use the following basic notation and conventions:

N: the set of all natural numbers.

Λ: the set of all complex numbers.

$\sigma' = \Lambda - \sigma$ for $\sigma \subset \Lambda$.

$C(K, r) = \{\lambda \in \Lambda;\ \text{dist}(\lambda, K) \leqq r\}$, where $K(\subset \Lambda)$ is compact and $r \geqq 0$.

$\mathfrak{F}(K)$: the set of all analytic complex functions whose domains of definition are open sets containing K, where K is a compact subset of Λ.

$\mathfrak{X}$: a Banach space over the complex field Λ.

$\mathfrak{L}(\mathfrak{X})$: the algebra of all linear bounded operators acting in $\mathfrak{X}$.

I: the unity of $\mathfrak{L}(\mathfrak{X})$.

$\sigma(T)$: the spectrum of $T \in \mathfrak{L}(\mathfrak{X})$.

Let $T \in \mathfrak{L}(\mathfrak{X})$ and $f \in \mathfrak{F}(\sigma(T))$. Then $f(T) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda; T)d\lambda$, where

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Γ is an admissible contour in the sense of [10], VII. 3.9 and $R(\lambda; T)$ is the resolvent of T.

LEMMA 1.1. *Let G be an open set and K a compact subset of G. If f is a continuous complex function on G, then for any compact subset F of $f(K)$ and for any $\varepsilon>0$ there is $\eta>0$ such that*

$$f^{-1}(C(F, r))\cap K\subset C(f^{-1}(F)\cap K, \varepsilon)$$

for any $r\leqq\eta$.

PROOF. Let us suppose that there is $\varepsilon_0>0$ such that for any $n\in N$ we can find $r_n\leqq 1/n$ with the property $f^{-1}(C(F, r_n))\cap K\not\subset C(f^{-1}(F)\cap K, \varepsilon_0)$. Let $\lambda_n\in f^{-1}(C(F, r_n))\cap K$, $\lambda_n\notin C(f^{-1}(F)\cap K, \varepsilon_0)$ and λ_0 be a limit point of the sequence $\{\lambda_n\}$. We have $\lambda_0\in K$ and because $f(\lambda_n)\in C(F, r_n)$, we also have $f(\lambda_0)\in F$. Thus $\lambda_0\in f^{-1}(F)\cap K$, which is impossible because $\lambda_n\notin C(f^{-1}(F)\cap K, \varepsilon_0)$, and the lemma results.

2. Algebra of functions

In this section we shall use the terminology and the definitions introduced in [15], [16].

The symbols D and $\bar{D}$ will denote the operators $\frac{1}{2}\left(\frac{\partial}{\partial s}+i\frac{\partial}{\partial t}\right)$ and $\frac{1}{2}\left(\frac{\partial}{\partial s}-i\frac{\partial}{\partial t}\right)$ respectively, where $s+it=\lambda\in\Lambda$.

If K is a compact set in Λ and ϕ is an n-times continuously differentiable complex function defined in a neighborhood of K, then we shall put

$$|\phi|_{n,K}=\sum_{p+q=0}^{n}\sup_{\lambda\in K}|(\bar{D}^p D^q\phi)(\lambda)|.$$

Let f be an analytic complex function defined on an open set G. We define the function m_f in G as follows:

$$m_f(\lambda)=\begin{cases}\text{the least integer } n \text{ such that } f^{(n)}(\lambda)\neq 0, \text{ if it exists;}\\ \infty, \text{ if } f^{(n)}(\lambda)=0 \text{ for any integer } n.\end{cases}$$

In fact $m_f(\lambda)$ is the order of multiplicity of λ as root of the equation $f(\mu)=0$.

Now we can introduce some algebras of functions which will be used in the sequel.

(1) $\mathcal{C}^n$: the algebra of all n-times continuously differentiable complex functions defined in Λ with the topology given by the family of semi-norms $\{|\cdot|_{n,K}; K \text{ compact}\}$, if n is finite; by $\{|\cdot|_{k,K}; k=0, 1, 2, \cdots, K \text{ compact}\}$, if $n=\infty$.

