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On the Decomposition of a Linearly Connected Manifold with Torsion.

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§ 1. Introduction

Let M be a differentiable manifold with a linear connection, and let Φ_x be the homogeneous holonomy group at a point $x \in M$. If the tangent vector space at x is decomposed into a direct sum of subspaces which are invariant under Φ_x, then by the parallel displacements along curves on M, parallel distributions are defined on M corresponding to those subspaces. If M is a Riemannian manifold and its connection is Riemannian, then by the de Rham decomposition theorem ([7] or [4] p. 185) the above parallel distributions are completely integrable and, at any point, M is locally isometric to the direct product of leaves through the point. Moreover, if M is simply connected and complete, it is globally isometric to the direct product of those leaves (see also [7] or [4] p. 192).

The above local and global decomposition theorems of de Rham are generalized to the case of pseudo-Riemannian manifold by H. Wu ([9]). On the other hand, in [2], S. Kashiwabara generalized the global decomposition theorem to the case of linearly connected manifold without torsion, under the assumption of local decomposability.

In the present paper, a linearly connected manifold with torsion will be treated and a condition of local decomposition will be given in terms of curvature and torsion (Theorem 1). Next, in §4, the results will be applied to a reductive homogeneous space with the canonical connection of the second kind, using the notion of algebra introduced by A. A. Sagle in [8].

Finally, in §5, we shall remark about the decomposition of a local loop with any point in M as its origin ([3]), corresponding to the local decomposition of the linearly connected manifold M.

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§ 2. Integrability of parallel distributions

Let (M, ∇) be a connected differentiable manifold with a linear connection, where ∇ means the covariant differentiation of the connection. The curvature tensor R and the torsion tensor S are defined by the formulas:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{2.1}$$

$$S(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{2.2}$$

for vector fields X, Y and Z on M.

DEFINITION 1. For a subspace T'_x of the tangent vector space $T_x(M)$ at x, the torsion tensor S is said to be *inducible* to T'_x at x if $S_x(X_x, Y_x) \in T'_x$ for any X_x and Y_x in T'_x. When $T_x(M)$ is decomposed into a direct sum of complementary subspaces T'_x and T''_x, the torsion S is said to be *completely inducible* with respect to the direct sum if S is inducible to each of T'_x and T''_x, and $S_x(X_x, Y_x) = 0$ for $X_x \in T'_x$ and $Y_x \in T''_x$. The complete inducibility of torsion with respect to a direct sum of finite number of subspaces will be defined similarly. The torsion S is said to be inducible or completely inducible to distributions if it is so at every point in M.

If, at a point x_0 in M, a subspace T'_{x_0} of the tangent vector space $T_{x_0}(M)$ is invariant under the homogeneous holonomy group Φ_{x_0}, the parallel displacements along curves joining x_0 to all points of M define a parallel distribution T' on M. In fact the result of parallel displacement of T'_{x_0} to a point x is independent of the choice of curves from x_0 to x.

PROPOSITION 1. A parallel distribution T is completely integrable if and only if the torsion tensor S is inducible to T.

PROOF. Let Y be a vector field in T. Since T is parallel, $\nabla_X Y$ also belongs to T for any vector field X on M. Hence for any pair of vector fields X, Y in T, vector fields $\nabla_X Y$ and $\nabla_Y X$ belong to T. Now, from (2.2) the bracket $[X, Y]$ of X and Y in T belongs to T if and only if $S(X, Y)$ belongs to T.

PROPOSITION 2. Let T' be a completely integrable parallel distribution on M, and N be a leaf (maximal integral manifold) of T'. Then N is a totally geodesic submanifold of (M, ∇). Moreover, (M, ∇) induces a linear connection ∇' on N whose curvature tensor R' and torsion tensor S' are tensors induced naturally in N from R and S respectively.

