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A Generalization of Kuhn's Theorem for an Infinite Game

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An extensive n-person game is usually described in terms of a finite tree in an oriented plane. The game involves in its structure the mixed and behavior strategies of each player closely related to some specified information sets. Under the assumption that each move possesses at least two alternatives, H.W. Kuhn [1] proved the theorem that the game has perfect recall if and only if, given any mixed strategies $\mu_1, \mu_2, \cdots, \mu_n$, there may be associated with them behavior strategies $\beta_1, \beta_2, \cdots, \beta_n$, each β_i depending only on the corresponding mixed strategy μ_i, so that they give rise to the equalities,

$$H_i(\mu_1, \mu_2, \cdots, \mu_n)=H_i(\beta_1, \beta_2, \cdots, \beta_n),\ i=1, 2, \cdots, n,$$

where H_i stands for any expected pay-off to the player i.

In this note we shall generalize the game to an infinite game, and show that there remains still valid an analogue to Kuhn's theorem just referred to. However, the term "*perfect recall*" should be understood in a more general sense in order to remove the assumption cited above.

§ 1. An infinite extensive n-person game

We shall introduce an infinite extensive game with which we shall be concerned in this note. To this end, we consider an ordered set $(E, \leq)$ with the properties:

(1) *E has the least element* x_0.

(2) *For any* $x, y \in E$, *if there exists a* $z \in E$ *such that* $x \leq z$ *and* $y \leq z$, *then* $x \leq y$ *or* $y \leq x$.

If $x \leq y$, we say that x is a predecessor of y, and y is a successor of x. If $x<y$ and there is no element z: $x<z<y$, then we say that x is an immediate predecessor of y and y an immediate successor of x.

(3) *Every* $x \in E$ *except* x_0 *has a unique immediate predecessor which will be denoted by* $f(x)$.

(4) *Every* $x \in E$ *has an immediate successor, and the set* $f^{-1}(x)$ *is finite.*

(5) *For each* $x \in E$, *there is an integer* $m \geqq 0$ *such that* $f^m(x)=x_0$, *where* f^0 *denotes the identical mapping.*

From these assumptions one can easily verify that $y \leq x$ holds if and only if there exists a non-negative integer k such that $y = f^k(x)$.

In what follows, any element of E will be referred to as a position. The position x_0 is called the initial position, and $y \in f^{-1}(x)$ an alternative of the position x. A play π is understood as an infinite sequence of positions $\{x_r\}$ such that $f(x_{r+1}) = x_r$. We shall then write $\pi = (x_0, x_1, \cdots)$ and $\pi(m) = x_m$. Denote by P the set of all the plays and by $P(x)$ the set of all the plays containing x. Put $E_m = \{x \mid f^m(x) = x_0\}$. We shall say that x is of rank m if $x \in E_m$. The left section at x will be understood as the set $\{f^k(x)\}_{0 \leq k \leq m}$ if $x \in E_m$. Since E_m is a finite set, it may be considered a compact space with discrete topology, so the product space $\prod_{m=0}^{\infty} E_m$ will be compact and metrizable. Then, as a closed subset of $\prod_{m=0}^{\infty} E_m$, P is also compact.

The set E with the following specifications (I) and (II) will be called an infinite extensive n-person game.

(I) *A partition of the positions into n indexed sets $S_1, \cdots, S_n$. Each S_i admits a partition $\{S_i^j\}_{j=1,2,\dots}$ satisfying the conditions*:

(i) *No two positions of S_i^j lie on the same play.*

(ii) *Every $x \in S_i^j$ has the same finite number of alternatives depending only on S_i^j. We associate with S_i^j the indexing set $I_i^j = \{1, 2, \cdots, |I_i^j|\}$ and fix once for all the ordering of the alternatives of x: $x_{(1)}, x_{(2)}, \cdots, x_{(|I_i^j|)}$.* The set S_i^j is called the information set of the player i. We shall say that S_i^j is trivial if $|I_i^j| = 1$.

