J. Sci. Hiroshima Univ. Ser. A-I
32 (1968) 441-447

Integral Domains which are Almost Krull

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(Received July 6, 1968)

1. Introduction.

In [1] Gilmer introduced the notion of almost-Dedekind domain. Every Dedekind domain is almost-Dedekind (*AD*) and *AD*-domains in general have many of the properties of Dedekind domains. Every Dedekind domain is a Krull domain in which proper nonzero prime ideals are maximal. Hence it seems natural to look for the proper generalization of almost-Dedekind domains to almost-Krull domains.

2. Definition and general properties.

In what follows, R denotes a commutative integral domain with identity and K denotes the quotient field of R. Proper prime ideals of R are nonzero prime ideals which are not equal to R. The notation is that found in [3] and [4].

We state the following definition from [1].

Definition 2.1. *R is AD iff R_M is a Dedekind domain for each maximal ideal M of R.*

It follows that proper prime ideals in an *AD*-domain are maximal. Now if R is *AD* and X is an indeterminate, then $R[X]$ is not *AD*. However, we do have the following proposition which motivates the definition of almost-Krull domain.

Proposition 2.2. *Let R be an AD-domain and let X be an indeterminate. Then for every proper prime ideal P of $R[X]$, the ring $R[X]_P$ is a Krull domain.*

Proof. Put $Q=P\cap R$ and let $M=R[X]-P$. Let $M_1=R-Q$, $M_2=M-M_1$. Now Q is a prime ideal of R so that M_1 is a multiplicative system in R and hence in $R[X]$. M_2 is the set of nonconstant polynomials in M and hence M_2 is also a multiplicative system in $R[X]$. It follows that $(R[X])_P=(R_{M_1}[X])_{M_2}$. Since R is *AD*, R_{M_1} is either a field or a Dedekind domain (If $P\cap R=(0)$ then $R_{M_1}=K$.). Thus $R_{M_1}[X]$ is a Krull domain. Since M_2 is a multiplicative system in $R_{M_1}[X]$, $(R_{M_1}[X])_{M_2}$ is a Krull domain.

The above proposition suggests the following definition.

DEFINITION 2.3. *R is called an almost-Krull (AK) domain iff R_P is a Krull domain for each proper prime ideal P of R.*

PROPOSITION 2.4. *R is AK iff R_M is a Krull domain for each maximal ideal M of R.*

PROOF. The proof is a straight forward application of the general properties of quotient ring formation found in [3].

PROPOSITION 2.5. *Let R be an AK-domain. Then R is integrally closed, and hence a Noetherian AK-domain is a Krull domain.*

PROOF. By [4], page 94, we have that $R=\cap R_M$, where M runs over all maximal ideals of R. Each R_M is integrally closed since each R_M is a Krull domain. It follows that R is integrally closed. It is well known that a Noetherian integrally closed domain is a Krull domain.

In the remainder of this paper we shall assume that R is an AK-domain unless otherwise stated. Thus let Δ denote the set of nonzero minimal primes of R. $\Delta \neq \emptyset$, for if M is a maximal ideal of R then the Krull domain R_M contains a minimal prime P. It follows that $Q=P\cap R$ is a minimal prime of R.

PROPOSITION 2.6. $R=\bigcap_{P\in\Delta} R_P$

PROOF. Let M be any maximal ideal of R and let B_M denote the collection of nonzero minimal primes of R that are contained in M. Since R_M is a Krull domain, $R_M=\bigcap_{P\in B_M}(R_M)_{PR_M}=\bigcap_{P\in B_M}R_P$. Then $R=\cap R_M=\bigcap_M(\bigcap_{P\in B_M}R_P)=\bigcap_{P\in\Delta}R_P$, where M runs over all maximal ideals of R.

COROLLARY 2.7. *R is completely integrally closed.*

PROOF. Each R_P is a discrete rank one valuation ring and hence is completely integrally closed.

COROLLARY 2.8. *Let F denote the family of valuations on K induced by the family of nonzero minimal primes of R. Then F satisfies the following:*

(i) *Each $v\in F$ has rank one and is discrete.*

(ii) $R=\bigcap_{v\in F}R_v$

(iii) *For each $v\in F$, $R_v=R_{P(v)}$, where $P(v)$ denotes the center of v on R.*

(iv) *For each maximal ideal M of R and for each nonzero $x\in K$, $v(x)\neq 0$ for only a finite number of $v\in F$ such that $P(v)\subset M$.*

PROOF. (i), (ii), (iii) are clear. (iv) follows from the fact that for each maximal ideal M, R_M is a Krull domain with quotient field K.