PROOF. A geodesic which passes through a point x in N and is tangent to N at x has its tangent vectors in the parallel distribution T'. Since N is a leaf of T' through x, this geodesic is a curve in N([4], p. 86). Hence by definition N is a totally geodesic submanifold of M. Next, let X, Y be tangent vector fields on N. If $X_{x_0} \neq 0$, denote by $\tau(s)$ the parallel displacement of vectors along a trajectory $c(s)$ of X in a neighborhood of $x_0 = c(0)$ in N. Since any vector obtained by parallel displacement of a tangent vector of N also is tangent to N, $\tau(s)^{-1} Y_{c(s)}$ is contained in $T_{x_0}(N)$. Thus we can define an operation ∇' by the formula:

$$(\nabla'_x Y)_{x_0} = \lim_{s \to 0} \frac{1}{s} (\tau(s)^{-1} Y_{c(s)} - Y_{x_0}) \tag{2.3}$$

for any vector fields $X(X_{x_0}\neq 0)$ and Y on N, and by setting $(\nabla'_X Y)_{x_0}=0$ for $X_{x_0}=0$. Then ∇' defines a linear connection on N. In fact, let $(u^1, u^2, \cdots, u^n, u^{n+1}, \cdots, u^m)$ be a system of coordinates valid in a neighborhood V of x_0 in M such that $u^{n+1}=0, \cdots, u^m=0$ define N in a neighborhood of x_0 and $(u^1, \cdots, u^n)$ is a system of coordinates in a neighborhood U of x_0 in N. Then the second term of (2.3) has an expression

$$\left(\frac{dY^a}{ds}+\Gamma^a_{bc}\frac{dc^b}{ds}Y^c\right)_{s=0}\left(\frac{\partial}{\partial u^a}\right)_{x_0}, \quad 1\leqq a, b, c\leqq n, \tag{2.4}$$

in the local coordinates, where Γ^i_{jk}'s are the coefficients of given connection expressed in V ([1], p. 41). From (2.4) we see that the operation $\nabla' : \mathfrak{X}'\times\mathfrak{X}'\to\mathfrak{X}'$ satisfies the conditions of covariant differentiation on N, where $\mathfrak{X}'$ denotes the module of vector fields on N. For any vector fields X, Y on N, if we choose vector fields X^*, Y^* on M which coincide with X, Y respectively on an open subset U of N, we have $(\nabla_{X^*}Y^*)_x=(\nabla'_X Y)_x$ at each point x in U. Therefore, from (2.1) and (2.2), the curvature R' and the torsion S' of (N, ∇') are equal to the tensors induced naturally by restricting R and S to N respectively.

§ 3. Decomposition of linearly connected manifolds

Let (M', ∇') and (M'', ∇'') be connected manifolds each of which has a linear connection. We choose a covering of $M=M'\times M''$ by coordinate neighborhoods adapted to the direct product, that is, each coordinate neighborhood U is a direct product of coordinate neighborhoods U' in M' and U'' in M'' with a system of coordinates $('u^1, 'u^2, \cdots, 'u^a, \cdots, 'u^{m'}, ''u^1, \cdots, ''u^\alpha, \cdots, ''u^{m''})$ where $('u^a)$ and $(''u^\alpha)$ are systems of local coordinates on U' and U'' respectively and $m'=\dim M'$, $m''=\dim M''$.

We shall define a linear connection on M by associating a family of functions $\Gamma_{(U)}=\{\Gamma^i_{jk}('u^a, ''u^\alpha)\}$ with each adapted coordinate neighborhood U as follows

$$\Gamma^a_{bc}('u, ''u)={}'\Gamma^a_{bc}('u) \quad \text{for} \quad 1\leqq a, b, c\leqq m', \tag{3.1}$$

$$\Gamma^i_{jk}('u, ''u)={}''\Gamma^\alpha_{\beta\gamma}(''u) \quad \text{for} \quad 1\leqq \alpha, \beta, \gamma\leqq m'', \quad \text{and} \quad i=m'+\alpha, \; j=m'+\beta, \; k=m'+\gamma, \tag{3.2}$$

