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Some Properties of the Kuramochi Boundary

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Introduction

It has been shown that the Kuramochi boundary of a Riemann surface or of a Green space has many useful potential-theoretic properties (see [9], [4], [11], etc.). In this paper, we shall give a few more properties of the Kuramochi boundary.

We consider a Green space Ω in the sense of Brelot-Choquet [3] and denote by Ω^* its Kuramochi compactification of Ω (see [4], [9] and [14] for the definition). Let Γ be the harmonic boundary on $\Delta=\Omega^*-\Omega$, i.e., the support of a harmonic measure $\omega\equiv\omega_{x_0}$ ($x_0 \in \Omega$). By definition, Γ is a non-empty closed subset of Δ.

Let K_0 be a fixed compact ball in Ω. For any resolutive function φ on Δ, let H_φ be the Dirichlet solution on $\Omega-K_0$ with boundary values φ on Δ and 0 on ∂K_0 (=the relative boundary of K_0). For the existence of H_φ, see e.g. [11]. If φ is a function on Γ and is the restriction of a resolutive function $\tilde{\varphi}$ on Δ, then $H_{\tilde{\varphi}}$ is uniquely determined by φ; we denote it also by H_φ. With this convention, we consider the space $\boldsymbol{R}_D(\Gamma)$ of functions φ on Γ which are restrictions of resolutive functions on Δ and for which $H_\varphi \in \boldsymbol{HD}_0$. Here, $\boldsymbol{HD}_0$ is the space of all harmonic functions u on $\Omega-K_0$ having finite Dirichlet integral $D[u]$ on $\Omega-K_0$ and vanishing on ∂K_0. Identifying two functions which are equal ω-almost everywhere, we can define a norm $\|\cdot\|$ on $\boldsymbol{R}_D(\Gamma)$ by

$$\|\varphi\|^2=D[H_\varphi]$$

for $\varphi \in \boldsymbol{R}_D(\Gamma)$.

In this paper, we shall show the following three properties: (1) The space $\boldsymbol{R}_D(\Gamma)$ is a Dirichlet space in the sense of Beurling-Deny [1] on Γ; (2) The capacity on Γ associated with this Dirichlet space coincides with the Kuramochi capacity ([9] and [4]); (3) The solution of a boundary value problem (of Neumann type) is expressed in terms of the Kuramochi kernel.

1. Dirichlet space $\boldsymbol{R}_D(\Gamma)$

The following lemma is a consequence of Lemma 5.3 in [13] (also cf. [11]):

LEMMA 1. *There exists a constant $M>0$ such that*

$$\int \varphi^2 \, d\omega \leqq M \|\varphi\|^2$$

for all $\varphi \in \boldsymbol{R}_D(\Gamma)$.

Let $D[u_1, u_2]$ be the mutual Dirichlet integral of $u_1, u_2 \in \boldsymbol{HD}_0$ over $\Omega - K_0$. We define an inner product $<\cdot, \cdot>$ on $\boldsymbol{R}_D(\Gamma)$ by

$$<\varphi_1, \varphi_2> = D[H_{\varphi_1}, H_{\varphi_2}]$$

for $\varphi_1, \varphi_2 \in \boldsymbol{R}_D(\Gamma)$. Then, using Lemma 1, we easily obtain (cf. the proof of Lemma 5.2 in [13] or Theorem 1 of [11]):

LEMMA 2. *$\boldsymbol{R}_D(\Gamma)$ is a Hilbert space with respect to the inner product* $<\cdot, \cdot>$.

Also we have

LEMMA 3. *If $\varphi \in \boldsymbol{R}_D(\Gamma)$, then $\psi \equiv \min(\max(\varphi, 0), 1) \in \boldsymbol{R}_D(\Gamma)$ and* $\|\psi\| \leqq \|\varphi\|$.

