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Asymptotic Behavior of Solutions of Parabolic Differential Equations with Unbounded Coefficients

Dedicated to Professor Tokui Satō on the occasion of his retirement

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Introduction

Let $x=(x_1, \dots, x_n)$ denote points in the real Euclidean n-space E^n and t denote points on the real line E^1. The distance of a point x of E^n to the origin is defined by $|x|=(\sum_{i=1}^{n} x_i^2)^{1/2}$.

Consider the Cauchy problem

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}+(-k^2|x|^2+l)u-\frac{\partial u}{\partial t}=0 \text{ in } E^n\times(0, \infty),$$

$$u(x, 0)=M\exp(a|x|^2) \text{ on } E^n,$$

where $k>0$, l, a and M are constants. It is shown in [5] that if $2a<k$ the solution of this problem exists and is given explicitly by

$$u(x, t)=M\left(\frac{k}{k\cosh 2kt-2a\sinh 2kt)}\right)^{n/2}$$

$$\times\exp\left[\frac{k(2a\cosh 2kt-k\sinh 2kt)}{2(k\cosh 2kt-2a\sinh 2kt)}|x|^2+lt\right].$$

This formula shows that if $l-kn$ is negative, then $u(x, t)$ tends to zero as $t\to\infty$, the convergence being of exponential order and uniform with respect to $x\in E^n$.

The purpose of the present paper is to prove similar results for general second order parabolic equations with unbounded coefficients. In Section 1 we investigate under what conditions the solutions of

$$\text{(A)} \qquad \sum_{i,j=1}^{n} a_{ij}(x, t)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b_i(x, t)\frac{\partial u}{\partial x_i}+c(x, t)u-\frac{\partial u}{\partial t}=0$$

with unbounded initial values decay exponentially to zero as $t\to\infty$. In Section 2 the results of Section 1 are extended to weakly coupled parabolic systems of the form

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(B) $$\sum_{i,j=1}^{n} a_{ij}^{\mu}(x, t)\frac{\partial^2 u^{\mu}}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i^{\mu}(x, t)\frac{\partial u^{\mu}}{\partial x_i}+\sum_{\nu=1}^{N} c^{\mu\nu}(x, t)u^{\nu}-\frac{\partial u^{\mu}}{\partial t}=0,$$

$$\mu=1, \ldots, N.$$

1. Exponential decay of solutions of (A)

(**a**) *Statement of results.* Throughout this section it is assumed that there exist constants $K_1>0$, $K_2\geqq 0$, $K_3>0$ and K_4 such that

(1.1) $$0\leqq \sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j\leqq K_1|\xi|^2,$$

(1.2) $$|b_i(x, t)|\leqq K_2(|x|^2+1)^{1/2},\ i=1, \ldots, n,$$

(1.3) $$c(x, t)\leqq -K_3|x|^2+K_4,$$

for all $(x, t)\in E^n\times[0, \infty)$ and $\xi=(\xi_1, \ldots, \xi_n)\in E^n$. We put

(1.4) $$\alpha=\min_{i=1,\ldots,n}\left[\inf_{(x,t)\in E^n\times[0,\infty)} a_{ii}(x, t)\right]$$

and let λ be the positive root of the equation

(1.5) $$4K_1\lambda^2+2K_2n\lambda-K_3=0.$$

One of the main results of this paper is the following

THEOREM 1. *Let $u(x, t)$ be a regular solution of* (A) *in $E^n\times(0, \infty)$ such that*

(1.6) $$|u(x, 0)|\leqq M\exp(a|x|^2)\ \text{for}\ x\in E^n,$$

where M and a are positive constants. Suppose that the following inequalities are satisfied:

(1.7) $$4K_1a^2+2K_2na-K_3<0,$$

(1.8) $$K_4+2(K_2-\alpha)n\lambda<0.$$

Then $\lim_{t\to\infty} u(x, t)=0$, the convergence being of exponential order and uniform with respect to x in E^n.

By a regular solution of (A) we mean a function $u(x, t)$ with the properties: (i) $u(x, t)$ is continuous in $E^n\times[0, \infty)$, (ii) $u(x, t)$ has the continuous partial derivatives which appear in (A) and fulfils (A) in $E^n\times(0, \infty)$, and (iii) for each $T>0$ there are positive numbers M_T and a_T such that $|u(x, t)|\leqq M_T\exp(a_T|x|^2)$ for $(x, t)\in E^n\times[0, T]$.

