J. Sci. Hiroshima Univ. Ser. A-I
33 (1969), 11-27

On the Fine Cauchy Problem for the System of Linear Partial Differential Equations

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(Received March 4, 1969)

In our previous paper [4, p. 406] we have introduced the notion of a canonical extension of a distribution in studying the distributional boundary values of holomorphic functions. The notion will be treated in this paper with considerable detail so as to be applied to the study of the fine Cauchy problem for the system of linear partial differential equations. Let Ω be a non-empty open subset $\subset R^n$ and T a positive number which may be $+\infty$. Let u be a distribution on $\Omega\times(0, T)$. We shall say that a distribution $\tilde{u}$ on $\Omega\times(-\infty, T)$ is a canonical extension of u if $\tilde{u}=\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}u$, where $\rho(t)$ is an arbitrary function with certain properties (cf. Definition 1 below) and $\rho_{(\varepsilon)}(t)=\rho\left(\frac{t}{\varepsilon}\right)$. If u happens to have the boundary value $\lim_{t\downarrow 0} u=\alpha$, then the identity:

$$\frac{\partial}{\partial t}(\rho_{(\varepsilon)}u)=\rho_{(\varepsilon)}\frac{\partial u}{\partial t}+\rho'_{(\varepsilon)}u$$

will imply that

$$\frac{\partial \tilde{u}}{\partial t}=\left(\widetilde{\frac{\partial u}{\partial t}}\right)+\alpha\otimes\delta_t.$$

The fact will be used to bring the initial conditions into the differential system as done in L. Schwartz [6, p. 133].

Section 1 is devoted to the discussions centering around the canonical extension $\tilde{u}$. In Section 2 we consider the canonical extension in a narrow sense and develop the same consideration as in Section 1. In Section 3 we deal with the fine Cauchy problem for the system of linear partial differential equations (cf. [6, p. 133]). For instance, consider the Cauchy problem for the system (we use the vector notation):

$$\frac{\partial u}{\partial t}=P(x, t, D_x)u+f$$

with the initial condition $\lim_{t\downarrow 0} u(x, t)=\alpha\in\mathcal{D}'(\Omega)$, where f is a given vector of distributions in $\mathcal{D}'(\Omega\times(0, T))$. Suppose f has the canonical extension $\tilde{f}$. We shall show in Theorem 1 below that to solve the Cauchy problem just considered is to find a vector v of distributions which satisfies the system:

$$\frac{\partial v}{\partial t}=P(x,\ t,\ D_x)v+\tilde{f}+\alpha\otimes\delta_t$$

and vanishes for $t<0$. The same discussions are also made about the case where u becomes a $\mathcal{D}'(\Omega)$-valued continuous function of t.

1. Canonical extensions

First we recall the notion of a canonical extension of a distribution which was introduced in our previous paper [4, p. 410]. In what follows Ω denotes a non-empty open subset of an n-dimensional Euclidean space R^n with points $x=(x_1,\ \cdots,\ x_n)$ and T a positive number which may be $+\infty$. An open set $\Omega\times(0,\ T)$ will be a cylinder in R^{n+1}. $\mathcal{D}'(\Omega\times(0,\ T))$ means the space of distributions on $\Omega\times(0,\ T)$ with the usual topology of L. Schwartz. Let $u\in\mathcal{D}'(\Omega\times(0,\ T))$ and let $\rho(t)$ be any real-valued C^∞ function of the real variable t equal to 1 for $t\geqq 2$ and 0 for $t\leqq 1$. We define $\rho_{(\varepsilon)}(t)=\rho\left(\frac{t}{\varepsilon}\right)$ for $\varepsilon>0$. The multiplicative product $\rho_{(\varepsilon)}u$ will be understood as a distribution extended over $\Omega\times(-\infty,\ T)$ so as to vanish for $t<\varepsilon$.

DEFINITION 1. *A distribution $\tilde{u}\in\mathcal{D}'(\Omega\times(-\infty,\ T))$ will be called to be a canonical extension of u over $t=0$ if $\rho_{(\varepsilon)}u$ converges to $\tilde{u}$ in $\mathcal{D}'(\Omega\times(-\infty,\ T))$ for any ρ as $\varepsilon\downarrow 0$.*

Observe that $\rho_{(\varepsilon)}$ can be written in the form $\rho_{(\varepsilon)}=Y*\chi_\varepsilon$, Y being the Heaviside function and χ is a real-valued C^∞ function with support contained in $[1,\ 2]$ such that $\int\chi(t)dt=1$, where we put $\chi_\varepsilon(t)=\frac{1}{\varepsilon}\chi\left(\frac{t}{\varepsilon}\right)$. We have proved the following [4, p. 409]

PROPOSITION 1. *$u\in\mathcal{D}'(\Omega\times(0,\ T))$ has a canonical extension whenever there exists the distributional boundary value* $\lim_{t\downarrow 0}u=\alpha\in\mathcal{D}'(\Omega)$, i.e., $\lim_{\varepsilon\downarrow 0}\langle u,\ \chi_\varepsilon\rangle_t=\alpha$ *for every χ mentioned above.*

However, the converse is not true in general. In connexion with the canonical extension, we introduce

DEFINITION 2. *$w\in\mathcal{D}'(\Omega\times(-\infty,\ T))$ is called to be canonical whenever $w=\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}w$ for every ρ with the property mentioned before.*

We note that $\tilde{u}$ in Definition 1 is canonical because of the fact that $\tilde{u}=\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}u$ and $\tilde{u}=u$ for $t>0$. Clearly this is the case for $a(x,\ t)\tilde{u}$, where $a(x,\ t)$ is an arbitrary C^∞ function on $\Omega\times(-T_1,\ T)$ with $T_1>0$.

LEMMA 1. *Suppose $v\in\mathcal{D}'(\Omega\times(-\infty,\ T))$ vanishes for $t<0$. v is canoni-*

cal provided there exists the section $v(x, 0)$ for $t=0$.

PROOF. In what follows, $A\subset\subset B$ will mean once for all that A is relatively compact in B. In view of a theorem of S. Łojasiewicz [5, p. 21], given any non-empty open subset $G\subset\subset\Omega$ and any ε_0 with $0<\varepsilon_0<T$, we can find a non-negative integer k, a multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ of non-negative integers and a continuous function $F(x, t)$ on $G\times(-\infty, \varepsilon_0)$ such that $v=D_x^\alpha D_t^k F(x, t)$ on $G\times(-\varepsilon_0, \varepsilon_0)$ and $F=o(t^k)$ uniformly on G as $t\to 0$, where $D_x^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ with $D_j=\dfrac{\partial}{\partial x_j}$ and $D_t=\dfrac{\partial}{\partial t}$. Now we can write

$$\rho_{(\varepsilon)}v=D_x^\alpha\rho_{(\varepsilon)}D_t^kF=D_x^\alpha\sum_{j=0}^{k}\binom{k}{j}(-1)^jD_t^{k-j}(D_t^j\rho_{(\varepsilon)}\cdot F)$$

on $G\times(-\infty, \varepsilon_0)$. $D_t^k(\rho_{(\varepsilon)}\cdot F)$ tends to D_t^kF in $\mathcal{D}'(G\times(-\varepsilon_0, \varepsilon_0))$ as $\varepsilon\downarrow 0$, and since $|D_t^j\rho_{(\varepsilon)}|=O\left(\dfrac{1}{\varepsilon^j}\right)$, $D_t^j\rho_{(\varepsilon)}\cdot F$, $0<j\leqq k$, tends therefore to 0 in $\mathcal{D}'(G\times(-\varepsilon_0, \varepsilon_0))$ as $\varepsilon\downarrow 0$. Consequently $\lim\limits_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}v=D_x^\alpha D_t^kF(x, t)=v$ in $\mathcal{D}'(G\times(-\varepsilon_0, \varepsilon_0))$. We conclude therefore that v is canonical, as desired.