PROOF. Applying Lemma 4.9 in [13] to $X = \Omega - K_0$, we have $\varphi^+ \equiv \max(\varphi, 0) \in \boldsymbol{R}_D(\Gamma)$ and $\|\varphi^+\| \leqq \|\varphi\|$. Proposition 3.1 in [13] implies that $\psi = \min(\varphi^+, 1)$ is resolutive and H_ψ is the greatest harmonic minorant of $\min(H_{\varphi^+}, 1)$. It follows (cf. Lemma 4.5 in [13]) that $\psi \in \boldsymbol{R}_D(\Gamma)$ and $D[H_\psi] \leqq D[H_{\varphi^+}]$, i.e., $\|\psi\| \leqq \|\varphi^+\|$.

Now, let $\boldsymbol{C}(\Gamma)$ be the space of all continuous functions on Γ with the uniform convergence topology and let $\boldsymbol{C}_D(\Gamma) = \boldsymbol{C}(\Gamma) \cap \boldsymbol{R}_D(\Gamma)$. By Stone-Weierstrass theorem, we have (cf. [10] and [11])

LEMMA 4. *$\boldsymbol{C}_D(\Gamma)$ is dense in $\boldsymbol{C}(\Gamma)$.*

Next we prove

LEMMA 5. *$\boldsymbol{C}_D(\Gamma)$ is dense in $\boldsymbol{R}_D(\Gamma)$.*

PROOF. Let $\varphi \in \boldsymbol{R}_D(\Gamma)$ be given and let $u = H_\varphi$. We consider a sequence $\{K_n\}$ of compact sets in Ω, $n = 1, 2, \dots$, such that the interior of K_n contains K_{n-1} for each $n = 1, 2, \dots$ and $\bigcup_{n=1}^{\infty} K_n = \Omega$. Let $u_n \equiv u^{K_n}$ in the notation of [4] or [12]. Then $D[u_n] \leqq D[u]$ and $u_n = u$ q.p.[1] on K_n. Hence $D[u - u_n] \leqq 2D_{\Omega - K_n}[u]$.[2] By the definition of the Kuramochi boundary, each u_n has continuous extension to Δ. Let φ_n be its restriction to Γ. It is easy to see that H_{φ_n} is the harmonic part in the Royden decomposition of u_n on $\Omega - K_0$. It follows that $H_{\varphi_n} \in \boldsymbol{HD}_0$, i.e., $\varphi_n \in \boldsymbol{C}_D(\Gamma)$. Since $D[u_n, u_n - H_{\varphi_n}] = D[u_n - H_{\varphi_n}]$ and $D[u, u_n - H_{\varphi_n}] = 0$, we have

$$0 \leqq D[u - H_{\varphi_n}] = D[(u - u_n) - (u_n - H_{\varphi_n})]$$

1) q.p. (quasi-partout) means "except for a set of capacity zero".

2) $D_{\Omega - K_n}[u]$ is the Dirichlet integral of u over $\Omega - K_n$.

$$=D[u-u_n]-D[u_n-H_{\varphi_n}]$$
$$\leqq 2D_{\Omega-K_n}[u]\rightarrow 0\,(n\rightarrow\infty).$$

Therefore $\|\varphi-\varphi_n\|\rightarrow 0\,(n\rightarrow\infty)$.

THEOREM 1. *The space $\boldsymbol{R}_D(\Gamma)$ is a Dirichlet space with respect to the measure ω.*

PROOF. By Lemmas 1, 2, 4 and 5, we see that $\boldsymbol{R}_D(\Gamma)$ is a regular functional space with respect to ω (see [5] and [8]). Lemma 3 shows that the unit contraction operates on $\boldsymbol{R}_D(\Gamma)$. Thus, by Theorem 2 in [8], we see that $\boldsymbol{R}_D(\Gamma)$ is a Dirichlet space.

2. Capacity on the Kuramochi boundary

2.1. In case Ω is a Riemann surface, Kuramochi himself defined a capacity on his boundary ([9]), which coincides with the capacity defined by Constantinescu-Cornea [4]. According to [4] (p. 185), the Kuramochi capacity $\tilde{C}(\delta)$ of a closed set δ on Δ is defined by

$$\tilde{C}(\delta)=\sup\left\{\mu(\delta);\ \begin{array}{l}\mu \text{ is a canonical measure on } \Delta \text{ such that}\\ \int_\Delta N(\xi,a)d\mu(\xi)\leqq 1 \quad \text{for all } a\in\Omega-K_0\end{array}\right\},$$