Under the additional hypothesis that there exists a positive constant β such that

(1.9) $$\sum_{i=1}^{n}(a_{ii}(x, t)+b_i(x, t)x_i)\geqq\beta \text{ for } (x, t)\in E^n\times[0, \infty),$$

we can prove the following theorem.

THEOREM 2. *Let $u(x, t)$ be a regular solution of* (A) *in $E^n\times(0, \infty)$ satisfying* (1.6). *Assume the following inequalities to hold*:

(1.10) $$4K_1a^2+2K_2na-K_3<0,$$
$$K_4-\beta\sqrt{K_3/K_1}<0.$$

Then $\lim_{t\to\infty} u(x, t)=0$, the convergence being of exponential order and uniform with respect to x in E^n.

It will be of interest to compare our theorems with an earlier result of Il'in, Kalashnikov and Oleinik [2] (§12, Theorem 6).

(**b**) *Proof of Theorem* 1. At first we shall show that under assumptions (1.1)–(1.3) a finite time can be found at which the solution $u(x, t)$ becomes a bounded function of x in E^n. For this purpose we employ the method as described in [6]. We introduce the auxiliary function

(1.11) $$v(x, t)=M\exp\left[a|x|^2\rho^{-\theta_0 t}+\frac{2(K_1+K_2)an+2K_4}{\theta_0\log\rho}(1-\rho^{-\theta_0 t})\right],$$

where $\rho(1<\rho<2)$ is a parameter and

$$\theta_0=\frac{K_3a^{-1}-2K_2n-4K_1a}{\log\rho}.$$

θ_0 is positive by assumption (1.7). Using (1.1)–(1.3) it is easy to verify that $v(x, t)$ satisfies the differential inequality

$$\sum_{i,j=1}^{n}a_{ij}(x, t)\frac{\partial^2 v}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i(x, t)\frac{\partial v}{\partial x_i}+c(x, t)v-\frac{\partial v}{\partial t}\leqq 0$$

in $E^n\times(0, \theta_0^{-1}]$. Setting $w_\pm(x, t)=v(x, t)\pm u(x, t)$ and applying the maximum principle of Krzyżański [3] to $w_\pm(x, t)$ we have $w_\pm(x, t)\geqq 0$, or equivalently $|u(x, t)|\leqq v(x, t)$ in $E^n\times(0, \theta_0^{-1}]$. Substituting, in particular, $t=\theta_0^{-1}$ we obtain

(1.12) $$|u(x, \theta_0^{-1})|\leqq M_1\exp(a\rho^{-1}|x|^2) \text{ for } x\in E^n,$$

where

$$M_1=M\exp\left[\frac{2(K_1+K_2)an+2K_4}{\log\rho}(1-\rho^{-1})\theta_0^{-1}\right].$$

Now regarding $t=\theta_0^{-1}$ as the initial time and (1.12) as the bound for the initial values of $u(x, t)$, we can use the same argument as above to derive the inequality

$$|u(x,t)|\leqq M_1\exp\Big[a\rho^{-1}|x|^2\rho^{-\theta_1(t-\theta_0^{-1})}$$
$$+\frac{2(K_1+K_2)a\rho^{-1}n+2K_4}{\theta_1\log\rho}(1-\rho^{-\theta_1(t-\theta_0^{-1})})\Big]$$

for $(x,t)\in E^n\times(\theta_0^{-1},\theta_0^{-1}+\theta_1^{-1}]$, where

$$\theta_1=\frac{K_3a^{-1}\rho-2K_2n-4K_1a\rho^{-1}}{\log\rho}>0.$$

In particular

$$|u(x,\theta_0^{-1}+\theta_1^{-1})|\leqq M_2\exp(a\rho^{-2}|x|^2)\text{ for }x\in E^n$$

where

$$M_2=M\exp\Big[\frac{2(K_1+K_2)an}{\log\rho}(1-\rho^{-1})(\theta_0^{-1}+\rho^{-1}\theta_1^{-1})$$
$$+\frac{2K_4}{\log\rho}(1-\rho^{-1})(\theta_0^{-1}+\theta_1^{-1})\Big].$$