(2) $\mathcal{C}^n_\lambda=\{\phi\in\mathcal{C}^n; (D^j\phi)(\lambda)=0 \text{ for } 1\leqq j, j<n\}$ with the topology induced

by $\mathcal{C}^n$.

(3) $\mathcal{C}^n(r)$: the algebra of all n-times continuously differentiable complex functions defined in $C(0, r)$ with the topology given by the norm

$$|\phi|_n = \sum_{p+q=0}^{n} \sup_{\lambda \in C(0,r)} |(\bar{D}^p D^q \phi)(\lambda)|.$$

where $n < \infty$ and $r > 0$.

(4) $\mathcal{C}_0^n(r) = \{\phi \in \mathcal{C}^n(r);\ (D^j\phi)(0) = 0 \text{ for } 1 \leqq j,\ j < n\}$ with the topology induced by $\mathcal{C}^n(r)$.

(5) Let K be a compact set and $f \in \mathcal{F}(K)$. Then we consider the algebra

$$\Phi(f, K) = \bigcap_{\lambda \in K} \mathcal{C}_\lambda^{m_{f'}(\lambda)}$$

with the topology induced by $\mathcal{C}^{n_0}$, where $n_0 = \max_{\lambda \in K} m_{f'}(\lambda)$.

Remark 1. $\mathcal{C}_\lambda^0 = \mathcal{C}^0$, $\mathcal{C}_\lambda^1 = \mathcal{C}^1$, $\mathcal{C}_0^0(r) = \mathcal{C}^0(r)$ and $\mathcal{C}_0^1(r) = \mathcal{C}^1(r)$.

Proposition 2.1. *Let K be a compact set and $f \in \mathcal{F}(K)$ be non-constant in each connected component of its domain which intersects K. Then $\Phi(f, K)$ is a basic algebra and, for any n and $g \in \mathcal{C}_0^n(r)$ such that $g(0) \in K$, $0 \leqq n \leqq m_{f'}(g(0))$, the function g is $\Phi(f, K)$-proper with respect to $\mathcal{C}_0^n(r)$.* (See Def. 1.1 of [15] and Def. 1.1 of [16].)

Proof. Because f is non-constant in each connected component of its domain which intersects K, the function $m_{f'}|_K$ is different from 0 only in a finite set. Thus $\Phi(f, K)$ is a finite intersection of $\mathcal{C}_\lambda^n$; if $f'(\lambda) \neq 0$ for any $\lambda \in K$, then $\Phi(f, K) = \mathcal{C}^0$, if f' has only simple zeros in K, then $\Phi(f, K) = \mathcal{C}^1$ and if $\mathcal{M} = \{\lambda \in K;\ f'(\lambda) = f''(\lambda) = 0\} \neq \emptyset$, then $\Phi(f, K) = \bigcap_{\lambda \in \mathcal{M}} \mathcal{C}^{m_{f'}(\lambda)}$.

The non-trivial case is when $\mathcal{M} \neq \emptyset$. But the properties (ii) and (iii) of Def. 1.1 of [15] are evidently verified. Thus we shall prove only (i) of Def. 1.1 of [15]. For this, let $F(\subset \Lambda)$ be a compact set and G be an open set containing F. Choose $\varepsilon > 0$ such that $C(F, \varepsilon) \subset G$, $C(F, \varepsilon) \cap \mathcal{M} = F \cap \mathcal{M}$ and let $F_0 = C(F, \varepsilon/2)$, $G_0 = \operatorname{int} C(F, \varepsilon)$ and $n_0 = \max_{\lambda \in K} m_{f'}(\lambda)$. Taking $\phi \in \mathcal{C}^{n_0}$ such that $\phi(\lambda) = 1$ for $\lambda \in F$ and $\phi(\lambda) = 0$ for $\lambda \notin G_0$ we have evidently $\phi \in \Phi(f, K)$. Consequently $\Phi(f, K)$ is a basic algebra.