Using the above corollary, we can in fact characterize AK-domains in terms of families of valuations.

THEOREM 2.9. *R is an AK-domain iff there exists a family F of valuations on K with the following properties*:

(i) *Each $v \in F$ has rank one and is discrete.*

(ii) *For each maximal ideal M of R there is a subfamily F_M of F such that $R_M = \bigcap_{v \in F_M} R_v$.*

(iii) *For each $v \in F$, $R_v = R_{P(v)}$.*

(iv) *For each maximal ideal M of R and for each nonzero $x \in K$, there are only a finite number of $v \in F_M$ such that $v(x) \neq 0$.*

PROOF. The "only if" part follows from corollary 2.8. The "if" part follows from the fact that the existence of a family F satisfying (i) through (iv) implies that R_M is a Krull domain for each maximal ideal M of R. Thus R is AK by proposition 2.4.

A family F of valuations satisfying (i) through (iv) of the above theorem is called a family of essential valuations for R. The next proposition shows that, as is the case for Krull domains, a family of essential valuations for R is uniquely determined by R.

PROPOSITION 2.10. *Let F be a family of essential valuations for R. Then the quotient rings R_P, where P runs over the family of all minimal primes of R, are identical with the valuation rings R_v, $v \in F$.*

PROOF. Let $v \in F$ and let $P(v)$ denote the center of v on R. Then $R_v = R_{P(v)}$. Since R_v is a discrete rank one valuation ring, we must have that $P(v)$ is a minimal prime in R.

Conversely, let P be any minimal prime in R. We must show that P is the center of some valuation $v \in F$. Let M be any maximal ideal containing P. By theorem 2.9 there is a family F_M of valuations which is the family of essential valuations for the Krull domain R_M and $F_M \subset F$. Now PR_M is a minimal prime in R_M and since R_M is a Krull domain there is $v \in F_M$ such that PR_M is the center of v on R_M. So if $x \in R_M$ then $v(x) > 0$ iff $x \in PR_M$ iff $x = \frac{t}{m}$, $t \in P$, $m \in R - M$ iff $v(t) > 0$ since m is a unit in R_M and hence $v(m) = 0$. Thus $P \subset P(v)$ where $P(v)$ denotes the center of v on R. Since $P(v)$ is minimal we must have $P = P(v)$. Then $R_v = R_{P(v)} = R_P$.

We note that if R is not a Krull domain, there may be a proper subfamily G of F such that $R = \bigcap_{v \in G} R_v$. For example if R is an AD-domain which is not a Dedekind domain such a family G always exists [2, theorem 3].

The next few theorems show some ways to obtain AK-domains from a given AK-domain R.

THEOREM 2.11. *Let R be an AK-domain and let $X_1, X_2, \cdots, X_n$ be indeterminates. Then $R[X_1, X_2, \cdots, X_n]$ is an AK-domain.*

PROOF. It is sufficient to prove the case $n=1$. The proof for this case is similar to the proof of proposition 2.2 with only a few minor changes.

Now, let X be an indeterminate. For $f(X)=\sum_{i=0}^{n} a_i X^i \in R[X]$, and $v \in F$, define $v'(f(X))=\min\{v(a_i) \mid 0 \leq i \leq n\}$. v' may be extended to a valuation on $K(X)$, and is called the canonical extension of v to $K(X)$. Let G denote the family of $a(X)$-adic valuations on $K(X)$ where $a(X)$ is a nonconstant irreducible polynomial in $K[X]$ and let F' denote the family of canonical extensions of elements of F to valuations on $K(X)$.