In our previous paper [2, p. 170] we have discussed the partial multiplication of distributions. Given $h\in\mathcal{D}'(R_t)$ and $w\in\mathcal{D}'(\Omega\times(-\infty, T))$, the multiplicative product hw is a distribution on $\Omega\times(-\infty, T)$ defined as the distributional limit $\lim\limits_{\varepsilon\downarrow 0}(h*\psi_\varepsilon)w$, if it exists, for every choice of $\psi\in\mathcal{D}(R_t)$ such that $\int\psi(t)dt=1$ and $\psi_\varepsilon(t)=\dfrac{1}{\varepsilon}\psi\left(\dfrac{t}{\varepsilon}\right)$. When the limit $\lim\limits_{k\to\infty}(h*g_k)w$ exists for every choice of δ-sequence $\{g_k\}$ instead of $\{\psi_\varepsilon\}$, the limit will be referred to as the multiplicative product in the strict sense and denoted by $h\cdot w$.

With the aid of Lemma 1 we shall show the following

PROPOSITION 2. *Let v, $w\in\mathcal{D}'(\Omega\times(-\infty, T))$ be such that $v=0$ for $t<0$ and $w=\dfrac{\partial v}{\partial t}$. The following conditions are equivalent:*

(a) *w is canonical.*
(b) *v is canonical and has the boundary value $\lim\limits_{t\downarrow 0}v$ equal to 0.*
(c) *The section $v(x, 0)$ exists.*
(d) *The partial multiplicative product $w(x, t)Y(t)$ exists.*
(e) *The partial multiplicative product $v(x, t)\delta_t$ exists.*

PROOF. (a)$\Rightarrow$(b). Let α be an arbitrary function in $\mathcal{D}((-\infty, T))$ and χ an arbitrary function in $\mathcal{D}((0, \infty))$ such that $\int\chi(t)dt=1$. Put $\rho_{(\varepsilon)}=Y*\chi_\varepsilon$. Since w is canonical, $\lim\limits_{\varepsilon\downarrow 0}<\rho_{(\varepsilon)}w, \alpha>_t=<w, \alpha>_t$. Observing now the relations

$$\begin{aligned}<\rho_{(\varepsilon)}w, \alpha>_t&=<(Y*\chi_\varepsilon)w, \alpha>_t\\&=<\chi_\varepsilon, \check{Y}*_t(\alpha w)>_t\end{aligned}$$

$$= <\chi_\varepsilon, 1*_t(\alpha w)>_t - <\chi_\varepsilon, Y*_t(\alpha w)>_t$$

$$= <w, \alpha>_t - <\chi_\varepsilon, Y*_t(\alpha w)>_t,$$

we find that $\lim_{\varepsilon\downarrow 0} <\chi_\varepsilon, Y*_t(\alpha w)>_t = 0$ and therefore $\lim_{t\downarrow 0}(Y*_t(\alpha w))=0$. Choosing α to be equal to 1 in a neighbourhood of 0, we can conclude that $\lim_{t\downarrow 0} v = \lim_{t\downarrow 0}(Y*_t w)=0$, which, when combined with the equation $\rho_{(\varepsilon)}w=\frac{\partial}{\partial t}(\rho_{(\varepsilon)}v)-\chi_\varepsilon v$, yields $w=\lim_{\varepsilon\downarrow 0}\frac{\partial}{\partial t}(\rho_{(\varepsilon)}v)$ and therefore $v=Y*_t w=\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}v$.

(b)$\Rightarrow$(c). Let G and ε_0 be taken as in Lemma 1. Since $\lim_{t\downarrow 0} v=0$, we can write $v=D_x^\alpha D_t^k F(x, t)$ on $G\times(0, \varepsilon_0)$, where F is a continuous function on $G\times(-\infty, \varepsilon_0)$ vanishing for $t<0$ such that $F=o(t^k)$ uniformly on G as $t\downarrow 0$. Noting that v is canonical and proceeding as in the proof of Lemma 1, we see that $v=\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}v=\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}D_x^\alpha D_t^k F=D_x^\alpha D_t^k F$ on $G\times(-\infty, \varepsilon_0)$. Hence we get $v(x, 0)=0$.

(c)$\Leftrightarrow$(d)$\Leftrightarrow$(e). Recalling the definition of the section of a distribution for $t=0$ [5, p. 15], we see that the section $v(x, 0)$ exists if and only if the distribution $<v, \phi>_x$, $\phi\in\mathcal{D}(\Omega)$, has the value $<v, \phi>_x(0)$ at $t=0$. Here we have the relation $<v(x, 0), \phi>_x=<v, \phi>_x(0)$. Similar situations hold also for the partial multiplication [2, p. 170]. The equivalence (c)$\Leftrightarrow$(e) follows then from Lemma 4 in [2, p. 166]. Similarly, since the multiplication is normal in the sense of [2, p. 153] it follows that (d) and (e) are equivalent.

(c)$\Rightarrow$(a). v has the section for $t=0$, hence owing to Lemma 1, v is canonical. Since, further, $v(x, 0)=0$, it follows that $\lim_{\varepsilon\downarrow 0}\rho'_{(\varepsilon)}v=0$. From the relation $\rho_{(\varepsilon)}w=\frac{\partial}{\partial t}(\rho_{(\varepsilon)}v)+\rho'_{(\varepsilon)}v$ we see that $\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}w=\frac{\partial v}{\partial t}=w$, which shows that w is canonical. This completes the proof.

PROPOSITION 3. *Let $u, v\in\mathcal{D}'(\Omega\times(0, T))$ be such that $u=\frac{\partial v}{\partial t}$. The following conditions are equivalent*:

(a) *u has the canonical extension.*

(b) *The distributional boundary value $\lim_{t\downarrow 0} v$ exists.*

PROOF. (a)$\Rightarrow$(b). Let $\tilde{u}$ be the canonical extension of u. We know that $\tilde{u}$ is canonical. Owing to the preceding proposition, it follows that $Y*_t\tilde{u}$ is canonical and $\lim_{t\downarrow 0}(Y*_t\tilde{u})=0$. Since $\frac{\partial}{\partial t}(Y*_t\tilde{u})=\tilde{u}=u=\frac{\partial v}{\partial t}$ for $t>0$, we may write $v-Y*_t\tilde{u}=\alpha(x)\otimes Y(t)$ for some $\alpha\in\mathcal{D}'(\Omega)$, which gives $\lim_{t\downarrow 0} v=\alpha$.

(b)$\Rightarrow$(a). As noted before, v admits the canonical extension $\tilde{v}$. From the relation $\rho_{(\varepsilon)}u=\frac{\partial}{\partial t}(\rho_{(\varepsilon)}v)-\rho'_{(\varepsilon)}v$ we obtain that $\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}u=\frac{\partial\tilde{v}}{\partial t}-\alpha\otimes\delta_t$ with

$\alpha = \lim_{t \downarrow 0} v$, which implies (a).