Now if $g \in \mathcal{C}_0^n(r)$ and $n \leqq n_0$, then it is well-known that the map $\phi \to \phi \circ g$ from $\mathcal{C}^{n_0}$ to $\mathcal{C}^n(r)$ is continuous. By the definition of topologies in $\Phi(f, K)$ and in $\mathcal{C}_0^n(r)$, we have to prove only that, in the hypothesis of our proposition, we have $\phi \circ g \in \mathcal{C}_0^n(r)$ for any $\phi \in \Phi(f, K)$. But if $n = 0, 1$, this happens as a trivial consequence of Remark 1. In the contrary case, using the formula $D(\phi \circ g) = ((\bar{D}\phi) \circ g)Dg + ((D\phi) \circ g)D\bar{g}$, we obtain by a simple calculus again $\phi \circ g \in \mathcal{C}_0^n(r)$, which proves the proposition.

Let $n \in N$ and $0 \leqq k \leqq n$. For any $\phi \in \mathcal{C}^n(r)$ we shall put

$$\phi_k(\lambda)=\begin{cases}\dfrac{1}{(n+1)\lambda^k}\sum_{p=0}^{n}(i_p)^{-k}\phi(i_p\lambda) & \text{if } \lambda\neq 0\\ 0 & \text{if } \lambda=0,\end{cases}$$

where $i_p=\exp[2\pi ip/(n+1)]$.

LEMMA 2.2. *For any $n\in N$, $\phi\in\mathcal{O}_0^n(r)$ and $0\leqq k\leqq n$, we have*

$$\sup_{\lambda\in C(0,r)}|\phi_k(\lambda)|\leqq 4|\phi|_n$$

PROOF. If $k=0$, then the inequality becomes evident, thus we shall suppose $k\geqq 1$. By Taylor's formula, we have in this case, for any $\mu\in C(0,r)$,

$$\phi(\mu)=\sum_{j=0}^{k-1}\frac{(\mu\bar{D}+\bar{\mu}D)^j\phi}{j!}\Big|_{\lambda=0}+\tilde{\phi}_k(\mu),$$

where

$$\tilde{\phi}_k(\mu)=\frac{(\mu\bar{D}+\bar{\mu}D)^k(\phi+\bar{\phi})}{2k!}\Big|_{\lambda=\theta_1\mu}+\frac{(\mu\bar{D}+\bar{\mu}D)^k(\phi+\bar{\phi})}{2k!}\Big|_{\lambda=\theta_2\mu}$$

for some θ_1, θ_2; $0<\theta_1<1$, $0<\theta_2<1$. It follows

$$|\tilde{\phi}_k(\mu)|\leqq|\mu|^k\frac{2^{k+1}|\phi|_k}{k!}\leqq 4|\mu|^k|\phi|_n.$$

Because $(D^j\phi)(0)=0$ for $1\leqq j$, $j<n$ we obtain

$$\phi(\mu)=\sum_{j=0}^{k-1}\frac{\mu^j}{j!}(\bar{D}^j\phi)(0)+\tilde{\phi}_k(\mu)$$

and using the equality $\sum_{p=0}^{n}(i_p)^{j-k}=0$ for $0\leqq j<k$ we have, for any $\lambda\neq 0$,

$$\phi_k(\lambda)=\frac{1}{n+1}\sum_{p=0}^{n}\frac{\tilde{\phi}_k(i_p\lambda)}{(i_p\lambda)^k}.$$

It follows

$$|\phi_k(\lambda)|\leqq\frac{1}{n+1}\sum_{p=0}^{n}\frac{|\tilde{\phi}_k(i_p\lambda)|}{|(i_p\lambda)^k|}\leqq 4|\phi|_n.$$

LEMMA 2.3. *Let ϕ, $\psi\in\mathcal{O}_0^n(r)$, $n\in N$. Then for any m, $0\leqq m\leqq n$, we have*

$$(\phi\psi)_m(\lambda)=\sum_{k+j=m}\phi_k(\lambda)\psi_j(\lambda)+\lambda^{n+1}\sum_{\substack{k+j=m+n+1\\0\leqq k,j\leqq n}}\phi_k(\lambda)\psi_j(\lambda).$$