(II) *Pay-off functions $h_1, \cdots, h_n$. They are real-valued continuous functions defined on P, where $h_i(\pi)$ denotes the pay-off to the player i when the play π is realized.*

The product $\sum_i = \prod_j I_i^j$ may be regarded as the set of mappings σ_i: $S_i^j \to \sigma_i^j \in I_i^j$, $j = 1, 2, \cdots$. A mapping σ_i is called a pure strategy for the player i. We write $\sigma_i = (\sigma_i^j)_{j=1,2,\dots}$, and define the mapping $\underline{\sigma}_i$: $x \in S_i^j \to x_{(\sigma_i^j)}$, $j = 1, 2, \cdots$. The set $\sum_i$ becomes a compact space with the usual product topology. For any compact space Ω we shall use the notation $M_1^+(\Omega)$ to denote the set of all the probability laws on Ω, the positive Radon measures of mass 1. Any $\mu_i \in M_1^+(\sum_i)$ is called a mixed strategy for the player i. A class of mixed strategies μ_i, $i = 1, \cdots, n$, determines a product probability law $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n \in M_1^+(\sum)$, where $\sum$ is the compact space $\sum_1 \times \sum_2 \times \cdots \times \sum_n$.

Let $\beta_i^j \in M_1^+(I_i^j)$, $j = 1, 2, \cdots$. The family $\{\beta_i^j\}_{j=1,2,\dots}$ determines a product probability law $\beta_i = \prod_j \beta_i^j \in M_1^+(\sum_i)$. Thus β_i is a special mixed strategy for the player i and called a behavior strategy for the player i.

Given any $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in \sum$, we define $\underline{\sigma}(x) = \underline{\sigma}_j(x)$ for $x \in S_j$. Noting that $f \circ \sigma$ is the identical mapping, we see that σ determines a unique play $\pi_\sigma = (x_0, \underline{\sigma}(x_0), \underline{\sigma}^2(x_0), \cdots)$. Let u be the mapping $\sigma \to \pi_\sigma$ from $\sum$ onto P. We can conclude that u must be continuous. This is because of the facts that the family $\{P(x)\}_{x \in E}$ forms a basis of the space P and that $u^{-1}(P(x))$ is written as the product $A_1 \times A_2 \times \cdots \times A_n$ of open cylinder sets $A_1, A_2, \cdots, A_n$, contained respectively in $\sum_1, \sum_2, \cdots, \sum_n$. Here, by a cylinder set in $\sum_k$ we mean $\sum_k$ or a set of the form $\{\sigma_k \in \sum_k | \sigma_k^{l_1} = \nu_1, \sigma_k^{l_2} = \nu_2, \cdots, \sigma_k^{l_m} = \nu_m\}$. The image measure $u_*(\mu)$ of $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$, $\mu_i \in M_1^+(\sum_i)$, $i = 1, 2, \cdots, n$, is defined by the formula

$$(\alpha) \qquad \int_P h(\pi) du_*(\mu) = \int_\Sigma h(u(\sigma)) d\mu$$

for any continuous function h on P. If h is a pay-off function to the player i, then the integral $\int_P h(\pi) du_*(\mu)$ is an expected value of the pay-off to the player i corresponding to the given mixed strategies $\mu_1, \mu_2, \cdots, \mu_n$. Since $P(x)$ is open and closed, we may take the characteristic function of $P(x)$ for h. We see from the equation (α) that $u_*(\mu)(P(x)) = \mu(u^{-1}(P(x)))$.