2. Canonical extensions in the strict sense

Let $\{g_k\}$ be an arbitrary δ-sequence with $\operatorname{supp} g_k \subset (0, \infty)$ and put $\rho_k = Y * g_k$. A distribution $\tilde{u} \in \mathcal{D}'(\Omega \times (-\infty, T))$ is called to be a canonical extension of $u \in \mathcal{D}'(\Omega \times (0, T))$ over $t=0$ in the strict sense if $\tilde{u} = \lim_{k \to \infty} \rho_k u$ holds. Also we shall say that $w \in \mathcal{D}'(\Omega \times (-\infty, T))$ is canonical in the strict sense if we have $w = \lim_{k \to \infty} \rho_k w$ for every $\{\rho_k\}$. Note that $1_x \otimes \sin \frac{1}{t} \in \mathcal{D}'(\Omega \times (0, T))$ has the canonical extension but in the strict sense it does not.

By a similar argument as in the proof of Lemma 3 in [7] we can easily prove the following

LEMMA 2. *Let $u \in \mathcal{D}'(\Omega \times (0, T))$. If $\lim_{t \downarrow 0} u$ exists in the strict sense*, i. e., *if $\lim_{k \to \infty} \langle u, g_k \rangle_t$ exists for every choice of a δ-sequence of $\{g_k\}$ stated above, then there exists for any given $\phi \in \mathcal{D}(\Omega)$ an interval $[0, \varepsilon_0]$ where $\langle u, \phi \rangle_x$ is equivalent to a bounded function continuous at 0, and therefore u has the canonical extension in the strict sense.*

On the basis of this lemma and Remark 1 in [7, p. 229] we can show the following two Propositions 4 and 5, which are analogues to Propositions 2 and 3 already proved in the preceding section. The proofs are omitted since the reasoning is very similar to that in the corresponding proofs.

PROPOSITION 4. *Let $v, w \in \mathcal{D}'(\Omega \times (-\infty, T))$ be such that $v=0$ for $t<0$ and $w = \frac{\partial v}{\partial t} \in \mathcal{D}'(\Omega \times (-\infty, T))$. The following conditions are equivalent:*

(a) *w is canonical in the strict sense.*

(b) *v is canonical, and has the boundary value $\lim_{t \downarrow 0} v = 0$ in the strict sense.*

(c) *$\lim_{k \to \infty} \langle v, g_k \rangle_t$ exists for every δ-sequence $\{g_k\}$.*

(d) *The partial multiplicative product $w(x, t) \cdot Y(t)$ exists.*

(e) *The partial multiplicative product $v(x, t) \cdot \delta_t$ exists.*

(f) *$\langle v, \phi \rangle$, $\phi \in \mathcal{D}(\Omega)$, is a bounded function in a neighbourhood of 0 continuous at 0.*

PROPOSITION 5. *Let $u, v \in \mathcal{D}'(\Omega \times (0, T))$ be such that $u = \frac{\partial v}{\partial t}$. The following conditions are equivalent:*

(a) *u has the canonical extension in the strict sense.*

(b) *The distributional boundary value $\lim_{t \downarrow 0} v$ in the strict sense exists.*

Now, let $\mathcal{H}_{(m,s)}(\bar{R}_+^{n+1})$ be the space of all distributions $u \in \mathcal{D}'(R_+^{n+1})$ such that there exists a distribution $U \in \mathcal{H}_{(m,s)}(R^{n+1})$ with $U=u$ on R_+^{n+1}. The norm of u is defined by $\|u\|_{(m,s)}=\inf\|U\|_{(m,s)}$, the infimum being taken over all such U. Here R_+^{n+1} denotes the half space $\{(x, t) \in R^{n+1}: x \in R^n \text{ and } t>0\}$, m a non-negative integer and s a real number (cf. Chapter II of L. Hörmander [1]).

$u \in \mathcal{H}_{(m,s)}(\bar{R}_+^{n+1})$ has always the canonical extension $\tilde{u}$ in the strict sense because of the fact that

$$\lim_{k,l\to\infty}\|\rho_k u-\rho_l u\|_{(0,s)}=\frac{1}{(2\pi)^n}\lim_{k,l\to\infty}\iint|(\rho_k-\rho_l)\hat{U}_{n+1}|^2(1+|\xi|^2)^s d\xi\,dt=0,$$

where $\hat{U}_{n+1}$ denotes the partial Fourier transform of U (cf. [1, p. 24]). Note that if u has an extension w in $\mathcal{H}_{(m,s)}(R^{n+1})$ such that $w=0$ for $t<0$, w must be $\tilde{u}$. In fact, $w-\tilde{u} \in \mathcal{H}_{(0,s)}(R^{n+1})$ will vanish for $t\neq 0$, which, together with the definition of $\mathcal{H}_{(0,s)}(R^{n+1})$, yields $w=\tilde{u}$.

From these considerations we can show the following

PROPOSITION 6. *The following conditions are equivalent for* $u\in\mathcal{H}_{(m,s)}(\bar{R}_+^{n+1})$:

(a) *There exists a distribution* $w \in \mathcal{H}_{(m,s)}(R^{n+1})$ *with* $\operatorname{supp} w\subset \bar{R}_+^{n+1}$ *such that* $w=u$ *on* R_+^{n+1}.

(b) $\tilde{u} \in \mathcal{H}_{(m,s)}(R^{n+1})$.

(c) $\lim_{t\downarrow 0} u=\lim_{t\downarrow 0}\frac{\partial u}{\partial t}=\cdots=\lim_{t\downarrow 0}\frac{\partial^{m-1}u}{\partial t^{m-1}}=0.$

(d) $\lim_{t\downarrow 0} u=\lim_{t\downarrow 0}\frac{\partial u}{\partial t}=\cdots=\lim_{t\downarrow 0}\frac{\partial^{m-1}u}{\partial t^{m-1}}=0$ *in the strict sense.*

PROOF. We have only to show the implications (b)$\Rightarrow$(d) and (c)$\Rightarrow$(b).

(b)$\Rightarrow$(d). We know that $\frac{\partial^j\tilde{u}}{\partial t^j}$, $j=0, \dots, m-1$, has the trace for $t=0$ (cf. Theorem 2.5.6 in [1]). Since $\mathcal{H}_{(m,s)}(R^{n+1})$ possesses the approximation property by reguralization, it follows from the corollary to Proposition 4 in [3] that the trace coincides with the section for $t=0$. Noting that $\tilde{u}=u$ for $t>0$, we can confer that $\lim_{t\downarrow 0} u=\lim_{t\downarrow 0}\frac{\partial u}{\partial t}=\cdots=\lim_{t\downarrow 0}\frac{\partial^{m-1}u}{\partial t^{m-1}}=0$.