PROOF. The equality is non-trivial if $\lambda\neq 0$. But if $\lambda\neq 0$ we have

$$\sum_{k+j=m}\phi_k(\lambda)\psi_j(\lambda)+\lambda^{n+1}\sum_{\substack{k+j=m+n+1\\0\leqq k,j\leqq n}}\phi_k(\lambda)\psi_j(\lambda)$$

$$=\sum_{k=0}^{m}\phi_k(\lambda)\psi_{m-k}(\lambda)+\lambda^{n+1}\sum_{k=m+1}^{n}\phi_k(\lambda)\psi_{m+n+1-k}(\lambda)$$

$$=\frac{1}{(n+1)^2\lambda^m}\sum_{p,q=0}^{n}\left\{\sum_{k=0}^{n}(i_p)^{-k}(i_q)^{k-m}\right\}\phi(i_p\lambda)\psi(i_q\lambda)$$

$$=\frac{1}{(n+1)\lambda^m}\sum_{p=0}^{n}(i_p)^{-m}\phi(i_p\lambda)\psi(i_p\lambda)=(\phi\psi)_m(\lambda).$$

3. Functional calculus

Let Φ be a closed subalgebra of some $\mathcal{C}^n$ and at the same time a basic algebra. If $\mathcal{U}$ is a Φ-spectral representation (Def. 1.3 of [15]), then it has compact support in view of the topology of Φ. Therefore we shall write $\mathcal{U}(\lambda)$, $\mathcal{U}(1)$ for $\mathcal{U}(\phi_1)$ and $\mathcal{U}(\phi_0)$ respectively, where $\phi_1, \phi_0 \in \Phi$ and $\phi_1(\lambda)=\lambda$, $\phi_0(\lambda)=1$ in some neighborhood of supp $\mathcal{U}$. In fact we have supp$\,\mathcal{U}=\sigma(\mathcal{U}(\lambda))$ and $\mathcal{U}(1)=I$.

Definition 3.1. A Φ-spectral representation $\mathcal{U}$ is called *regular* if it is valued in the bicommutant (=the commutant of the commutant; see [12]) of $\mathcal{U}(\lambda)$. An operator $T\in\mathcal{L}(\mathfrak{X})$ is called a *regular Φ-scalar operator* if there is a regular Φ-spectral representation $\mathcal{U}$ such that $T=\mathcal{U}(\lambda)$.

Proposition 3.2. *Let $T\in\mathcal{L}(\mathfrak{X})$ and $\{P_j\}_{j=1}^m$ be a finite set of projections in the bicommutant of T such that $P_jP_k=0$ if $j\neq k$ and $\sum_{j=1}^m P_j=I$. Let Φ be a basic closed subalgebra of some $\mathcal{C}^n$. Then T is a regular Φ-scalar operator if and only if $T|P_j\mathfrak{X}$ is a regular Φ-scalar operator for each j.*

Proof. If T is a regular Φ-scalar operator, then there is a spectral representation $\mathcal{U}$ valued in the bicommutant of T. The map $\mathcal{U}_j=\mathcal{U}|P_j\mathfrak{X}$ is again a Φ-spectral representation in $\mathcal{L}(P_j\mathfrak{X})$ and $\mathcal{U}_j(\lambda)=T|P_j\mathfrak{X}$.

If $V_j(T|P_j\mathfrak{X})=(T|P_j\mathfrak{X})V_j$, then we have $(V_jP_j)T=T(V_jP_j)$ and because $V_jP_j\in\mathcal{L}(\mathfrak{X})$ we obtain $\mathcal{U}_jV_j=\mathcal{U}V_jP_j=V_jP_j\mathcal{U}=V_j\mathcal{U}P_j=V_j\mathcal{U}_j$ in $P_j\mathfrak{X}$, which proves that $\mathcal{U}_j$ is regular.