By induction we have in general

(1.13) $$|u(x,\theta_0^{-1}+\theta_1^{-1}+\cdots+\theta_k^{-1})|\leqq M_{k+1}\exp(a\rho^{-k-1}|x|^2)\text{ for }x\in E^n,$$

where

$$\theta_k=\frac{K_3a^{-1}\rho^k-2K_2n-4K_1a\rho^{-k}}{\log\rho}>0,$$

(1.14) $$M_{k+1}=M\exp\Big[\frac{2(K_1+K_2)an}{\log\rho}(1-\rho^{-1})(\theta_0^{-1}+\rho^{-1}\theta_1^{-1}+\cdots+\rho^{-k}\theta_k^{-1})$$
$$+\frac{2K_4}{\log\rho}(1-\rho^{-1})(\theta_0^{-1}+\theta_1^{-1}+\cdots+\theta_k^{-1})\Big],\ k=0,1,2,\ldots.$$

We form the convergent series

$$f(\rho)=\sum_{i=0}^{\infty}\rho^{-i}\theta_i^{-1}=\sum_{i=0}^{\infty}\frac{\rho^{-i}\log\rho}{K_3a^{-1}\rho^i-2K_2n-4K_1a\rho^{-i}},$$

$$g(\rho)=\sum_{i=0}^{\infty}\theta_i^{-1}=\sum_{i=0}^{\infty}\frac{\log\rho}{K_3a^{-1}\rho^i-2K_2n-4K_1a\rho^{-i}}$$

and observe that the following relations hold:

(1.15) $$f(\rho)\leqq\frac{1}{K_3a^{-1}-2K_2n-4K_1a}\frac{\log\rho}{1-\rho^{-1}},$$

(1.16) $$\lim_{\rho\to1}g(\rho)=\lim_{\rho\to1}\int_0^{\infty}\frac{\log\rho}{K_3a^{-1}\rho^s-2K_2n-4K_1a\rho^{-s}}ds$$

$$=\frac{1}{2\sqrt{K_2^2 n^2+4K_1K_3}}\log\frac{K_3a^{-1}-K_2n+\sqrt{K_2^2n^2+4K_1K_3}}{K_3a^{-1}-K_2n-\sqrt{K_2^2n^2+4K_1K_3}}.$$

From (1.14) and (1.15) it follows that

$$M_k\leqq \bar{M}\exp\left[\frac{2K_4}{\log\rho}(1-\rho^{-1})\sum_{i=0}^{\infty}\theta_i^{-1}\right],\ k=1, 2, \dots, \tag{1.17}$$

where we have set

$$\bar{M}=M\exp\left[\frac{2(K_1+K_2)an}{K_3a^{-1}-2K_2n-4K_1a}\right],$$

and on account of (1.16) it is possible to choose $\rho_0\,(1<\rho_0<2)$ so that the right-hand side of (1.17) does not exceed a constant, say $M_0=2\bar{M}\exp(2K_4T_0)$ provided $1<\rho<\rho_0$, where T_0 stands for the limit $\lim_{\rho\to 1} g(\rho)$ given in (1.16). Therefore it follows from (1.13) that

$$|u(x, \sum_{i=0}^{k}\theta_i^{-1})|\leqq M_0\exp(a\rho^{-k-1}|x|^2)\ \text{for}\ x\in E^n \tag{1.18}$$

provided ρ is sufficiently close to 1.

Let $x\in E^n$ be arbitrary but fixed. Given an $\varepsilon>0$, by (1.16) and the continuity of $u(x, t)$ there exists a number $\rho_1(1<\rho_1<2)$ such that $|u(x, T_0)-u(x, g(\rho))|<\varepsilon/2$ for $1<\rho<\rho_1$. On the other hand, for a fixed ρ with $1<\rho<\min(\rho_0, \rho_1)$ an integer N can be found such that $|u(x, g(\rho))-u(x, \sum_{i=0}^{k}\theta_i^{-1})|<\varepsilon/2$ for $k>N$. Thus we obtain

$$|u(x, T_0)|<|u(x, \sum_{i=0}^{k}\theta_i^{-1})|+\varepsilon\ \text{for}\ k>N,$$

whence in view of (1.18)

$$|u(x, T_0)|<M_0\exp(a\rho^{-k-1}|x|^2)+\varepsilon\ \text{for}\ k>N.$$

Letting $k\to\infty$ and $\varepsilon\to 0$ we have $|u(x, T_0)|\leqq M_0$. Since x is arbitrary, this inequality holds throughout E^n.