$$\Gamma^i_{jk}('u, ''u)=0 \quad \text{for the rest}, \tag{3.3}$$

where ${}'\Gamma^a_{bc}('u)$'s (resp. ${}''\Gamma^\alpha_{\beta\gamma}(''u)$'s) are the coefficients of the connection on M' (resp. M'') with respect to the local coordinates $('u^a)$ (resp. $(''u^\alpha)$). If $U=U'\times U''$ and $V=V'\times V''$ are coordinate neighborhoods adapted to the direct product and if they have common points, two families $\Gamma_{(U)}$ and $\Gamma_{(V)}$ are related to each other in the law of transformation of coefficients of a linear connection;

$$\Gamma_{(V)qr}^{\ p}=\sum_{i,j,k=1}^{m'+m''}\frac{\partial u^j}{\partial v^q}\frac{\partial u^k}{\partial v^r}\frac{\partial v^p}{\partial u^i}\Gamma_{(U)jk}^{\ i}+\sum_{i=1}^{m'+m''}\frac{\partial^2 u^i}{\partial v^q \partial v^r}\frac{\partial v^p}{\partial u^i},\quad 1\leqq p,\ q,\ r\leqq m'+m''. \tag{3.4}$$

Thus a linear connection is defined on $M=M'\times M''$.

DEFINITION 2. The product manifold $M=M'\times M''$ with the linear connection defined above will be called the *affine product* of (M', ∇') and (M'', ∇'') ([2]).

THEOREM 1. Let (M, ∇) be a connected differentiable manifold with a linear connection. Suppose that the homogeneous holonomy group Φ_{x_0} leaves complementary subspaces T'_{x_0} and T''_{x_0} of the tangent space at x_0 invariant and denote by T' and T'' the corresponding parallel distributions. If (1) the curvature R satisfies $R(X, Y)=0$ for $X\in T'$ and $Y\in T''$, (2) the torsion S is completely inducible to these distributions, then T' and T'' are both completely integrable, and at each point of M, (M, ∇) is locally affinely isomorphic to the affine product of (M', ∇') and (M'', ∇'') where M' (resp. M'') is a leaf of T' (resp. T'') with the connection ∇' (resp. ∇'') induced naturally from (M, ∇).

PROOF. Since the torsion S is inducible to T' and T'', these distributions are completely integrable (Proposition 1), and every leaf M' (resp. M'') of T' (resp. T'') has a naturally induced connection (Proposition 2). Let x_0 be any point in M and assume that M' and M'' contain x_0 in common. There exists a coordinate neighborhood U' in M' (resp. U'' in M'') with a system of coordinates $(u^1, u^2, \cdots, u^a, \cdots, u^{m'})$ (resp. $(u^{m'+1}, \cdots, u^\alpha, \cdots, u^{m'+m''})$) such that $U'\times U''$ is diffeomorphic to a neighborhood U of x_0 in M and that $\left(\frac{\partial}{\partial u^1}, \cdots, \frac{\partial}{\partial u^a}, \cdots, \frac{\partial}{\partial u^{m'}}\right)$ and $\left(\frac{\partial}{\partial u^{m'+1}}, \cdots, \frac{\partial}{\partial u^\alpha}, \cdots, \frac{\partial}{\partial u^{m'+m''}}\right)$ form local bases for T' and T'' respectively if we choose $(u^1, \cdots, u^a, \cdots, u^{m'}, u^{m'+1}, \cdots, u^{m'+m''})$ as local coordinates in U. Denote by $\Gamma_{jk}^h(u^i)$ $(h, i, j, k=1, 2, \cdots, m'+m'')$ the coefficients of the connection with respect to the above coordinates in U. In the rest of the proof we shall adopt notational conventions of indices as $1\leqq a, b, c, \cdots\leqq m'$; $m'+1\leqq \alpha, \beta, \gamma, \cdots\leqq m'+m''$ and $1\leqq i, j, k, \cdots\leqq m'+m''$.