Now if $T|P_j\mathfrak{X}$ is a regular Φ-scalar operator for each j and $\mathcal{U}_j(\lambda)=T|P_j\mathfrak{X}$, then $\mathcal{U}=\bigoplus_{j=1}^m\mathcal{U}_j$ is a Φ-spectral representation, regular if $\mathcal{U}_j$ is regular for each j. Indeed, if $V\in\mathcal{L}(\mathfrak{X})$, $VT=TV$, then putting $V_j=V|P_j\mathfrak{X}$ we have $V_j(T|P_j\mathfrak{X})=(T|P_j\mathfrak{X})V_j$, thus $V\mathcal{U}=\bigoplus_{j=1}^m V_j\mathcal{U}_j=\bigoplus_{j=1}^m\mathcal{U}_jV_j=\mathcal{U}V$.

Theorem 3.3. *Let $T\in\mathcal{L}(\mathfrak{X})$ and suppose that for some $n\in N$ the operator $T^{n+1}=S$ is of scalar type. If T is one-to-one, then for any $r>0$ such that $\sigma(T)\subset C(0, r)$, the map $\mathcal{U}$ from $\mathcal{C}_0^n(r)$ to $\mathcal{L}(\mathfrak{X})$ defined by the equation*

$$\mathcal{U}(\phi)=\sum_{k=0}^{n}T^k\phi_k(\sqrt[n+1]{S})$$

is a continuous homomorphism valued in the bicommutant of T such that $\mathcal{U}(1)=I$ and $\mathcal{U}(\lambda)=T$.

Proof. The operator $\sqrt[n+1]{S}$ is a scalar operator with the spectrum in $C(0, r)$ (see [9], Lemma 6) and any operator which commutes with T commutes also with $\phi_k(\sqrt[n+1]{S})$, because such an operator commutes with S and $\phi_k(\sqrt[n+1]{S})$ is a function of S. Thus $\mathcal{U}(\phi)$ is in the bicommutant of T. The linearity of $\mathcal{U}$ is trivial and the continuity results by [9], Lemma 6 and our Lemma 2.2.

If E is the spectral measure of $\sqrt[n+1]{S}$, then using Lemma 2.3 we have

$$\mathcal{U}(\phi)\mathcal{U}(\psi) = \sum_{m=0}^{n} T^m \int \Big\{ \sum_{k+j=m} \phi_k(\lambda)\psi_j(\lambda) + \lambda^{n+1} \sum_{k+j=m+n+1} \phi_k(\lambda)\psi_j(\lambda) \Big\} E(d\lambda)$$

$$= \sum_{m=0}^{n} T^m \int (\phi\psi)_m(\lambda) E(d\lambda) = \mathcal{U}(\phi\psi).$$

Now if $m=0, 1$ and $\phi(\lambda)=\lambda^m$, then by a simple calculus we obtain $\phi_k(\lambda)=1$ for $k=m$, $\lambda\neq 0$ and $\phi_k(\lambda)=0$ for $k\neq m$. Because T is one-to-one, S and $\sqrt[n+1]{S}$ have the same property. Consequently $E(\{0\})=0$. Thus we have

$$\mathcal{U}(1) = \int_{\{0\}'} E(d\lambda) = I \quad \text{and} \quad \mathcal{U}(\lambda) = T\int_{\{0\}'} E(d\lambda) = T.$$

Lemma 3.4. *Let K be a compact set, $f \in \mathcal{F}(K)$ be non-constant in each connected component of its domain G which intersects K. Suppose, for $\lambda_0 \in K$ and $r>0$ such that $C(\lambda_0, r)\subset G$, f is expressed as $f(\lambda)=f(\lambda_0)+((\lambda-\lambda_0)h(\lambda))^{n+1}(n\geqq 1)$ with a holomorphic function h in a neighborhood G' of $C(\lambda_0, r)$ such that $h(\lambda_0)\neq 0$ and $g(\lambda)=(\lambda-\lambda_0)h(\lambda)$ is one-to-one in G'. Given $T\in\mathcal{L}(\mathfrak{X})$, if $f(T)$ is a scalar operator and $\sigma(T)\subset C(\lambda_0, r)\cap K$, then T is a regular $\Phi(f, K)$-scalar operator.*

