J. Sci. Hiroshima Univ. Ser. A-I
33 (1969), 209-212

A Theorem on Characteristically Nilpotent Algebras

Shigeaki Tôgô

(Received September 20, 1969)

1. The notion of characteristic nilpotency of Lie algebras has been introduced by Dixmier and Lister [3] and studied by Leger and Tôgô [4]. Let L be a Lie algebra over a field Φ and $\mathfrak{D}(L)$ be the set of all derivations of L. Put $L^{[1]}=L\mathfrak{D}(L)=\{\sum a_iD_i \mid a_i \in L,\ D_i \in \mathfrak{D}(L)\}$ and define inductively $L^{[n]}=L^{[n-1]}\mathfrak{D}(L)$ for $n\geq 2$. L is called characteristically nilpotent provided there exists an integer k such that $L^{[k]}=(0)$. Then every characteristically nilpotent Lie algebra is nilpotent. However, few similarities are known between the properties of nilpotent and characteristically nilpotent algebras, e.g., a subalgebra and a quotient algebra of a characteristically nilpotent Lie algebra are not necessarily characteristically nilpotent, contrary to the case of nilpotent algebras.

Recently, in [2], C.-Y. Chao has shown a characterization of nilpotent Lie algebras: Let L be a Lie algebra over a field Φ and N be a nilpotent ideal of L. Then L is nilpotent if and only if L/N^2 is nilpotent.

The purpose of this note is to show a similar characterization of characteristically nilpotent Lie algebras, as a matter of fact, more generally of characteristically nilpotent nonassociative algebras.

By a nonassociative algebra we mean an algebra which is not necessarily associative, that is, a distributive algebra [6]. The definition of characteristic nilpotency of a nonassociative algebra A is obtained by replacing A instead of L in that of a Lie algebra stated above and this is due to the first version of the paper [5] of T.S. Ravisankar. However, it has not yet been known that there actually exists a characteristically nilpotent nonassociative algebra which is not a Lie algebra. We shall show the existence of such an algebra in Section 3. All the algebras considered in this note are assumed to be finite dimensional over their base fields.

2. For a nonassociative algebra N, all the products of n elements in N, irrespective of how they are associated, span a subspace of N, which is a subalgebra of N and denoted by N^n. N is called nilpotent if $N^k=(0)$ for some k [6].

Lemma. *Let A be a nonassociative algebra over a field Φ and N be a characteristic subalgebra of A. If $N\mathfrak{D}(A)^m\subset N^n$, then for every integer $r\geq 1$*

$$N^r\mathfrak{D}(A)^{rm-r+1}\subset N^{n+r-1}.$$

Proof. We prove the statement by induction on r. It holds for $r=$

1 by the hypothesis. Assume it holds for every $r \leq k$ and we consider the case where $r=k+1$. Put

$$t=(k+1)m-(k+1)+1.$$

Then it follows that

$$N^{k+1}\mathfrak{D}(A)^t = \sum_{\substack{p+q=k+1\\ p,q\geq 1}} (N^p N^q)\mathfrak{D}(A)^t$$

$$\subset \sum_{\substack{p+q=k+1\\ p,q\geq 1}} \sum_{i=0}^{t} {}_tC_i (N^p\mathfrak{D}(A)^{t-i})(N^q\mathfrak{D}(A)^i).$$

If $i \geq qm-q+1$, by induction hypothesis we have

$$N^q\mathfrak{D}(A)^i \subset N^q\mathfrak{D}(A)^{qm-q+1} \subset N^{n+q-1}.$$

If $i \leq qm-q$, then

$$t-i \geq t-qm+q=pm-p+1$$

and therefore by induction hypothesis

$$N^p\mathfrak{D}(A)^{t-i} \subset N^p\mathfrak{D}(A)^{pm-p+1} \subset N^{n+p-1}.$$

Hence in any case we have

$$N^{k+1}\mathfrak{D}(A)^t \subset N^{n+p+q-1}=N^{n+k}=N^{n+(k+1)-1},$$

which shows that the statement holds for $r=k+1$. Thus the proof is complete.