Since $\nabla_{\frac{\partial}{\partial u^b}}\frac{\partial}{\partial u^c}$ belong to T', all coefficients of the type $\Gamma_{bc}^\alpha(u^i)$ vanish and $\Gamma_{\alpha\beta}^b$'s similarly vanish. By the assumption (2) we have

$$\nabla_{\frac{\partial}{\partial u^a}}\frac{\partial}{\partial u^\alpha}-\nabla_{\frac{\partial}{\partial u^\alpha}}\frac{\partial}{\partial u^a}=0. \tag{3.5}$$

On the other hand since the distributions T' and T'' are parallel, $\nabla_{\frac{\partial}{\partial u^a}}\frac{\partial}{\partial u^\alpha}$ is in T'' as long as $\frac{\partial}{\partial u^\alpha}$ is in T'', and $\nabla_{\frac{\partial}{\partial u^\alpha}}\frac{\partial}{\partial u^a}$ is in T' simultaneously with $\frac{\partial}{\partial u^a}$.

Therefore, from (3.5) we have $\nabla_{\frac{\partial}{\partial u^a}} \frac{\partial}{\partial u^\alpha} = \nabla_{\frac{\partial}{\partial u^\alpha}} \frac{\partial}{\partial u^a} = 0$, which gives $\Gamma^b_{a\alpha} = \Gamma^b_{\alpha a} = 0$ and $\Gamma^\beta_{a\alpha} = \Gamma^\beta_{\alpha a} = 0$.

By the assumption (1), $R\left(\frac{\partial}{\partial u^a}, \frac{\partial}{\partial u^\alpha}\right) = 0$, from which we have $\frac{\partial}{\partial u^a} \Gamma^\alpha_{\beta\gamma}(u^i) = 0$ and $\frac{\partial}{\partial u^\alpha} \Gamma^a_{bc}(u^i) = 0$. From the definition of connection on a leaf M', we have $\Gamma^b_{cd}(u^i) = {}'\Gamma^b_{cd}(u^a)$ on M' where the coefficients of the connection ∇' on M' are denoted by ${}'\Gamma^b_{cd}(u^a)$. Similarly we have $\Gamma^\beta_{\gamma\delta}(u^i) = {}''\Gamma^\beta_{\gamma\delta}(u^\alpha)$ on M'', where ${}''\Gamma^\beta_{\gamma\delta}(u^\alpha)$'s are coefficients of the connection ∇'' on M''.

After all, we can conclude that (M, ∇) is locally affinely isomorphic to the affine product of (M', ∇') and (M'', ∇'') by the diffeomorphism of U onto $U' \times U''$.

Remark. From the definition of the affine product (Definition 2), it is clear that the conditions (1) and (2) are necessary for (M, ∇) to be locally affinely isomorphic at each point to an affine product of leaves of parallel distributions containing the point.

Definition 3. A linearly connected differentiable manifold (M, ∇) is said to be *locally reductive* if the curvature tensor R and the torsion tensor S are both parallel with respect to the connection, i. e., $\nabla R = 0$ and $\nabla S = 0$.

Corollary. Let (M, ∇) be a connected differentiable manifold with a linear connection which is locally reductive. Suppose that, at a point x_0 in M, the following conditions are satisfied:

(1) The tangent space T_{x_0} to M is decomposed into a direct sum of subspaces invariant under the homogeneous holonomy group, such as $T_{x_0} = T'_{x_0} + T''_{x_0}$;

(2) $R_{x_0}(X, Y) = 0$ for $X \in T'_{x_0}$ and $Y \in T''_{x_0}$;

(3) S_{x_0} is completely inducible (Definition 1) with respect to the direct sum.

Then any point of M has a neighborhood which is locally affinely isomorphic to an affine product of locally reductive spaces.