(c)$\Rightarrow$(b). We shall proceed by induction on m. The case $m=0$ is trivial. Assuming then the implication holds with m replaced by $m-1$ for $m>0$, and taking into account the facts that $u \in \mathcal{H}_{(m-1,s+1)}(\bar{R}_+^{n+1})$ and $\frac{\partial u}{\partial t}\in\mathcal{H}_{(m-1,s)}(\bar{R}_+^{n+1})$, we obtain from (c) that $\tilde{u} \in \mathcal{H}_{(m-1,s+1)}(R^{n+1})$ and $\widetilde{\left(\frac{\partial u}{\partial t}\right)} \in \mathcal{H}_{(m-1,s)}(R^{n+1})$ hold. Hence the proof will be complete if we can show the equality $\widetilde{\left(\frac{\partial u}{\partial t}\right)}=\frac{\partial \tilde{u}}{\partial t}$. However, this follows from the calculations:

$$\left(\widetilde{\frac{\partial u}{\partial t}}\right)=\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}\frac{\partial u}{\partial t}$$
$$=\lim_{\varepsilon\downarrow 0}\frac{\partial}{\partial t}(\rho_{(\varepsilon)}u)-\lim_{\varepsilon\downarrow 0}\rho'_{(\varepsilon)}u=\frac{\partial\tilde{u}}{\partial t}.$$

3. Fine Cauchy problem

Consider the Cauchy problem for a system of partial differential equations in the unknown distributions $u_1, u_2, \cdots, u_s \in \mathcal{D}'(\Omega\times(0, T))$:

$$\frac{\partial^{n_i}u_i}{\partial t^{n_i}}=\sum_{j=1}^{s}\sum_{k=0}^{n_j-1}a^i_{k,j}(x, t, D_x)\frac{\partial^k u_j}{\partial t^k}+f_i, \qquad i=1, 2, \cdots, s \tag{1}$$

under the initial conditions

$$\lim_{t\downarrow 0}\frac{\partial^k u_i}{\partial t^k}=\alpha_{i,k+1}\in\mathcal{D}'(\Omega), \qquad i=1, 2, \cdots, s \quad k=0, 1, \cdots, n_i-1 \tag{2}$$

where $a^i_{k,j}(x, t, D_x)=\sum_\alpha a^i_{k,j,\alpha}(x, t)D^\alpha_x$ and $a^i_{k,j,\alpha}\in C^\infty(\Omega\times(-T_1, T))$.

Substituting $u_{i,k}=\dfrac{\partial^{k-1}u_i}{\partial t^{k-1}}$, $i=1, \cdots, s$, $k=1, \cdots, n_i$, we obtain an equivalent system of partial differential equations:

$$\begin{cases}\dfrac{\partial u_{i,1}}{\partial t}=u_{i,2}\\ \quad\vdots\\ \dfrac{\partial u_{i,n_i-1}}{\partial t}=u_{i,n_i}\\ \dfrac{\partial u_{i,n_i}}{\partial t}=\displaystyle\sum_{j=1}^{s}\sum_{k=0}^{n_j-1}a^i_{k,j}(x, t, D_x)u_{j,k+1}+f_i, \qquad i=1, 2, \cdots, s\end{cases} \tag{3}$$

with the initial conditions

$$\lim_{t\downarrow 0}(u_{i,1}, \cdots, u_{i,n_i})=(\alpha_{i,1}, \cdots, \alpha_{i,n_i}), \qquad i=1, 2, \cdots, s \tag{4}$$

which is a special case of the Cauchy problem for a system of partial differential equations:

$$\frac{\partial u}{\partial t}=P(x, t, D_x)u+f \tag{5}$$

$$\lim_{t\downarrow 0}u=\alpha, \tag{6}$$

where we used the vector notation: $u=(u_1, \cdots, u_m)$, $f=(f_1, \cdots, f_m)$ $\alpha=(\alpha_1, \cdots,$

α_m) with the obvious meanings and $P(x, t, D_x)$ denotes an $m\times m$ matrix $\|P_{ij}(x, t, D_x)\|$ such that

$$P_{i,j}(x, t, D_x)=\sum_{\alpha} a_{i,j,\alpha}(x, t)D_x^{\alpha}, \quad a_{i,j,\alpha}(x, t)\in C^{\infty}(\Omega\times(-T_1, T)).$$

We shall write $u\in\mathcal{D}'(\Omega\times(0, T))$ if each component u_j belongs to $\mathcal{D}'(\Omega\times(0, T))$ and we shall say that u has the canonical extension if each component u_j has the canonical extension. The term canonical will also be used in a similar way.

Put $Y_l=\dfrac{1}{(l-1)!}t_+^{l-1}$, l being a non-negative integer, where we understand $Y_0=\delta_t$. Note that Y_1 is the Heaviside function Y. Then we have

LEMMA 3. *Let $\phi\in C^{\infty}(\Omega\times(-T_1, T))$. Then, for any $v\in\mathcal{D}'(\Omega\times(-\infty, T))$, we have*

$$Y_l*_t(\phi v)=\sum_{j=0}^{l}\binom{l}{j}(-1)^j Y_j*_t((D_t^j\phi)(Y_l*_t v)), \tag{7}$$

where the notation $*_t$ means the partial convolution with respect to the variable t.

PROOF. We can write for $w=Y_l*_t v$

$$\phi D_t^l w=\sum_{j=0}^{l}\binom{l}{j}(-1)^j D_t^{l-j}(D_t^j\phi\cdot w).$$

Then, observing that $v=D_t^l w$, we obtain

$$\begin{aligned} Y_l*_t(\phi v)&=\sum_{j=0}^{l}\binom{l}{j}(-1)^j(Y_l*_t D_t^{l-j}(D_t^j\phi\cdot w))\\ &=\sum_{j=0}^{l}\binom{l}{j}(-1)^j(Y_j*_t(D_t^j\phi\cdot w)),\end{aligned}$$

where we have used the equalities

$$Y_l*_t D_t^{l-j}(D_t^j\phi\cdot w)=Y_j*_t(D_t^j\phi\cdot w), \qquad j=1, 2, \dots, l.$$

THEOREM 1. *Suppose there exists a solution $u\in\mathcal{D}'(\Omega\times(0, T))$ of the system* (5) *with the initial condition* (6). *Then f has the canonical extension $\tilde{f}$. If we denote by v the restriction of the canonical extension of u to $\Omega\times(-T_1, T)$, then v is a solution of the following system of partial differential equations:*

$$\frac{\partial v}{\partial t}=P(x, t, D_x)v+\tilde{f}+\alpha\otimes\delta_t. \tag{8}$$

Conversely, if $v\in\mathcal{D}'(\Omega\times(-T_1, T))$ is a solution of the system (8) *and*

$\operatorname{supp} v \subset \bar{\Omega} \times [0, T]$, *then the restriction* $u = v \mid \Omega \times (0, T)$ *is a solution of the system* (5) *with the initial condition* (6) *and the canonical extension of* u *coincides with* v *on* $\Omega \times (-T_1, T)$.

PROOF. Let $u \in \mathcal{D}'(\Omega \times (0, T))$ be a solution of the Cauchy problem for the system (5) with the initial condition (6). Then $\lim_{\varepsilon \downarrow 0} \rho'_{(\varepsilon)} u = \alpha \otimes \delta_t$ and, owing to Proposition 2, u has the canonical extension $\tilde{u}$. Now from the equalities:

$$\begin{aligned} \rho_{(\varepsilon)} f &= \rho_{(\varepsilon)} \frac{\partial u}{\partial t} - \rho_{(\varepsilon)} P(x, t, D_x) u \\ &= \frac{\partial}{\partial t}(\rho_{(\varepsilon)} u) - \rho'_{(\varepsilon)} u - P(x, t, D_x) \rho_{(\varepsilon)} u, \end{aligned} \tag{9}$$

passing to the limit $\varepsilon \downarrow 0$, we see that f has the canonical extension $\tilde{f}$ and that the restriction v satisfies the system (8).