where $N(\xi, a)$ ($\xi\in\Delta$, $a\in\Omega-K_0$) is the Kuramochi kernel relative to K_0 (cf. [9] and [14]). This definition is also valid in case Ω is a Green space (cf. [12]) and the whole theory in section 17 of [4] can be verified for a Green space (cf. the results and methods in [2], [7], [10] and [12]). Note that $\varphi_a(\xi)=N(\xi, a)$ is a continuous function on Δ for each $a\in\Omega-K_0$ and in fact $\varphi_a\in\boldsymbol{C}_D(\Gamma)$. For a non-negative measure μ on Δ, we denote by u_μ the N-potential of μ:

$$u_\mu(a)=\frac{1}{c_d}\int_\Delta N(\xi, a)d\mu(\xi) \quad (a\in\Omega-K_0),$$

where c_d is the constant given in [13]. The set of all N-potentials is denoted by $\mathcal{P}_b$ (see [12] for this notation).

We know that $\tilde{C}$ is a Choquet capacity and $\tilde{C}(\Delta-\Gamma)=0$ (see Folgesatz 17.24 of [4]). Also, by Satz 17.3 and statements in p. 188 of [4], we have

LEMMA 6. *If μ is a non-negative canonical measure such that $u_\mu\in\boldsymbol{HD}_0$, then $\tilde{C}(\sigma)=0$ implies $\mu(\sigma)=0$ for $\sigma\subset\Delta$; in particular, the support of μ is contained in Γ.*

2.2. As we have shown that $\boldsymbol{R}_D(\Gamma)$ is a Dirichlet space, we have another notion of capacity on Γ through the theory of Dirichlet space (cf. [1], [5], [6] and [8]): For a closed set δ in Γ,

$$C(\delta)=\inf\{\|\varphi\|^2;\ \varphi \in \boldsymbol{C}_D(\Gamma),\ \varphi \geqq 1 \text{ on } \delta\}.$$

Now, let $\mathfrak{S}(\Gamma)$ be the set of all signed Radon measures ν on Γ such that the mapping $\varphi \rightarrow \int \varphi d\nu$ is continuous on $\boldsymbol{C}_D(\Gamma)$ with respect to the norm $\|\cdot\|$. For each $\nu \in \mathfrak{S}(\Gamma)$, there exists a unique element ρ_ν in $\boldsymbol{R}_D(\Gamma)$ such that

$$<\rho_\nu, \varphi>=\int \varphi\, d\nu$$

for all $\varphi \in \boldsymbol{C}_D(\Gamma)$. ρ_ν is called the potential of ν in the theory of Dirichlet space (see [1] and [6]). The following results are generally known (see [5] and [8]):

LEMMA 7. *Let δ be a closed subset of Γ. Then there exists a unique non-negative measure $\nu_\delta \in \mathfrak{S}(\Gamma)$ such that $\nu_\delta(\Gamma)=\nu_\delta(\delta)=\|\rho_{\nu_\delta}\|^2=C(\delta)$. Furthermore, $0 \leqq \rho_{\nu_\delta} \leqq 1\,(\omega$-a.e.) and there exists a sequence $\{\varphi_n\}$ in $\boldsymbol{C}_D(\Gamma)$ such that $0 \leqq \varphi_n \leqq 1$ on Γ, $\varphi_n=1$ on δ for each n and $\|\varphi_n-\rho_{\nu_\delta}\| \rightarrow 0\,(n\rightarrow\infty)$.*

LEMMA 8. *If $C(\delta)=0$, then $\omega(\delta)=0$.*

3. Equality of C and $\tilde{C}$

First we prove

PROPOSITION 1. *If $\nu \in \mathfrak{S}(\Gamma)$, then*

$$H_{\rho_\nu}(a)=\frac{1}{c_d}\int_\Gamma N(\xi, a)\,d\nu(\xi) \quad (a \in \Omega-K_0).$$