Proof. Let us remark first that by Proposition 2.1, $\Phi(f, K)$ is a basic algebra. If $f(T)-f(\lambda_0)=S$, then $(g(T))^{n+1}=S$. Let E be the spectral measure of S and $E(\{0\})=P_1$, $E(\{0\}')=P_2$. By Proposition 3.2, we have to prove that $T|P_1\mathfrak{X}$ and $T|P_2\mathfrak{X}$ are $\Phi(f, K)$-scalar operators. Because we have $S|P_1\mathfrak{X}=0 =(h(T|P_1\mathfrak{X}))^{n+1}((T|P_1\mathfrak{X})-\lambda_0)^{n+1}$ and $h(\lambda)\neq 0$ for $\lambda\in C(\lambda_0, r)$, it results $((T|P_1\mathfrak{X}) -\lambda_0)^{n+1}=0$. Thus if $Q_1=(T|P_1\mathfrak{X})-\lambda_0$ then the equation

$$\mathcal{U}_1(\phi) = \sum_{k=0}^{n} \frac{Q_1{}^k}{k!}(\bar{D}^k\phi)(\lambda_0)$$

defines evidently a regular $\Phi(f, K)$-spectral representation such that $\mathcal{U}_1(\lambda)= T|P_1\mathfrak{X}$. Now if we take $r'>0$ such that $g(C(\lambda_0, r))\subset \operatorname{int} C(0, r')$, then we have also $\sigma(g(T))\subset \operatorname{int} C(0, r')$ ([10], VII. 3.11). Let $\mathcal{V}$ be the map $\mathcal{V}$: $\mathcal{C}_0^n(r')\to \mathcal{L}(P_2\mathfrak{X})$ given in Theorem 3.3 associated to $g(T|P_2\mathfrak{X})$. Because g is one-to-one there is $\psi\in\mathcal{C}^n(r')$ such that $(\psi\circ g)(\lambda)=\lambda$ in a neighborhood of $C(\lambda_0, r)$. By Proposition 2.1, ψ is $\Phi(f, K)$-proper with respect to $\mathcal{C}_0^n(r')$, because it is analytic in a neighborhood of 0. Consequently $\mathcal{U}_2(\phi)=\mathcal{V}(\phi\circ\psi)$ defines a $\Phi(f, K)$-spectral representation ([16], Prop. 1.2), and $\mathcal{U}_2(\lambda)=\mathcal{V}(\psi)=\psi(g(T|P_2\mathfrak{X}))=T|P_2\mathfrak{X}$. The regularity of $\mathcal{U}_2$ results from the properties of $\mathcal{V}$.

Theorem 3.5. *Let $T\in\mathcal{L}(\mathfrak{X})$ and $f\in\mathcal{F}(\sigma(T))$ be non-constant in each con-*

nected component of its domain which intersects $\sigma(T)$. If $f(T)$ is a scalar operator then T is a regular $\Phi(f, \sigma(T))$-scalar operator.