Proof. If $\nabla S = 0$, we have $S_x(\tau X_{x_0}, \tau Y_{x_0}) = \tau S_{x_0}(X_{x_0}, Y_{x_0})$ for any X_{x_0} and Y_{x_0} in $T_{x_0}(M)$, where τ is the parallel displacement of tangent vectors along a curve starting at x_0 and ending to any point x in M. Hence S_x is inducible to the tangent subspace $T'_x = \tau T'_{x_0}$ at x if S_{x_0} is inducible to the subspace T'_{x_0} at x_0. Complete inducibility at any point x follows similarly. In the same way, if $\nabla R = 0$ we have $R_x(\tau X_{x_0}, \tau Y_{x_0}) \tau Z_{x_0} = \tau (R_{x_0}(X_{x_0}, Y_{x_0}) Z_{x_0})$. Hence $R(T'_x, T''_x) = 0$ at any point x if and only if $R(T'_{x_0}, T''_{x_0}) = 0$, where $R(T'_x, T''_x) = \{R_x(X, Y); X \in T'_x \text{ and } Y \in T''_x\}$. Therefore the corresponding parallel distributions T' and T'' satisfy the conditions (1) and (2) in Theorem 1, and any point in M has a neighborhood which is locally affinely isomorphic to the affine product of leaves M' and M'' of T' and T'' respectively. Since $\nabla R = 0$ and $\nabla S = 0$ on M, the naturally induced tensors R' and S' (resp. R'' and S'') are also parallel with respect to the induced connection on M' (resp. M'').

THEOREM 2. Let (M, ∇) be a connected and simply connected differentiable manifold with a complete linear connection. Then under the assumptions same as in Theorem 1 we have the global affine isomorphism of (M, ∇) to the affine product of (M', ∇') and (M'', ∇'').

For the proof, see [2]. In [2] a global decomposition of a linearly connected manifold without torsion has been treated and it is also valid in our case.

§ 4. Application to reductive homogeneous spaces[1].

DEFINITION 4. A homogeneous space G/H of a connected Lie group G is called *reductive* if the following condition is satisfied; in the Lie algebra $\mathfrak{G}$ of G there exists a subspaces $\mathfrak{M}$ such that $\mathfrak{G}$ is decomposed into a direct sum $\mathfrak{G}=\mathfrak{M}+\mathfrak{H}$ and $\mathrm{ad}(H)\mathfrak{M}\subset\mathfrak{M}$, where $\mathfrak{H}$ is the subalgebra of $\mathfrak{G}$ corresponding to H.

Let $M=G/H$ be a reductive homogeneous space with a fixed Lie algebra decomposition $\mathfrak{G}=\mathfrak{M}+\mathfrak{H}$ (direct sum), and let ∇ denote the canonical connection (of the second kind in the sense of Nomizu [6]). The connection ∇ is G-invariant on M whose curvature tensor R and torsion tensor S are parallel on M. Let π denote the natural projection of G onto M. By identifying $\mathfrak{M}$ with the tangent space at the origin $x_0=\pi(e)$ (e is the identity of G), we have

$$R_{x_0}(X, Y)=\mathrm{ad}\,(-[X, Y]_{\mathfrak{H}}) \tag{4.1}$$

$$S_{x_0}(X, Y)=-[X, Y]_{\mathfrak{M}} \tag{4.2}$$

for any X, Y in $\mathfrak{M}$, where $[X, Y]_{\mathfrak{H}}$ and $[X, Y]_{\mathfrak{M}}$ denote the $\mathfrak{H}$-component and the $\mathfrak{M}$-component of the bracket with respect to the direct sum $\mathfrak{G}=\mathfrak{M}+\mathfrak{H}$ ([6] Theorem 10. 3).

We shall define two mappings ([8]) $\varphi(X, Y)=-[X, Y]_{\mathfrak{M}}$ and $h(X, Y)=-[X, Y]_{\mathfrak{H}}$ for any X, Y in $\mathfrak{M}$. The mapping φ is an anti-symmetric and bilinear binary operation on $\mathfrak{M}$ which defines an algebra $(\mathfrak{M}, \varphi)$. The subalgebra $\mathfrak{M}'$ of $\mathfrak{M}$ is said to be simple if $\varphi(\mathfrak{M}', \mathfrak{M}')\neq 0$ and $\mathfrak{M}'$ has no proper ideal of $\mathfrak{M}'$. The subalgebra $\mathfrak{M}'$ is said to be semi-simple if it is a direct sum

$$\mathfrak{M}'=\mathfrak{M}_1+\mathfrak{M}_2+\cdots+\mathfrak{M}_p \qquad \text{(direct sum)} \tag{4.3}$$

of ideals $\mathfrak{M}_i$ $(i=1, 2, \cdots, p)$ each of which is simple.