PROPOSITION 2.12. *$F' \cup G$ is the family of essential valuations for the AK-domain $R[X]$.*

PROOF. We shall show that $F' \cup G$ satisfies the conditions of theorem 2.9. It is clear the valuations in $F' \cup G$ have rank one and are discrete so that (i) is satisfied. To see that $F' \cup G$ satisfies (ii) of 2.9, let M be any maximal ideal of $R[X]$. We will show that there exists a subfamily H_M of $F' \cup G$ such that $R[X]_M = \bigcap_{w \in H_M} R[X]_w$. To construct H_M put $M \cap R = P$. Then P is a prime ideal of R. Let $F_M = \{v \in F \mid P(v) \subset P\}$, and let F'_M denote canonical extensions of elements of F_M to $K(X)$. Let $G_M = \{v_a \in G \mid Q(v_a) \subset M\}$, where v_a denotes the valuation induced by the nonconstant irreducible polynomial $a(X)$, and $Q(v_a)$ denotes the center of v_a on $R[X]$. It can be shown that $R[X]_M = \bigcap_{w \in F'_M \cup G_M} R[X]_w$. Thus we take $H_M = F'_M \cup G_M$. The proof that $F' \cup G$ satisfies (iii) of 2.9 is the same as the proof of theorem 29(b), page 85 of [4]. It follows from the construction of H_M that if $y \in K(X)$, $y \neq 0$, there are only a finite number of $w \in H_M$ such that $w(y) \neq 0$. Thus $F' \cup G$ satisfies the conditions of 2.9.

THEOREM 2.13. *Let R be an AK-domain with quotient field K and let L be a finite algebraic extension of K. Let R' denote the integral closure of R in L. Then R' is an AK-domain.*

PROOF. Let M' be any maximal ideal of R'. Since R' is integral over R we have that $M' \cap R = M$ is a maximal ideal of R. Put $S = R - M$, so that $R_S = R_M$ is a Krull domain. Since R' is the integral closure of R in L, R'_S is the integral closure of R_S in L. Since R_S is a Krull domain, R'_S is a Krull domain. Now, $S \subset R' - M'$, so $(R'_S)_{R'-M'} = R'_{R'-M'} = R'_{M'}$. Thus $R'_{M'}$ is a Krull domain since R'_S is a Krull domain and $R' - M'$ is a multiplicative system in R'_S.

PROPOSITION 2.14. *Let R, K, L, R', F be as in 2.13 above. Let F' denote the family of valuations on L which are extensions of members of F. F' is the family of essential valuations of the AK-domain R'.*

PROOF. We will show that $\{R'_{v'} \mid v' \in F'\} = \{R'_P \mid P$ is a minimal prime of

$R'\}$. It is clear that the left hand side is contained in the right hand side. To see that the right hand side is contained in the left hand side, let P be any minimal prime of R'. Then $R'_P=R'_u$ and $P=P(u)$ for some essential valuation u of the AK-domain R'. Let $Q=P\cap R$. Then Q is a minimal prime in R since R' is integral over R. Thus $Q=Q(v)$ for some $v \in F$. Let u_0 denote the restriction of u to K. Then u_0 is nonnegative on R with center $Q(u_0)=Q(v)$. Thus u is an extension of v, i.e., $u \in F'$. Since $R'_P=R'_u$ we have $\{R'_P \mid P$ is a minimal prime of $R'\} \subset \{R'_{v'} \mid v' \in F'\}$.

THEOREM 2.15. *Let R be an AK-domain with quotient field K and family F of essential valuations. Let S be a multiplicative system in R. Then R_S is an AK-domain with $G=\{v \in F \mid P(v)\cap S=\emptyset\}$ the family of essential valuations of R_S.*

PROOF. Let M be a maximal ideal of R_S. Then $M\cap R=P$ is a prime ideal of R and $M=PR_S$. Let $T=R-P$ so that T is a multiplicative system in R with $T\subset R_S-M$. Now $P\cap S=\emptyset$ so that $S\subset R-P=T$. Also $(R_S)_T=(R_T)_S$. Since $T\subset R_S-PR_S$ we have $(R_S)_{R_S-PR_S}=[(R_S)_T]_{R_S-PR_S}$. So $(R_S)_M=(R_S)_{R_S-PR_S}=[(R_S)_T]_{R_S-PR_S}=[(R_T)_S]_{R_S-PR_S}$. The result follows from the fact that $R_T=R_P$ is a Krull domain. Now let G denote the family of essential valuations of R_S. If $v \in G$, then $Q(v)$, the center of v on R_S, is a minimal prime of R_S. Then $Q(v)\cap R=P$ is a minimal prime of R and hence $P=P(v)$ is the center of v on R. Since $Q(v)=P(v)R_S\neq R_S$, we must have $P(v)\cap S=\emptyset$, since otherwise $P(v)R_S=R_S$. On the other hand, let $v \in F$ be such that $P(v)\cap S=\emptyset$. Then $P(v)R_S\neq R_S$ and so $P(v)R_S$ is a minimal prime of R_S. Thus $v \in G$.