As an immediate consequence of the equation (α), we have

Lemma 1. *Let $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$, $\mu' = \mu_1' \times \mu_2' \times \cdots \times \mu_n'$ where $\mu_i, \mu_i' \in M_1^+(\sum_i)$, $i = 1, 2, \cdots, n$. Then $u_*(\mu) = u_*(\mu')$ if and only if $\mu(u^{-1}(P(x))) = \mu'(u^{-1}(P(x)))$ for every $x \in E$.*

§2. A generalization of Kuhn's theorem

We shall continue to use the same notation as before. For our later purpose we begin with the definition of perfect recall.

$$\text{Put } S_{i\nu}^j = \{y \in E | y \geq x_{(\nu)} \text{ for some } x \in S_i^j\} \quad \text{for } 1 \leq \nu \leq |I_i^j|.$$

We shall say that the player i has perfect recall if $S_{i\nu}^j \cap S_i^p \neq \phi$ implies $S_{i\nu}^j \supset S_i^p$ for any two non trivial S_i^j, S_i^p, and that the game has perfect recall if each player has perfect recall. The notation $S_i^p > S_i^j$ will be used if there exists a $\nu \in I_i^j$ such that $S_{i\nu}^j \supset S_i^p$.

Lemma 2. *Suppose a player i has perfect recall. Given a non trivial $S_i^{j_0}$, there exists a family of non trivial information sets $\{S_i^{j_0}, S_i^{j_1}, \cdots, S_i^{j_k}\}$ with the properties:*

(1) $S_i^{j_0} > S_i^{j_1} > \cdots > S_i^{j_k}$.

(2) *For any $x \in S_i^{j_0}$, $\{S_i^{j_0}, S_i^{j_1}, \cdots, S_i^{j_k}\}$ is the maximal family of non trivial information sets for the player i which intersect the left section at x.*

Proof. Let $F = \{S_i^{j_0}, S_i^{j_1}, \cdots, S_i^{j_k}\}$ be the maximal family determined by the property (2) for an $x \in S_i^{j_0}$. Since the player i has perfect recall, it is clear that F is independent of the choice of the position $x \in S_i^{j_0}$ and $j_0, \cdots, j_k$ can be so arranged that (1) holds. Thus the proof is completed.

Let $S_i^{j_0}$ be a non trivial information set and $x \in S_i^{j_0}$. Then $u^{-1}(P(x))$ is of the form $A_1 \times A_2 \times \cdots \times A_n$ where each A_k is a cylinder set in $\sum_k$. Suppose the player i has perfect recall. If we use the notation in Lemma 2, we can write $A_i = \{\sigma_i \in \sum_i | \sigma_i^{j_1} = \nu_1, \cdots, \sigma_i^{j_k} = \nu_k\}$ where $\nu_1, \cdots, \nu_k$ are chosen so that $S_i^{j_0} \subset S_{i\nu_1}^{j_1}, \cdots, S_i^{j_{k-1}} \subset S_{i\nu_k}^{j_k}$. This shows that A_i is independent of the choice of $x \in S_i^{j_0}$. This fact will be used in the proof of the following theorem 1.

Let us denote by $\mu \| \beta_i$, $\beta_i \in M_1^+(\sum_i)$, the product probability law obtained from $\mu = \mu_1 \times \cdots \times \mu_i \times \cdots \times \mu_n$ by replacing μ_i by β_i.

THEOREM 1. *A player i has perfect recall if and only if for any $\mu = \mu_1 \times \cdots \times \mu_n$, $\mu_j \in M_1^+(\sum_j)$, $j = 1, 2, \cdots, n$, there exists a behavior strategy β_i depending only on μ_i such that*

$$u_*(\mu) = u_*(\mu \| \beta_i).$$

PROOF. *Necessity.* Take any S_i^j. If it is trivial, we put $\beta_i^j(1) = 1$. Otherwise, let $x \in S_i^j$ and $u^{-1}(P(x)) = A_1 \times A_2 \times \cdots \times A_n$. As remarked above, A_i is independent of the choice of $x \in S_i^j$. Put $A_{i\nu} = A_i \cap \{\sigma_i \in \sum_i | \sigma_i^j = \nu\}$ and define

$$\beta_i^j(\nu) = \begin{cases} \dfrac{\mu_i(A_{i\nu})}{\mu_i(A_i)} & \text{if } \mu_i(A_i) \neq 0, \\[2ex] \dfrac{1}{|I_i^j|} & \text{if } \mu_i(A_i) = 0. \end{cases}$$