Conversely, let v be a solution of (8) stated in the theorem. Now let w be an extension of v to $\Omega \times (-\infty, T)$ such that $w = 0$ for $t < 0$. Then, applying the equality (7) with l replaced by $k+1$, we obtain the equality

$$\begin{aligned} Y_k *_t w = \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^j Y_j *_t \left(\left(\frac{\partial^j}{\partial t^j} P(x, t, D_x) \right) \left(Y_{k+1} *_t w \right) \right) \\ + Y_{k+1} *_t \tilde{f} + \alpha \otimes Y_{k+1}, \end{aligned} \tag{10}$$

where k is any non-negative integer and $\frac{\partial^j}{\partial t^j} P(x, t, D_x)$ stands for the matrix $\| \sum_{\alpha} \left(\frac{\partial^j}{\partial t^j} a_{p,q,\alpha}(x, t) \right) D_x^{\alpha} \|$. We know that $\tilde{f}$ is canonical. It follows therefore from Proposition 2 that $Y_{k+1} *_t \tilde{f}$, $k \geqq 0$, is canonical and $\lim_{t \downarrow 0} (Y_{k+1} *_t \tilde{f}) = 0$. Evidently $\alpha \otimes Y_{k+1}$ is canonical for $k \geqq 0$ and $\lim_{t \downarrow 0} (\alpha \otimes Y_{k+1}) = 0$ for $k > 0$ and $\lim_{t \downarrow 0} (\alpha \otimes Y) = \alpha$. Let G be an arbitrary non-empty open subset such that $G \subset\subset \Omega$. We can find a non-negative integer k_0 with the property that $Y_{k_0+1} *_t w$ is canonical on $G \times (-\infty, T)$. Using the equality (10) with k replaced by k_0, we see that $Y_{k_0} *_t w$ is also canonical on $G \times (-\infty, T)$. Repeating the same argument and proceeding step by step, we can conclude, since G is arbitrary, that w is canonical. Again using the equality (10) for $k = 0$:

$$w = P(x, t, D_x)(Y *_t w) - \left(\frac{\partial}{\partial t} P(x, t, D_x) \right) (Y *_t w) + Y *_t \tilde{f} + \alpha \otimes Y, \tag{11}$$

we have $\lim_{t \downarrow 0} u = \lim_{t \downarrow 0} w = \alpha$, completing the proof.

COROLLARY 1. *Suppose there exists a solution* $(u_1, \cdots, u_s)$, $u_i \in \mathcal{D}'(\Omega \times (0, T))$, *of the Cauchy problem for the system* (1) *with the initial condition* (2). *Then* f_i *has the canonical extension* $\tilde{f}_i$ *for each* i. *If we denote by* v_i *the restriction* *of* $\tilde{u}_i$ *to* $\Omega\times(-T_1, T)$, *then* $(v_1, \cdots, v_s)$, $v_i\in\mathcal{D}'(\Omega\times(-T_1, T))$, *is a solution of the following system of partial differential equations:*

$$\frac{\partial^{n_i}v_i}{\partial t^{n_i}}=\sum_{j=1}^{s}\sum_{k=0}^{n_j-1}a_{k,j}^{i}(x, t, D_x)\frac{\partial^k v_j}{\partial t^k}+\tilde{f}_i+H_i, \tag{12}$$

where

$$H_i=\sum_{\nu=1}^{n_i}\alpha_{i,\nu}\otimes\delta_t^{(n_i-\nu)}-\sum_{j=1}^{s}\sum_{k=1}^{n_j-1}a_{k,j}^{i}(x, t, D_x)\sum_{\nu=1}^{k}\alpha_{j,\nu}\otimes\delta_t^{(k-\nu)}.$$

Conversely, suppose there exists a solution $(v_1, \cdots, v_s)$, $v_i\in\mathcal{D}'(\Omega\times(-T_1, T))$ *with* $\operatorname{supp}v_i\subset\bar{\Omega}\times[0, T]$, *of the system* (12). *If we denote by* u_i *the restriction of* v_i *to* $\Omega\times(0, T)$, *then* $(u_1, \cdots, u_s)$ *is a solution of the system* (1) *with the initial condition* (2) *and* $\tilde{u}_i$ *coincides with* v_i *on* $\Omega\times(-T_1, T)$.

PROOF. Let $(u_1, \cdots, u_s)$ be a solution of (1) with the initial condition (2). To prove (12) we use the identities:

$$\rho_{(\varepsilon)}\frac{\partial^l u_i}{\partial t^l}=\frac{\partial^l}{\partial t^l}(\rho_{(\varepsilon)}u_i)-\frac{\partial^{l-1}}{\partial t^{l-1}}(\phi_\varepsilon u_i)-\frac{\partial^{l-2}}{\partial t^{l-2}}\left(\phi_\varepsilon\frac{\partial u_i}{\partial t}\right)-\cdots-\phi_\varepsilon\frac{\partial^{l-1}u_i}{\partial t^{l-1}},$$
$$i=1, 2, \cdots, s \tag{13}$$

where $\phi_\varepsilon=\rho'_{(\varepsilon)}$. Since, owing to Proposition 2, u_i has the canonical extension $\tilde{u}_i$, and since $\lim_{t\downarrow 0}\frac{\partial^k u_i}{\partial t^k}=\alpha_{i,k+1}$ implies $\lim_{\varepsilon\downarrow 0}\phi_\varepsilon\frac{\partial^k u_i}{\partial t^k}=\alpha_{i,k+1}\otimes\delta_t$ for $k=0, \cdots, n_i-1$, it follows from the identities (13) that $\frac{\partial^l u_i}{\partial t^l}$, $i=1, 2, \cdots, s$, $l=0, \cdots, n_i$, has the canonical extension $\frac{\partial^l\tilde{u}_i}{\partial t^l}-\sum_{\nu=1}^{l}\alpha_{i,\nu}\otimes\delta_t^{(l-\nu)}$. This, together with (1), yields that f_i has the canonical extension $\tilde{f}_i$. Multiplying both sides of (1) by $\rho_{(\varepsilon)}$ and passing to the limit $\varepsilon\downarrow 0$, we can conclude that the restriction $(v_1, \cdots, v_s)$ of $(u_1, \cdots, u_s)$ to $\Omega\times(-T_1, T)$ satisfies (12).