PROOF. It is easy to see that $U_a \equiv H_{\varphi_a}(a \in \Omega-K_0)$ is the reproducing function defined in [12] ($=u_a$ in [4]; cf. [11], Th. 10). Therefore

$$\begin{aligned} c_d H_{\rho_\nu}(a) &= D[H_{\rho_\nu}, U_a] \\ &= D[H_{\rho_\nu}, H_{\varphi_a}] \\ &= <\rho_\nu, \varphi_a>=\int \varphi_a\, d\nu=\int_\Gamma N(\xi, a)\,d\nu(\xi). \end{aligned}$$

As a converse, we have

PROPOSITION 2. *If μ is a non-negative canonical measure on Δ such that $u_\mu \in \boldsymbol{HD}_0$, then $\mu \in \mathfrak{S}(\Gamma)$; in fact*

$$\int \varphi\, d\mu=D[H_\varphi, u_\mu]$$

for any $\varphi \in \boldsymbol{C}_D(\Gamma)$.

PROOF. By Lemma 6, μ is a measure on Γ. Since the set of irregular points on Δ has $\tilde{C}$-capacity zero (Folgesatz 17.26 in [4]), the extension of H_φ by φ on Γ (and arbitrary on $\Delta-\Gamma$) is quasi-continuous (with respect to $\tilde{C}$) on Ω^* for each $\varphi \in \boldsymbol{C}_D(\Gamma)$. Hence, by Hilfssatz 17.3 in [4], $\int\varphi\, d\mu=D[H_\varphi, u_\mu]$. Thus $|\int\varphi\, d\mu|\leqq\sqrt{D[u_\mu]}\cdot\|\varphi\|$, and hence $\mu \in \mathfrak{S}(\Gamma)$.

PROPOSITION 3. *Any non-negative $\mu \in \mathfrak{S}(\Gamma)$ is a canonical measure.*

PROOF. By Proposition 1, $H_{\rho_\mu}=u_\mu \in \mathcal{P}_b$. Hence there exists a non-negative canonical measure μ' on Δ such that $u_{\mu'}=H_{\rho_\mu}$ (see [4], [9], [12] or [14]). By the above proposition, $\mu' \in \mathfrak{S}(\Gamma)$ and

$$\int\varphi\, d\mu'=D[H_\varphi, H_{\rho_\mu}]=<\varphi, \rho_\mu>=\int\varphi\, d\mu$$

for all $\varphi \in \boldsymbol{C}_D(\Gamma)$. Since $\boldsymbol{C}_D(\Gamma)$ is dense in $\boldsymbol{C}(\Gamma)$ (Lemma 4), it follows that $\mu'=\mu$ on Γ. Since both measures belong to $\mathfrak{S}(\Gamma)$, we conclude that $\mu=\mu'$, so that μ is a canonical measure.

THEOREM 2. *$C(\delta)=\tilde{C}(\delta)$ for any closed subset δ of Γ.*

PROOF. Let χ_δ be the non-negative canonical measure on δ such that $\tilde{C}(\delta)=\chi_\delta(\delta)=D[u_{\chi_\delta}]$ (satz 17.6 in [4]). Let ψ_δ be the extension of u_{χ_δ} to Δ in the sense of [4]. Then, by the definition of χ_δ, $\psi_\delta=1$ q.p. (with respect to $\tilde{C}$) on δ. On the other hand, the non-negative measure ν_δ given in Lemma 7 is canonical by Proposition 3 and $u_{\nu_\delta} \in \boldsymbol{HD}_0$ by Proposition 1. Hence, by Hilfssatz 17.3 in [4], we have $D[u_{\chi_\delta}, u_{\nu_\delta}]=\int\psi_\delta\, d\nu_\delta$. Since $\psi_\delta=1$ q.p. on δ, Lemmas 6 and 7 imply that $\int\psi_\delta\, d\nu_\delta=\nu_\delta(\delta)=C(\delta)$. Hence

$$C(\delta)=D[u_{\chi_\delta}, u_{\nu_\delta}].$$

On the other hand $\chi_\delta \in \mathfrak{S}(\Gamma)$ by Proposition 2 and Proposition 1 implies $H_{\rho_{\chi_\delta}}=u_{\chi_\delta}$ as well as $H_{\rho_{\nu_\delta}}=u_{\nu_\delta}$. Hence