If $\mathfrak{M}'$ is semi-simple with direct sum decomposition (4.3) into simple ideals, then $\varphi(\mathfrak{M}_i, \mathfrak{M}_i)\subset\mathfrak{M}_i$ and $\varphi(\mathfrak{M}_i, \mathfrak{M}_j)=0$ for $i\neq j$, and since $S_{x_0}(X, Y)=\varphi(X, Y)$

1) For the details of reductive homogeneous space, see Nomizu [6] or Lichnerowicz [5] p. 48.

the torsion S is completely inducible with respect to the decomposition (4.3) of the tangent subspace at x_0 (by identifying $\mathfrak{M}'$ with $d\pi_e(\mathfrak{M}')$).

On the other hand, for two subalgebras $\mathfrak{M}'$ and $\mathfrak{M}''$ of $\mathfrak{M}$, if we set $\mathfrak{D}(\mathfrak{M}', \mathfrak{M}'')=\{\text{ad } h(X, Y); X \in \mathfrak{M}', Y \in \mathfrak{M}''\}$, $\mathfrak{D}(\mathfrak{M}, \mathfrak{M})$ can be regarded as the holonomy algebra of (M, ∇) since $R_{x_0}(X, Y)= \text{ad } h(X, Y)$ for any X, Y in $\mathfrak{M}$. Therefore, if $M=G/H$ is simply connected, the subalgebra $\mathfrak{M}'$ of $\mathfrak{M}$ is $\mathfrak{D}(\mathfrak{M}, \mathfrak{M})$-invariant if and only if the tangent subspace $d\pi_e(\mathfrak{M}')$ is invariant under the connected homogeneous holonomy group at x_0.

THEOREM 3. Let $M=G/H$ be a simply connected reductive homogeneous space of a connected Lie group G with a fixed Lie algebra decomposition $\mathfrak{G}=\mathfrak{M}+\mathfrak{H}$. Suppose that:

(1) The algebra $(\mathfrak{M}, \varphi)$ is a direct sum of the subspace $\mathfrak{M}_0=\{X \in \mathfrak{M}; \varphi(X, \mathfrak{M})=0\}$ and a semi-simple subalgebra $\mathfrak{M}'$ which is decomposed into a direct sum of simple ideals as (4.3).

(2) $\mathfrak{D}(\mathfrak{M}_i, \mathfrak{M}_j)=0$ for $i\neq j$.

(3) Each subspace $\mathfrak{M}_i$ $(i=0, 1, 2, \cdots, p)$ is $\mathfrak{D}(\mathfrak{M}, \mathfrak{M})$-invariant.

Then (M, ∇) is globally affinely isomorphic to an affine product of a locally affine symmetric space M_0 and locally reductive spaces $M_1, M_2, \cdots, M_p$.

PROOF. By identifying $\mathfrak{M}$ with the tangent space at $x_0=\pi(e)$, the assumptions (1), (2) and (3) correspond to those in Corollary to Theorem 1 respectively. In fact, as mentioned above, (1) and (3) are equivalent to the corresponding conditions of complete inducibility of the torsion and holonomy invariance of the subspaces respectively. The condition (2) implies that ad $h(X, Y)=0$ for $X \in \mathfrak{M}_i$ and $Y \in \mathfrak{M}_j$ $(i\neq j)$, which is equivalent to $R_{x_0}(\mathfrak{M}_i, \mathfrak{M}_j)=0$ for $i\neq j$.