Conversely, let $v=(v_1, \cdots, v_s)$ be a solution of (12) such that v vanishes for $t<0$. Putting

$$\begin{cases} v_{i,1}=v_i \\ v_{i,2}=\dfrac{\partial v_{i,1}}{\partial t}-\alpha_{i,1}\otimes\delta_t \\ \vdots \\ v_{i,n_i}=\dfrac{\partial v_{i,n_i-1}}{\partial t}-\alpha_{i,n_i-1}\otimes\delta_t, \qquad i=1, 2, \cdots, s \end{cases} \tag{14}$$

we can write

$$\begin{cases} \dfrac{\partial v_{i,1}}{\partial t} = v_{i,2} + \alpha_{i,1} \otimes \delta_t \\ \quad\vdots \\ \dfrac{\partial v_{i,n_i-1}}{\partial t} = v_{i,n_i} + \alpha_{i,n_i-1} \otimes \delta_t \\ \dfrac{\partial v_{i,n_i}}{\partial t} = \sum_{j=1}^{s} \sum_{k=0}^{n_j-1} a_{k,j}^{i}(x, t, D_x) v_{j,k+1} + \tilde{f}_i + \alpha_{i,n_i} \otimes \delta_t, \qquad i=1, 2, \cdots, s. \end{cases} \tag{15}$$

If we denote by $u_{i,j}$ the restriction of $v_{i,j}$ to $\Omega \times (0, T)$ for each i, j, we see, from the preceding theorem, that $(u_{i,1}, \cdots, u_{i,n_i})$, $i=1, \cdots, s$, is a solution of (3) with the initial condition (4). Since $u_i = u_{i,1}$, it follows that $(u_1, \cdots, u_s)$ is a solution of (1) with the initial condition (2). Owing to Theorem 1, the canonical extension of u_i coincides with v_i on $\Omega \times (-T_1, T)$, completing the proof.

From the discussions of Section 2 it is easily verified that the assertions of Theorem 1 and Corollary 1 valid even when the terms, such as the canonical extension, the canonical and the distributional limit, are taken in the strict sense.

REMARK 1. $H_i = 0$, $i=1, \cdots, s$, if and only if $\alpha_{i,k} = 0$, $i=1, \cdots, s$, $k=1, \cdots, n_i$. Indeed, if $\alpha_{i,k} = 0$ for each i, k, then clearly $H_i = 0$. Conversely, let $H_i = 0$ for $i=1, \cdots, s$. Without loss of generality we may assume that $n_1 \geqq n_2 \geqq \cdots \geqq n_s$. First consider the case $n_1 = n_2 = \cdots = n_s$. $\delta, \delta', \cdots$ are linearly independent. It follows therefore from the equations $H_i = 0$ that $\alpha_{i,1} = 0$ for $i=1, \cdots, s$. Substituting these into the expression of H_i, we can apply the same argument to obtain $\alpha_{i,2} = 0$ for $i=1, \cdots, s$. Repeating this process, finally we shall reach the conclusion. Let us turn to the general case. It will be sufficient to show that the case $n_1 = \cdots = n_i > n_{i+1} \geqq \cdots \geqq n_s$ can be reduced to the case $n_1 = \cdots = n_{i+1} > n_{i+2} \geqq \cdots \geqq n_s$. To do so, we note that, by the same argument as above, $H_l = 0$, $l=1, \cdots, i$, imply $\alpha_{l,m} = 0$ for $m=1, \cdots, n_l - n_{i+1}$. For our purpose it will then be sufficient to rewrite the equations $H_k = 0$, $k=1, \cdots, s$ by replacing $\alpha_{l, n_l - n_{i+1} + \nu}$ by $\beta_{l,\nu}$ for $l=1, \cdots, i$, $\nu=1, \cdots, n_{i+1}$.

PROPOSITION 7. *Let $u \in \mathcal{D}'(\Omega \times (0, T))$ be a solution of* (5):

$$\frac{\partial u}{\partial t} = P(x, t, D_x) u + f.$$

If $f \in \mathcal{D}'(\Omega \times (0, T))$ has the canonical extension $\tilde{f}$, then the following conditions are equivalent:

(a) *u can be extended over $t=0$.*

(b) *The distributional boundary value $\lim\limits_{t \downarrow 0} u(x, t)$ exists.*

PROOF. Owing to Proposition 1, we have only to show the implication (a)$\Rightarrow$(b). Let w be an extension of u over $t=0$. For any non-empty open subset $G \subset\subset \Omega$ and a small $\varepsilon_0 > 0$, we can find a non-negative integer k, a multi-index α and a vector $F=(F_1, \cdots, F_m)$, $F_i \in C(G\times(-\varepsilon_0, \varepsilon_0))$, such that $w=D_x^\alpha D_t^k F$ on $G\times(-\varepsilon_0, \varepsilon_0)$. If we put $v=D_x^\alpha D_t^k \tilde{F}$, where $\tilde{F}$ equals F for $t\geqq 0$ and 0 for $t<0$, then the support of $\frac{\partial v}{\partial t}-P(x, t, D_x)v-\tilde{f}$ lies in the hypersurface $t=0$ and therefore there exist a non-negative integer l and a vector $a_i \in \mathcal{D}'(G)$ such that

$$\frac{\partial v}{\partial t}-P(x, t, D_x)v=\sum_{i=0}^{l} a_i\otimes\delta_t^{(i)}+\tilde{f} \tag{16}$$

on $G\times(-\varepsilon_0, \varepsilon_0)$. Here we may assume that $l=0$. Indeed, if $l>0$, then $'v=v+a_l\otimes\delta_t^{(l-1)}$ is an extension of u such that $'v=0$ for $t<0$ and satisfies

$$\frac{\partial 'v}{\partial t}-P(x, t, D_x)'v=\sum_{i=0}^{l-1} a_i\otimes\delta_t^{(i)}-P(x, t, D_x)a_l\otimes\delta_t^{(l-1)}+\tilde{f},$$

where the right side can be written in the form $\sum_{i=0}^{l-1} b_i\otimes\delta_t^{(i)}+\tilde{f}$, $b_i \in \mathcal{D}'(G)$. Consequently we can apply Theorem 1. Thus (a) implies (b).

COROLLARY 2. *For a solution $u \in \mathcal{D}'(\Omega\times(0, T))$ of a homogeneous system $\frac{\partial u}{\partial t}=P(x, t, D_x)u$, u can be extended over $t=0$ if and only if the distributional limit $\lim_{t\downarrow 0} u$ exists.*

In what follows, if $u \in \mathcal{D}'(\Omega\times(0, T))$ belongs to $\mathcal{D}'(\Omega)\hat{\otimes} C((0, T))$, u is said to be continuous in t and identified in an obvious way with a $\mathcal{D}'(\Omega)$-valued continuous function denoted by $\boldsymbol{u}(t)$, $0<t<T$. Similarly for distributions on $\Omega\times(-\infty, T)$.

PROPOSITION 8. *Let $u \in \mathcal{D}'(\Omega\times(0, T))$ be a solution of* (5):

$$\frac{\partial u}{\partial t}=P(x, t, D_x)u+f.$$

The following conditions are equivalent:

(a) *u is continuous in t, $0<t<T$.*

(b) *For any g such that $f=\frac{\partial g}{\partial t}$, g is continuous in t, $0<t<T$.*

PROOF. (a)$\Rightarrow$(b). Owing to Proposition 5 in [7, p. 229], we have only to show that $g(x, t)\cdot\delta_{(t')}$ exists for every t', $0<t'<T$, where $\delta_{(t')}$ is the Dirac measure concentrated at t'. u is continuous in t. This implies that $u(x, t)\cdot\delta_{(t')}$ exists. From Remark 1 in [7, p. 229] it follows that there exist $\frac{\partial u}{\partial t}\cdot Y(t-t')$ and $u(x, t)\cdot Y(t-t')$. Since u satisfies (5), we see that $f(x, t)\cdot Y(t-t')$ exists. Using again Remark 1 just referred to, we can conclude that $g(x, t)\cdot\delta_{(t')}$ is well defined.