Denote by β_i the behavior strategy determined by $\{\beta_i^j\}$. We shall show that β_i satisfies the above condition. For any $x' \in E$ write

$$u^{-1}(P(x')) = A_1' \times A_2' \times \cdots \times A_n'$$

where A_i' is of the form $\{\sigma_i \in \sum_i | \sigma_i^{j_1} = \nu_1, \cdots, \sigma_i^{j_k} = \nu_k\}$. It will suffice to show $\mu_i(A_i') = \beta_i(A_i')$, for then

$$\begin{aligned} u_*(\mu)(P(x')) &= \mu_1(A_1') \cdots \mu_n(A_n') \\ &= (\mu \| \beta_i)(A_1' \times A_2' \times \cdots \times A_n') \\ &= u_*(\mu \| \beta_i)(P(x')). \end{aligned}$$

We write

$$B_i^{(l)} = \{\sigma_i \in \sum_i | \sigma_i^{j_1} = \nu_1, \cdots, \sigma_i^{j_l} = \nu_l\} \qquad (1 \leqq l \leqq k).$$

Then $B_i^{(l+1)} = B_{i\nu_{l+1}}^{(l)}$. If $\mu_i(B_i^{(l)}) = 0$ for some l, then either there is l', $1 \leqq l' < l$, such that $\mu_i(B_i^{(l')}) > 0$ and $\mu_i(B_i^{(l'+1)}) = 0$ or $\beta_i^1(\nu_1) = \mu_i(B_i^{(1)}) = 0$. In these cases, $\beta_i(A_i') = \mu_i(A_i') = 0$. If $\mu_i(B_i^{(l)}) > 0$ for each $l \leqq k$, then

$$\begin{aligned} \mu_i(A_i') = \mu_i(B_i^{(k)}) &= \mu_i(B_i^{(k-1)}) \beta_i^k(\nu_k) = \mu_i(B_i^{(k-2)}) \beta_i^{k-1}(\nu_{k-1}) \beta_i^k(\nu_k) \\ &= \beta_i^1(\nu_1) \cdots \beta_i^k(\nu_k) = \beta_i(A_i'). \end{aligned}$$

Sufficiency. Suppose the contrary. Then there would exist non trivial information sets S_i^p, S_i^j such that $S_i^j \neq S_{i\nu}^p$, while $S_{i\nu'}^p \cap S_i^j \neq \phi$ for some $\nu' \in I_i^p$. Let $\bar{x} \in S_i^j \cap S_{i\nu'}^p$ and $\bar{y} \in S_i^j \backslash S_{i\nu'}^p$. Let $\nu'' \in I_i^p \backslash \{\nu'\}$ be chosen so that $\bar{y} \in S_{i\nu''}^p$ whenever possible, otherwise ν'' is arbitrary. Put $\mu_k = \beta_k$ for $k \neq i$, where β_k are all uniform. To define μ_i, we consider β_i' and β_i'' with the properties: $\beta_i'^p(\nu') = \beta_i'^j(1) = \beta_i''^p(\nu'') = \beta_i''^j(2) = 1$ and $\beta_i'^l$, $\beta_i''^l$ are all uniform for $l \neq p, j$. Put $\mu_i = \frac{1}{2}(\beta_i' + \beta_i'')$. Then, by assumption, there exists a β_i such that $u_*(\mu) = u_*(\mu \| \beta_i)$. We have only to show that this leads to a contradiction. As already observed, $u^{-1}(P(x))$ can be written in the form $A_1 \times A_2 \times \cdots \times A_n$, A_k being a cylinder set in $\sum_k$ uniquely determined by the position x. Since $\prod_{k \neq i} \mu_k(A_k) \neq 0$, from the equalities $u_*(\mu)(P(x)) = \mu_i(A_i) \prod_{k \neq i} \mu_k(A_k)$ and $u_*(\mu \| \beta_i)(P(x)) = \beta_i(A_i) \prod_{k \neq i} \mu_k(A_k)$, it follows that $\mu_i(A_i) = \beta_i(A_i)$. Hence for $x \in S_i^j$, if $\mu_i(A_i) \neq 0$, we must have