Since $\nabla R=0$ and $\nabla S=0$ on M, Corollary of Theorem 1 implies that there pass through x_0 the leaves $M_0, M_1, \cdots, M_p$ of distributions obtained by parallel displacement of the subspaces $\mathfrak{M}_0, \mathfrak{M}_1, \cdots, \mathfrak{M}_p$ respectively and that M is locally affinely isomorphic to the affine product $M_0\times M_1\times\cdots\times M_p$. Since each $M_i (i=0, 1, 2, \cdots, p)$ has a connection induced naturally from the canonical connection ∇ on M, its curvature and torsion are both parallel on M_i, in particular, M_0 has zero torsion by the definition of the subspace $\mathfrak{M}_0$, i. e., M_0 is a locally affine symmetric space. Since the canonical connection of a reductive homogeneous space is complete, we have a global decomposition of $(G/H, \nabla)$ by Theorem 2.

REMARK. In the Theorem 3, if the condition (3) is replaced by (3′) each $\mathfrak{M}_i$ is ad $(\mathfrak{H})$-invariant, then (M, ∇) is locally affinely isomorphic to an affine product of reductive homogeneous spaces $M_i=G_i/H$ $(i=0, 1, \cdots, p)$ with canonical connections, where G_i is a connected subgroup of G corresponding to Lie subalgebra $\mathfrak{G}_i=\mathfrak{M}_i+\mathfrak{H}$.

In fact, since ad $(\mathfrak{H})$ contains $\mathfrak{D}(\mathfrak{M}, \mathfrak{M})$, (3′) implies (3). Moreover, under the assumption (3′) it is easy to see that $\mathfrak{G}_i=\mathfrak{M}_i+\mathfrak{H}$ is a Lie subalgebra of $\mathfrak{G}$.

Since H is connected $M_i'=G_i/H$ is well defined and reductive. Each locally reductive space $M_i (i=0, 1, \cdots, p)$ obtained in Theorem 3 has the same curvature and the torsion at x_0 as those of M_i' at the origin, with respect to the canonical connection. Therefore, M_i and M_i' are locally affinely isomorphic to each other at their origin ([5] p. 62). For any point x in M, there exists a local affine isomorphism of (M, ∇) which sends x to x_0. Thus, for any point x in M, there exists a local affine isomorphism of some neighborhood of x to a neighborhood of the origin of affine product $M_0'\times M_1'\times\cdots\times M_p'$.

§ 5. Some remarks about local loops

Any point p of a linearly connected manifold M has a neighborhood U in which a binary operation f_p can be defined so as to form a local loop $\mathcal{L}(U, f_p)$ ([3]). The binary operation f_p is defined as follows; let U be a normal neighborhood in which two points are joined by one and only one geodesic arc and let x and y be any two points of U, then there exist the unique geodesic arc $x(t)$ $(0\leqq t\leqq a)$ in U joining $p=x(0)$ to $x=x(a)$ and the unique geodesic arc $y(s)$ $(0\leqq s\leqq b)$ joining $p=y(0)$ to $y=y(b)$ (parameters are all affine). Let X_p be the vector tangent to $x(t)$ at p and X_y be the vector obtained by the parallel displacement of X_p to y along the arc $y(s)$, then we have the unique geodesic arc $z(t)$ $(0\leqq t\leqq a')$ in U starting from y and tangent to X_y. If $z(t)$ can be defined for $t=a$, we define $f_p(x, y)=z(a)$ and call it the product of x and y in U with respect to the origin p.

The product operation f_p defines a differentiable local loop $\mathcal{L}(U, f_p)$ on U, that is, (1) for any point x in U if we define $\rho_x(y)=f_p(x, y)$ and $\lambda_x(y)=f_p(y, x)$ $(y\in U)$, each of ρ_x and λ_x is a local diffeomorphism of a neighborhood of p onto a neighborhood of x; (2) $f_p(p, x)=f_p(x, p)=x$ for any $x\in U$, i. e., p is the unit.

The associative law does not hold in general.

Let T be a parallel distribution on M and suppose that the torsion tensor is inducible to T, then by Proposition 1 T is completely integrable. Let N be the maximal integral manifold of T containing p, then there exists a normal neighborhood U' of p in N (with respect to the naturally induced connection ∇') which is contained in the connected component of $N\cap U$, where U is the underlying neighborhood of a local loop $\mathcal{L}(U, f_p)$. The local loop $\mathcal{L}(U', f_p')$ is thereby defined in (N, ∇').