(b)$\Rightarrow$(a). First we shall show that f has the canonical extension $\tilde{f}_{t'}$ in the strict sense over $t=t'$, $0<t'<T$. Since g is continuous in t, it follows by the same argument as above that $f(x, t)\cdot Y(t-t')$ is well defined, and a fortiori f has the canonical extension $\tilde{f}_{t'}$ in the strict sense over $t=t'$. Applying Proposition 7, we see that the distributional boundary value $\lim_{t\downarrow 0} u(x, t)=\alpha_{t'}$ exists in the strict sense. If we denote by $\tilde{u}_{t'}$ the canonical extension of u over $t=t'$, then $\tilde{u}_{t'}$ satisfies

$$\frac{\partial \tilde{u}_{t'}}{\partial t}=P(x, t, D_x)\tilde{u}_{t'}+\tilde{f}_{t'}+\alpha_{t'}\otimes\delta_{(t')}.$$

Consider the equality:

$$\begin{aligned}Y_k*_t\tilde{u}_{t'}=&\sum_{j=0}^{k+1}\binom{k+1}{j}(-1)^j Y_j*_t\Big(\Big(\frac{\partial^j}{\partial t^j}P(x, t, D_x)\Big)(Y_{k+1}*_t\tilde{u}_{t'})\Big)\\&+Y_{k+1}*_t\tilde{f}_{t'}+\alpha_{t'}\otimes Y_{k+1}(t-t'),\end{aligned}$$

where k is a non-negative integer. $Y_{k+1}*_t\tilde{f}_{t'}$ and $\alpha_{t'}\otimes Y_{k+1}(t-t')$ are continuous in t, $t'<t<T$. Given any non-empty open subset $G\subset\subset\Omega$, we can find a non-negative integer k_0 such that $Y_{k_0+1}*_t\tilde{u}_{t'}$ is continuous in t, $-\infty<t<T$. Using the same consideration as in the proof of Theorem 1, we see that $\tilde{u}_{t'}$ is continuous in t, $t'<t<T$. Since G and t' are arbitrary, we can conclude that u is continuous in t, $0<t<T$, which completes the proof.

PROPOSITION 9. *Let $u\in\mathcal{D}'(\Omega\times(0, T))$ be a solution of* (5):

$$\frac{\partial u}{\partial t}=P(x, t, D_x)u+f.$$

The following conditions are equivalent:

(a) *u is continuous in t and $\boldsymbol{u}(t)$ is a $\mathcal{D}'(\Omega)$-valued continuously differentiable function of t, $0<t<T$.*

(b) *f is continuous in t, $0<t<T$.*

Moreover, under any one of these equivalent conditions, $\boldsymbol{u}(t)$ satisfies the equation:

$$\boldsymbol{u}'(t)=P(x, t, D_x)\boldsymbol{u}(t)+\boldsymbol{f}(t). \tag{17}$$

PROOF. The implication (a)$\Rightarrow$(b) is trivial.

(b)$\Rightarrow$(a). Since f is written in the form $f=\dfrac{\partial g}{\partial t}$, where g is continuous in t, it follows from Proposition 8 that the solution u is also continuous in t, $0<t<T$. If we put $w=\dfrac{\partial u}{\partial t}$ and differentiate the both sides of (5) with respect to t, we obtain

$$\frac{\partial w}{\partial t}=P(x, t, D_x)w+\Big(\Big(\frac{\partial}{\partial t}P(x, t, D_x)\Big)u+\frac{\partial f}{\partial t}\Big)$$

on $\Omega \times (0, T)$. By Proposition 8, w is continuous in t, $0<t<T$. u and w being continuous in t, it is easy to verify that $\boldsymbol{u}(t)$ has the continuous derivative $\boldsymbol{u}'(t)$ such that $\boldsymbol{u}'(\mathrm{t})=w(\hat{x}, t)$ (cf, [6, p. 57]), and satisfies the equation (17).

Applying Theorem 1 and Proposition 8, we shall show

THEOREM 2. *Let* $u \in \mathcal{D}'(\Omega \times (0, T))$ *be a solution of* (5):

$$\frac{\partial u}{\partial t} = P(x, t, D_x)u + f$$

with the initial condition (6): $\lim_{t \downarrow 0} u = \alpha$. *The following conditions are equivalent*:

(a) *u is continuous in t, $0<t<T$, and $\lim_{t \downarrow 0} u = \alpha$ in the strict sense.*

(b) *u is continuous in t and $\lim_{\varepsilon \downarrow 0} \boldsymbol{u}(\varepsilon) = \alpha$.*

(c) *$Y *_t \tilde{f}$ is continuous in t, $-\infty < t < T$.*

PROOF. We have only to show the implications (a)$\Rightarrow$(c) and (c)$\Rightarrow$(b).

(a)$\Rightarrow$(c). In virtue of Proposition 8, we see that $Y *_t \tilde{f}$ is continuous in t, $0<t<T$. Since $\tilde{f}$ is canonical in the strict sense, Proposition 4 implies that $\lim_{k \to \infty} < Y *_t \tilde{f}, g_k >_t = 0$ for every $\{g_k\}$ and therefore $Y *_t \tilde{f}$ is continuous in t, $-\infty < t < T$.

(c)$\Rightarrow$(b). Owing to Proposition 8, we know that u is continuous in t. Making use of the equality (10) with w replaced by $\tilde{u}$ and modifying the proof of Proposition 8, we can conclude from the equality

$$\tilde{u} = P(x, t, D_x)(Y *_t \tilde{u}) - \left(\frac{\partial}{\partial t} P(x, t, D_x)\right)(Y *_t \tilde{u}) + Y *_t \tilde{f} + \alpha \otimes Y$$

that $\lim_{\varepsilon \downarrow 0} \boldsymbol{u}(\varepsilon)$ exists and equals α.

COROLLARY 3. *Let* $u \in \mathcal{D}'(\Omega \times (0, T))$ *be a solution of* (5):

$$\frac{\partial u}{\partial t} = P(x, t, D_x)u + f.$$

Suppose f has the canonical extension $\tilde{f}$ such that $Y *_t \tilde{f}$ is continuous in t, $-\infty < t < T$. The following conditions are equivalent:

(a) $\lim_{t \downarrow 0} u$ *exists.*

(b) $\lim_{t \downarrow 0} u$ *exists in the strict sense.*

(c) *u is continuous in t and $\lim_{\varepsilon \downarrow 0} \boldsymbol{u}(\varepsilon)$ exists.*

(d) *u can be extended over $t=0$.*

PROOF. By Theorem 2, (a), (b) and (c) are equivalent, and by Proposition 7, (a) and (d) are equivalent.