$$D[u_{\chi_\delta}, u_{\nu_\delta}]=<\rho_{\chi_\delta}, \rho_{\nu_\delta}>.$$

Now, by Lemma 7, there exist $\varphi_n \in \boldsymbol{C}_D(\Gamma)$, $n=1, 2, \ldots$, such that $0\leqq\varphi_n\leqq 1$ on Γ, $\varphi_n=1$ on δ for each n and $\|\varphi_n-\rho_{\nu_\delta}\|\to 0$ $(n\to\infty)$. Then

$$<\rho_{\chi_\delta}, \rho_{\nu_\delta}>=\lim_{n\to\infty}<\rho_{\chi_\delta}, \varphi_n>=\lim_{n\to\infty}\int\varphi_n d\chi_\delta=\chi_\delta(\delta)=\tilde{C}(\delta).$$

Thus, we have the theorem.

4. Remarks on normal derivatives on the Kuramochi boundary

In [13], we said that a signed measure ν on Δ is a normal derivative of

$u \in \boldsymbol{HD}_0$ on Δ in the weak sense if

$$D[u, H_\varphi] = -\int \varphi\, d\nu$$

for all $\varphi \in \boldsymbol{C}(\Delta)$ such that $H_\varphi \in \boldsymbol{HD}_0$. It is easy to see that in this case ν is a measure on Γ and $\nu \in \mathfrak{S}(\Gamma)$, so that $\int \varphi\, d\nu = <\rho_\nu, \varphi> = D[H_{\rho_\nu}, H_\varphi]$ for any $\varphi \in \boldsymbol{C}_D(\Gamma)$. Hence Proposition 1 can be interpreted as follows (cf. Satz 17.26 and Satz 17.27 in [4]):

THEOREM 3. *If $u \in \boldsymbol{HD}_0$ has a normal derivative ν on Δ in the weak sense, then $\nu \in \mathfrak{S}(\Gamma)$ and*

$$u(a) = -\frac{1}{c_d}\int_\Gamma N(\xi, a)\, d\nu(\xi) \quad (a \in \Omega - K_0).$$

Conversely, if $\nu \in \mathfrak{S}(\Gamma)$, then there exists a unique $u \in \boldsymbol{HD}_0$ having a normal derivative ν on Δ in the weak sense; in fact u is given by the above formula.

COROLLARY 1. *If $u \in \boldsymbol{HD}_0$ has a normal derivative ν on Δ in the weak sense and if $\nu \leqq 0$, then $u = u_{-\nu} \in \mathcal{P}_b$.*

Conversely, using Proposition 2, we have

PROPOSITION 4. *Any function in $\mathcal{P}_b \cap \boldsymbol{HD}_0$ has a non-positive normal derivative on Δ in the weak sense.*

An ω-measurable function γ on Δ (or on Γ) is called a normal derivative of $u \in \boldsymbol{HD}_0$ if

$$D[u, H_\varphi] = -\int \varphi\gamma\, d\omega$$

for all $\varphi \in \boldsymbol{R}_{BD}(\Gamma)$ $(= \{\varphi \in \boldsymbol{R}_D(\Gamma);\ \text{bounded}\})$ (see [13]). Using Lemma 5, we can easily show that if $\gamma d\omega$ is a normal derivative of u on Δ in the weak sense then γ is a normal derivative of u on Δ (see Remark in p. 113 of [13]; cf. the proof of the corollary to Theorem 4. 1 in [13]). Thus Theorem 3 and Proposition 4 have the following consequences:

COROLLARY 2 to Theorem 3. *If γ is an ω-measurable function on Δ such that $\gamma d\omega \in \mathfrak{S}(\Gamma)$, then there exists a unique $u \in \boldsymbol{HD}_0$ having a normal derivative γ on Δ; in fact u is given by*

$$u(a) = -\frac{1}{c_d}\int_\Gamma N(\xi, a)\ \gamma(\xi)\, d\omega(\xi).$$

COROLLARY to Proposition 4. *If $u \in \mathcal{P}_b$ has a (function-valued) normal derivative on Δ, then it is non-positive (ω-a.e.).*

REMARK. The condition $\gamma d\omega \in \mathfrak{S}(\Gamma)$ coincides with condition (Γ) in [13] (p. 126).

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