We now determine all AK-domains between R and its quotient field K. Let A be a domain such that $R\subset A\subset K$. For any maximal ideal M of A, let $P=R\cap M$, and let $S=R-P$. We note that P is a prime ideal of R and $P\neq R$ since $1 \notin M$. For $v \in F$, $P(v)$ denotes the center of v on R. This notation is used in the following theorem.

THEOREM 2.16. *A is an AK-domain iff there is a subfamily G of F such that $A=\bigcap_{v \in G} R_v$ and for every maximal ideal M of A we have $\bigcap_{\substack{v \in G\\ P(v)\subset M}} R_v=A_M$.*

PROOF. The "only if" part is immediate. To see the "if" part, let G be a subfamily of F having the stated properties. Using the notation just given, for any maximal ideal M of A we have $R_S\subset A_S\subset \bigcap_{\substack{v \in G\\ P(v)\subset M}} R_v=A_M$. Since R_S is a Krull domain, A_M is also a Krull domain.

The following proposition gives a sufficient condition for an AK-domain to be a Krull domain. It generalizes a theorem of Gilmer in [1].

PROPOSITION 2.17. *Let R be an AK-domain. If every nonzero proper ideal of R is contained in only a finite number of maximal ideals then R is a*

Krull domain. Thus an AK-domain with only a finite number of maximal ideals is a Krull domain.

PROOF. Let $x \in R$, $x \neq 0$, and let F denote the family of essential valuations of R. It is sufficient to show that $v(x) \neq 0$ for only a finite number of $v \in F$. If $x \in R$ is a unit then $v(x)=0$ for all $v \in F$. If $x \in R$ is not a unit then Rx is a nonzero proper ideal of R and hence Rx is contained in only a finite number of maximal ideals, say $M_1, \cdots, M_n$. For any maximal ideal M of R let F_M denote the family of essential valuations of the Krull domain R_M. Then $v(x) \neq 0$ for only a finite number of $v \in \bigcup_{i=1}^{n} F_{M_i}$ since $v(x) \neq 0$ for only a finite number of $v \in F_{M_i}$, $i=1, 2, \cdots, n$. If M is any maximal ideal of R which does not contain Rx, then $x \notin M$. Thus x is a unit in R_M, and hence $v(x)=0$ for all $v \in F_M$. Thus $v(x)=0$ for all $v \in F - \cup \{F_{M_i} \mid i=1, \cdots, n\}$, and $v(x) \neq 0$ for only a finite number of $v \in F$.

The following shows that if R is an AK-domain which is not a Krull domain, then the same must be true of $R[X]$, where X is an indeterminate.

PROPOSITION 2.18. *Let R be an AK-domain with family F of essential valuations, and let X be an indeterminate. If $R[X]$ is a Krull domain then R is a Krull domain.*

PROOF. Let $F' \cup G$ denote the family of essential valuations of the AK-domain $R[X]$, where F' is the family of canonical extensions of members of F to valuations on $K(X)$. It is sufficient to show that if $r \in R$, $r \neq 0$, then $v(r) \neq 0$ for only a finite number of $v \in F$. Now $R \subset R[X]$, so if $v' \in F'$ and $r \in R$, then $v'(r)=v(r)$, and since $R[X]$ is a Krull domain, $v(r)=v'(r)$ is nonzero for only a finite number of $v \in F$.

COROLLARY 2.19. *If R is almost-Dedekind and if $R[X_1, \cdots, X_k]$ is a Krull domain for some k, then R is a Dedekind domain.*

The above corollary generalizes a result of Gilmer in [1]. Corollary 2.19 also shows that there exists a large class of AK-domains which are not Krull domains.

Author's note: This paper constitutes part of a Ph.D. dissertation written under the direction of Professor Paul J. McCarthy at the University of Kansas. The author wishes to express his appreciation to Professor McCarthy for his counsel and advice during the course of this work.

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