Applying Theorems 1, 2 and Proposition 9, we can show

COROLLARY 4. *Let* $u \in \mathcal{D}'(\Omega \times (0, T))$ *be a solution of* (5):

$$\frac{\partial u}{\partial t} = P(x, t, D_x)u + f$$

with the initial condition (6): $\lim_{t \downarrow 0} u = \alpha$. *If* f *is continuous in* t *and* $\lim_{\varepsilon \downarrow 0} \boldsymbol{f}(\varepsilon)$ *exists, then* u, $\frac{\partial u}{\partial t}$ *are continuous in* t, $\boldsymbol{u}(t)$ *is continuously differentiable in* t, $0 < t < T$, *and* $\lim_{\varepsilon \downarrow 0} \boldsymbol{u}(\varepsilon)$, $\lim_{\varepsilon \downarrow 0} \boldsymbol{u}'(\varepsilon)$ *exist.*

PROOF. Evidently $Y * \tilde{f}$ is continuous in t, $-\infty < t < T$. By Theorem 2, u is continuous in t and $\lim_{\varepsilon \downarrow 0} \boldsymbol{u}(\varepsilon) = \alpha$ exists. Differentiating (5) with respect to t we have

$$\frac{\partial^2 u}{\partial t^2} = P(x, t, D_x)\frac{\partial u}{\partial t} + \Big(\frac{\partial}{\partial t} P(x, t, D_x)\Big)u + \frac{\partial f}{\partial t},$$

where $g = \Big(\frac{\partial}{\partial t} P(x, t, D_x)\Big)u + \frac{\partial f}{\partial t}$ has the canonical extension equal to

$$\Big(\frac{\partial}{\partial t} P(x, t, D_x)\Big)\tilde{u} + \frac{\partial \tilde{f}}{\partial t} - (\lim_{\varepsilon \downarrow 0} \boldsymbol{f}(\varepsilon)) \otimes \delta_t.$$

Consequently

$$Y *_t \tilde{g} = Y *_t \Big(\frac{\partial}{\partial t} P(x, t, D_x)\Big)\tilde{u} + \tilde{f} - (\lim_{\varepsilon \downarrow 0} \boldsymbol{f}(\varepsilon)) \otimes Y,$$

which is continuous in $t, -\infty < t < T$. Therefore, by Theorem 2, $\frac{\partial u}{\partial t}$ is continuous in t, hence $\boldsymbol{u}(t)$ is continuously differentiable and $\lim_{\varepsilon \downarrow 0} \boldsymbol{u}'(\varepsilon)$ exists.

So far, we have considered the distributions on $\Omega \times (0, T)$ and their canonical extensions over $t=0$. It is clear that a similar argument can be applied to the distributions on $\Omega \times (-T_1, 0)$. A distribution $u \in \mathcal{D}'(\Omega \times (-T_1, 0))$ is said to have a canonical extension $\tilde{u}_- \in \mathcal{D}'(\Omega \times (-T_1, \infty))$ over $t=0$ if $\tilde{u}_- = \lim_{\varepsilon \downarrow 0} \check{\rho}_{(\varepsilon)} u$. For example, let $u \in \mathcal{D}'(\Omega \times (-T_1, 0))$ be a solution of a differential system:

$$\frac{\partial u}{\partial t} = P(x, t, D_x)u + f$$

with the initial condition

$$\lim_{t \uparrow 0} u = \alpha,$$

where f is a distribution on $\Omega\times(-T_1, 0)$. Then f has the canonical extension $\tilde{f}_-$ and the restriction v of $\tilde{u}_-$ to $\Omega\times(-T_1, T)$ will satisfy the equation:

$$\frac{\partial v}{\partial t}=P(x, t, D_x)v+\tilde{f}_- -\alpha\otimes\delta_t. \tag{18}$$

Conversely, any solution v of (18) vanishing for $t>0$ will be the restriction of the canonical extension of a solution for the above Cauchy problem. All the discussions given in Section 1 through Section 3 will remain true with necessary modifications.

EXAMPLE. Let u be a harmonic function on $\Omega\times(0, T)$. We note that the following conditions are equivalent:

(a) The distributional boundary value $\lim\limits_{t\downarrow 0} u$ exists.

(b) The distributional boundary value $\lim\limits_{t\downarrow 0}\frac{\partial u}{\partial t}$ exists.

(c) u can be extended to a distribution over $t=0$.

Suppose $\lim\limits_{t\downarrow 0} u$ and $\lim\limits_{t\downarrow 0}\frac{\partial u}{\partial t}$ exist: $\alpha_1=\lim\limits_{t\downarrow 0} u$, $\alpha_2=\lim\limits_{t\downarrow 0}\frac{\partial u}{\partial t}$. If either of α_1 and α_2 equals 0 then u can be extended to a harmonic function on $\Omega\times(-T, T)$. This was noted in our previous paper [4, p. 413] without proof in the case $n>1$. We shall here give the proof. Put $u_1=u$, $u_2=\frac{\partial u}{\partial t}$. Then u_1, u_2 have the canonical extensions $\tilde{u}_1$, $\tilde{u}_2$ and

$$\begin{cases}\dfrac{\partial\tilde{u}_1}{\partial t}=\tilde{u}_2+\alpha_1\otimes\delta_t\\[2ex] \dfrac{\partial\tilde{u}_2}{\partial t}=-\displaystyle\sum_{i=1}^{n}\frac{\partial^2\tilde{u}_1}{\partial x_i^2}+\alpha_2\otimes\delta_t\end{cases}$$

on $\Omega\times(-\infty, T)$. If we put $w(x, t)=u(x, -t)$ for $t<0$, then w is harmonic on $\Omega\times(-T, 0)$ and $w_1=w$, $w_2=\frac{\partial w}{\partial t}$ have the canonical extensions $\tilde{w}_1=\lim\limits_{\varepsilon\downarrow 0}\check{\rho}_{(\varepsilon)}w_1$, $\tilde{w}_2=\lim\limits_{\varepsilon\downarrow 0}\check{\rho}_{(\varepsilon)}w_2$ and

$$\begin{cases}\dfrac{\partial\tilde{w}_1}{\partial t}=\tilde{w}_2-\alpha_1\otimes\delta_t\\[2ex] \dfrac{\partial\tilde{w}_2}{\partial t}=-\displaystyle\sum_{i=1}^{n}\frac{\partial^2\tilde{w}_1}{\partial x_i^2}+\alpha_2\otimes\delta_t\end{cases}$$

on $\Omega\times(-T, \infty)$. Let $\alpha_1=0$. If we put $v_1=\tilde{u}_1-\tilde{w}_1$, $v_2=\tilde{u}_2-\tilde{w}_2$, then v_1, v_2 are harmonic on $\Omega\times(-T, T)$ and $\frac{\partial v_1}{\partial t}=v_2$, $\frac{\partial v_2}{\partial t}=-\sum_{i=1}^{n}\frac{\partial^2 v_1}{\partial x_i^2}$ and therefore $\frac{\partial^2 v_1}{\partial t^2}=-\sum_{i=1}^{n}\frac{\partial^2 v_1}{\partial x_i^2}$, i. e., $\Delta v_1=0$. Thus u_1 is continued to a harmonic function v_1. Similarly, in the case $\alpha_2=0$, if we put $v_3=\tilde{u}_1+\tilde{w}_1$, $v_4=\tilde{u}_2+\tilde{w}_2$, then v_3, v_4

are harmonic on $\Omega\times(-T, T)$ and $\frac{\partial v_3}{\partial t}=v_4$, $\frac{\partial v_4}{\partial t}=-\sum_{i=1}^{n}\frac{\partial^2 v_3}{\partial x_i^2}$ and therefore $\Delta v_3=0$. Thus u_1 is continued to a harmonic function v_3.

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