$$\beta_i^j(1) = \frac{\beta_i(A_{i1})}{\beta_i(A_i)} = \frac{u_*(\mu \| \beta_i)(P(x_{(1)}))}{u_*(\mu \| \beta_i)(P(x))} = \frac{u^*(\mu)(P(x_{(1)}))}{u_*(\mu)(P(x))}$$

$$= \frac{\mu(u^{-1}(P(x_{(1)})))}{\mu(u^{-1}(P(x)))} = \frac{\mu_i(A_{i1})}{\mu_i(A_i)}$$

where we have written $A_{i1} = A_i \cap \{\sigma_i \in \sum_i | \sigma_i^j = 1\}$, or more precisely $\beta_i^j(1) = \frac{\beta_i'(A_i)\beta_i'^j(1) + \beta_i''(A_i)\beta_i''^j(1)}{\beta_i'(A_i) + \beta_i''(A_i)} = \frac{\beta_i'(A_i)}{\beta_i'(A_i) + \beta_i''(A_i)}$. Using this formula, we calculate $\beta_i^j(1)$ in two ways with the aid of the positions $\bar{x}$, $\bar{y}$:

$$\beta_i^j(1) = \frac{\mu(u^{-1}(P(\bar{x}_{(1)})))}{\mu(u^{-1}(P(\bar{x})))} = 1,$$

$$\beta_i^j(1) = \frac{\mu(u^{-1}(P(\bar{y}_{(1)})))}{\mu(u^{-1}(P(\bar{y})))} = \begin{cases} 0 & \text{if } \bar{y} \in S_{i,\nu''}^p \\ \frac{1}{2} & \text{if } \bar{y} \notin S_{i,\nu''}^p, \end{cases}$$

which is a contradiction. Thus the proof is completed.

As a consequence of Theorem 1, we can show a generalization of Kuhn's theorem which can be stated in the following

THEOREM 2. *The game has perfect recall if and only if to every $\mu_i \in M_1^+(\sum_i)$ there corresponds a behavior strategy β_i such that $u_*(\mu) = u_*(\beta)$ where $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ and $\beta = \beta_1 \times \beta_2 \times \cdots \times \beta_n$.*

PROOF. *Necessity.* Every player i has perfect recall, so that by Theorem 1 there corresponds to every μ_i a behavior strategy β_i such that $u_*(\mu) = u_*(\mu \| \beta_i)$ for any given μ_k, $k \neq i$. Let $\mu_i \in M_1^+(\sum_i)$ be given for $i = 1, 2, \cdots, n$. We take β_i for each μ_i as stated just above. Then we have $u_*(\mu) = u_*(\mu \| \beta_1) = u_*((\mu \| \beta_1) \| \beta_2) = \cdots = u_*(\beta)$.

Sufficiency. We shall show that any assigned player i has perfect recall. For $k \neq i$, we take $\mu_k = \beta_k$ such that β_k^l are all uniform. For such a μ_k a corresponding behavior strategy must be μ_k itself. One can easily verify this statement. Then the proof of the second part of the preceding theorem shows us that the player i has perfect recall, which was to be proved.

Reference

[1] H. W. Kuhn, Extensive games and the problem of information, Annals of Mathematics Studies No. 28, Princeton University Press, Princeton, N. J., 1953, 193-216.

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