Proof. We shall suppose that the set $\mathcal{M}_0=\{\lambda;\lambda\in\sigma(T), f'(\lambda)=0\}$ is non-void, because in the contrary case the theorem is known to be true (see [3], Th. 3). $\mathcal{M}_0$ is a finite set, say $\{\lambda_j\}_{j=1}^n$. Let $\mathcal{N}=\{\mu_j\}_{j=1}^q=f(\mathcal{M}_0)$ and $\mathcal{M}=f^{-1}(\mathcal{N})\cap\sigma(T)=\{\lambda_j\}_{j=1}^m$, $n\leqq m<\infty$. For each j, $1\leqq j\leqq n$, there is $\varepsilon_j>0$ such that $f(\lambda)=f(\lambda_j)+((\lambda-\lambda_j)h_j(\lambda))^{n_j+1}$ with a holomorphic function h_j defined in a neighborhood G_j of $C(\lambda_j, \varepsilon_j)$ such that $h_j(\lambda)\neq 0$ and $g_j(\lambda)=(\lambda-\lambda_j)h_j(\lambda)$ is one-to-one in G_j, where $n_j=m_{f'}(\lambda_j)$. Choose $\varepsilon>0$ such that $\varepsilon<\min(\varepsilon_1, \cdots, \varepsilon_n)$, $C(\lambda_j, \varepsilon)\cap C(\lambda_k, \varepsilon)=\emptyset$ for $j\neq k$ and $C(\mathcal{M}, \varepsilon)\cap f^{-1}(\mathcal{N})=\mathcal{M}$. Using Lemma 1.1, we can find $\eta>0$ such that $C(\mu_j, \eta)\cap C(\mu_k, \eta)=\emptyset$ for $j\neq k$ and $f^{-1}(C(\mathcal{N}, \eta))\cap\sigma(T)\subset C(\mathcal{M}, \varepsilon)$.

Now if $f(T)=S$ and E is the spectral measure of S, then, denoting $\sigma_j=C(\mu_j, \eta)$, $1\leqq j\leqq q$, $\sigma_{q+1}=(C(N, \eta))'$, we have to prove the theorem for $T|E(\sigma_j)\mathfrak{X}$, $j=1, \cdots, q+1$ (see Proposition 3.2). Because E is in the bicommutant of T, we have $\sigma(T|E(\sigma_j)\mathfrak{X})\subset\sigma(T)$ and by [10], VII 3.11 $\sigma(T|E(\sigma_j)\mathfrak{X})\subset f^{-1}(\sigma_j)$. Also we have $f(T|E(\sigma_j)\mathfrak{X})=S|E(\sigma_j)\mathfrak{X}$. Therefore, applying [3], Th. 3, $T|E(\sigma_{q+1})\mathfrak{X}$ results to be scalar. If $j\leqq q$, then $\sigma(T|E(\sigma_j)\mathfrak{X})\subset\sigma(T)\cap C(\mathcal{M}, \varepsilon)$. Let us denote $E(\sigma_j)\mathfrak{X}=\mathfrak{X}_j$, $T|\mathfrak{X}_j=T_j$. We have $\sigma(T_j)\subset\bigcup_{\lambda_k\in\mathcal{M}_1}C(\lambda_k, \varepsilon)$, where $\mathcal{M}_1$ is a subset of $\mathcal{M}$. Let P_{λ_k} be the spectral projection of T_j corresponding to $C(\lambda_k, \varepsilon)$ (see [10], VII 3.17). Then $\sigma(T_j|P_{\lambda_k}\mathfrak{X}_j)\subset C(\lambda_k, \varepsilon)$ and because P_{λ_k} is in the bicommutant of T_j, the operator $f(T_j|P_{\lambda_k}\mathfrak{X}_j)=S|P_{\lambda_k}\mathfrak{X}_j$ is scalar. If $k>n$ then $f'(\lambda)\neq 0$ for $\lambda\in\sigma(T_j|P_{\lambda_k}\mathfrak{X}_j)$, thus $T_j|P_{\lambda_k}\mathfrak{X}_j$ is a scalar operator by [3], Th. 3, and if $k\leqq n$ then $T_j|P_{\lambda_k}\mathfrak{X}_j$ is a regular $\Phi(f, \sigma(T))$-scalar operator by Lemma 3.4. It follows from Proposition 3.2 that $T|E(\sigma_j)\mathfrak{X}$ is a regular $\Phi(f, \sigma(T))$-scalar operator, which proves our theorem.

Corollary 3.6. *In the hypothesis of Theorem 3.5, if f' has only simple zeros in $\sigma(T)$, then T is a regular generalized scalar operator in the sense of* [11] (i. e., a regular $\mathcal{C}^\infty$-scalar operator).

Proof. In this case $\Phi(f, \sigma(T))=\mathcal{C}^1$ and a $\mathcal{C}^1$-scalar operator is a $\mathcal{C}^\infty$-scalar operator.

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