PROPOSITION 3. The local loop $\mathcal{L}(U', f_p')$ is a local subloop of $\mathcal{L}(U, f_p)$.

PROOF. Let x and y be two points in $U'\subset U$ and let $x(t)$ $(0\leqq t\leqq a)$ and $y(s)$ $(0\leqq s\leqq b)$ be geodesic arcs in U joining p to x and y respectively. Since T is parallel any geodesic in (M, ∇) tangent to N at a point is a geodesic in (N, ∇') and the parallel displacement of a vector in $T_p(N)$ with respect to ∇' coincides with one with respect to ∇, along any curve in N. Therefore,

$f'_p(x, y)=f_p(x, y)$ if both sides are defined.

THEOREM 4. Let T' and T'' be complementary parallel distributions on a linearly connected manifold (M, ∇) and let $\mathfrak{L}(U, f_p)$ be a local loop in M with origin p. Suppose that the conditions (1) and (2) in theorem 1 are satisfied, then $\mathfrak{L}(U, f_p)$ is locally isomorphic to the direct product of local loops $\mathfrak{L}(U', f'_p)$ and $\mathfrak{L}(U'', f''_p)$ where U' and U'' are normal neighborhoods of p with respect to ∇' and ∇'' respectively introduced on the integral manifolds of T' and T'' containing p.

PROOF. By the above Proposition, local loops $\mathfrak{L}(U', f'_p)$ and $\mathfrak{L}(U'', f''_p)$ can be defined with respect to ∇' and ∇'' respectively, and they are local subloops of $\mathfrak{L}(U, f_p)$. Without loss of generality, we can suppose that $U'\times U''$ is affinely isomorphic to U and that they are coordinate neighborhoods such as considered in the proof of Theorem 1. Then any geodesic arc $x(t)$ $(0\leqq t\leqq a)$ in U is represented by $(x'(t), x''(t))$ in $U'\times U''$ where $x'(t)$ (resp. $x''(t)$) is a geodesic in U' (resp. U'') with respect to ∇' (resp. ∇''), and a parallel vector field $X(t)$ on the geodesic $x(t)$ is represented by $(X'(t), X''(t))$ where $X'(t)$ (resp. $X''(t)$) is the parallel vector field along $x'(t)$ (resp. $x''(t)$). In fact the above facts are clear at a glance of corresponding equations in local coordinates by taking account of the condition that the coefficients Γ^i_{jk} of ∇ containing some distinct sort of indices vanish. Therefore, identifying U with $U'\times U''$ by the affine isomorphism we have $f_p(x, y)=(f'_p(x', y'), f''_p(x'', y''))$ for two points $x=(x', x'')$ and $y=(y', y'')$ in U, if the left or the right side of the equation is defined.

References

[1] S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, 1962.
[2] S. Kashiwabara, *On the Reducibility of an Affinely Connected Manifold*, Tôhoku Math. J., **8** (1956) 13-28.
[3] M. Kikkawa, *On Local Loops in Affine Manifolds*, J. Sci. Hiroshima Univ., Ser. A-I, **28** (1964), pp. 199-207.
[4] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Interscience, 1963.
[5] A. Lichnerowicz, *Géométrie des Groupes de Transformations*, Dunod, 1958.
[6] K. Nomizu, *Invariant Affine Connections on Homogeneous Spaces*, Amer. J. Math., **76** (1954), 33-65.
[7] G. de Rham, *Sur la Réductibilité d'une Espaces de Riemann*, Comm. Math. Helv., **26** (1952), 328-344.
[8] A. A. Sagle, *On Anti-commutative Algebras and Homogeneous Spaces*, J. Math. and Mech., **16** (1967), 1381-1393.
[9] H. Wu, *On the de Rham Decomposition Theorem*, Illinois J. Math., **8** (1964), 291-311.

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