By making use of the lemma we shall now prove the following

Theorem. *Let A be a nonassociative algebra over a field Φ and N be a nilpotent characteristic ideal of A such that N^k, $k=2, 3, \dots$, are ideals of A. Then A is characteristically nilpotent if and only if A/N^n is characteristically nilpotent for some integer $n \geq 2$.*

Proof. Assume that A/N^n is characteristically nilpotent with $n \geq 2$. Since N^n is a characteristic ideal of A, there exists an integer m such that

$$A\mathfrak{D}(A)^m \subset N^n.$$

Putting

$$f(r)=\sum_{j=0}^{r} \{(jn-j+1)m-(jn-j)\}$$

for every integer $r \geq 0$, we assert that

$$A\mathfrak{D}(A)^{f(r)} \subset N^{(r+1)n-r}.$$

In fact, it holds for $r=0$ since

$$A\mathfrak{D}(A)^{f(0)}=A\mathfrak{D}(A)^{m}\subset N^{n}.$$

Assume it holds for $r=k-1$ and consider the case where $r=k$. We have

$$A\mathfrak{D}(A)^{f(k)}\subset A\mathfrak{D}(A)^{f(k-1)}\mathfrak{D}(A)^{(kn-k+1)m-(kn-k)}.$$

By the lemma and the induction hypothesis, it follows that

$$\begin{aligned}A\mathfrak{D}(A)^{f(k)}&\subset N^{kn-(k-1)}\mathfrak{D}(A)^{(kn-k+1)m-(kn-k)}\\&\subset N^{n+(kn-k+1)-1}\\&=N^{(k+1)n-k}.\end{aligned}$$

Hence by induction we see that our assertion holds.

Now take a sufficiently large integer k. Then, since $n\geq 2$, $N^{(k+1)n-k}=(0)$ and therefore

$$A^{[f(k)]}=A\mathfrak{D}(A)^{f(k)}=(0).$$

Thus we conclude that A is characteristically nilpotent.

The converse trivially holds since we can choose n such that $N^{n}=(0)$, and the proof is complete.

Corollary. *Let A be a Lie or alternative (associative) algebra over a field Φ and N be a nilpotent characteristic ideal of A. Then A is characteristically nilpotent if and only if A/N^{n} is characteristically nilpotent for some integer $n\geq 2$.*

Proof. In the case where A is Lie or alternative, if N is an ideal of A then N^{k}, $k=2, 3, \cdots$, are also ideals of A. Therefore the statement follows from the theorem.

3. We know several examples of characteristically nilpotent Lie algebras in [1, p. 123], [3] and [4]. However, it has not yet been known whether there exists a characteristically nilpotent algebra which is not a Lie algebra. We shall here give examples of such nonassociative algebras.

Let A be the nonassociative algebra over a field Φ described in terms of a basis x_1, x_2, x_3 by the following table:

$$\begin{aligned}&x_1x_1=x_2, &&x_1x_2=2x_3,\\&x_2x_1=x_3, &&x_2x_2=x_3,\\&x_3x_j=x_jx_3=0 &&\text{for } j=1, 2, 3.\end{aligned}$$

After calculation we see that $A^{[2]}=(0)$. Hence A is characteristically nilpotent. Furthermore it is evident that A is not a Lie algebra.

Next, let A be the nonassociative algebra over a field Φ of characteristic $\neq 2$ described in terms of a basis $x_1, x_2, \cdots, x_8$ by the following table:

$$x_1x_2=2x_3,\quad x_1x_3=x_4,\quad x_1x_4=x_5,\quad x_1x_5=x_6,$$

$$x_1x_6=x_8,\quad x_1x_7=x_8,\quad x_2x_3=x_5,\quad x_2x_4=x_6,$$

$$x_2x_5=x_7,\quad x_2x_6=2x_8,\quad x_3x_4=-x_7+x_8,\quad x_3x_5=-x_8.$$

In addition, $x_ix_j=-x_jx_i$ and for $i<j$ $x_ix_j=0$ if it is not in the above table. Then A is not a Lie algebra, since $(x_1x_2)x_4+(x_2x_4)x_1+(x_4x_1)x_2\neq 0$. We have $A^{[3]}=(0)$ and therefore A is characteristically nilpotent.

References

[1] N. Bourbaki, Groupes et Algèbres de Lie, Chap. I, Algèbres de Lie, Hermann, Paris, 1960.

[2] C.-Y. Chao, *Some characterizations of nilpotent Lie algebras*, Math. Z., **103** (1968), pp. 40-42.

[3] J. Dixmier and W. G. Lister, *Derivations of nilpotent Lie algebras*, Proc. Amer. Math. Soc., **8** (1957), pp. 155-158.

[4] G. Leger and S. Tôgô, *Characteristically nilpotent Lie algebras*, Duke Math. J., **26** (1959), pp. 623-628.

[5] T. S. Ravisankar, *Characteristically nilpotent algebras*, to appear in Canad. J. Math.

[6] R. D. Schafer, An introduction to nonassociative algebras, Academic Press, New York, 1966.

Department of Mathematics,
Faculty of Science,
Hiroshima University