Our next task is to study how $u(x, t)$ behaves for $t>T_0$. To do this, we make use of a result due to Krzyżański [4]. We introduce the function

$$\begin{aligned} w(x, t)=&M_0[\cosh 4K_1\lambda(t-T_0)]^{n(K_2-\alpha)/2K_1}\\ &\times\exp[-\lambda|x|^2\tanh 4K_1\lambda(t-T_0)+K_4(t-T_0)]. \end{aligned} \tag{1.19}$$

Then by assumptions (1.1) through (1.5) we can verify that

$$\sum_{i,j=1}^{n} a_{ij}(x, t)\frac{\partial^2 w}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b_i(x, t)\frac{\partial w}{\partial x_i}+c(x, t)w-\frac{\partial w}{\partial t}\leqq 0 \tag{1.20}$$

in $E^n\times(T_0, \infty)$. Thus, according to Krzyżański's maximum principle, we conclude that $|u(x, t)|\leqq w(x, t)$ in $E^n\times(T_0, \infty)$. Now the assertion of Theorem 1 follows from the observation that the asymptotic behavior of $w(x, t)$ as $t\to\infty$ is determined by the factor

$$[\cosh 4K_1\lambda(t-T_0)]^{(K_2-\alpha)n/2K_1}e^{K_4t},$$

which decays exponentially to zero as $t\to\infty$ provided (1.8) holds.

(c) *Proof of Theorem* 2. We are able to proceed entirely as in the first part of the proof of Theorem 1 to arrive at the estimate: $|u(x, T_0)|\leqq M_0$ for $x\in E^n$. In order to obtain information about the behavior of $u(x, t)$ for $t>T_0$ we employ a comparison function $w(x, t)$ slightly different from (1.19), namely

$$\begin{aligned} w(x, t)=M_0[\cosh 2\sqrt{K_1K_3}(t-T_0)]^{-\beta/2K_1} \\ \times\exp[-\sqrt{K_3/4K_1}|x|^2\tanh 2\sqrt{K_1K_3}(t-T_0)+K_4(t-T_0)]. \end{aligned}$$

Using the additional hypothesis (1.9) together with (1.1)–(1.3) we find that $w(x, t)$ satisfies the differential inequality (1.20) in $E^n\times(T_0, \infty)$ and hence that $|u(x, t)|\leqq w(x, t)$ for $(x, t)\in E^n\times(T_0, \infty)$. The conclusion of Theorem 2 follows immediately, for when $t\to\infty$ the function $w(x, t)$ behaves just like

$$[\cosh 2\sqrt{K_1K_3}(t-T_0)]^{-\beta/2K_1}e^{K_4t},$$

which tends exponentially to zero as $t\to\infty$ provided (1.10) holds.

2. Exponential decay of solutions of (B)

The system (B) of parabolic equations to which we shall extend the results of the preceding section can be written

$$L^\mu[u^\mu]+\sum_{\nu=1}^N c^{\mu\nu}(x, t)u^\mu=0, \ \mu=1, \dots, N,$$

where

$$L^\mu\equiv\sum_{i,j=1}^n a_{ij}^\mu(x, t)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^n b_i^\mu(x, t)\frac{\partial}{\partial x_i}-\frac{\partial}{\partial t}, \ \mu=1, \dots, N.$$

The system is coupled only in the terms which are not differentiated; so that a system of this form is said to be weakly coupled (see [7]).

It is assumed that there exist constants $K_1>0$, $K_2\geqq 0$, $K_3>0$ and K_4 such that

(2.1) $$0\leqq\sum_{i,j=1}^n a_{ij}^\mu(x, t)\xi_i\xi_j\leqq K_1|\xi|^2, \ \mu=1, \dots, N,$$

(2.2) $$|b_i^\mu(x, t)|\leqq K_2(|x|^2+1)^{1/2}, \ i=1, \dots, n, \ \mu=1, \dots, N,$$

(2.3) $$c^{\mu\nu}(x, t) \geqq 0 \text{ for } \mu \neq \nu,\ \mu,\ \nu = 1, \dots, N,$$

(2.4) $$\sum_{\nu=1}^{N} c^{\mu\nu}(x, t) \leqq -K_3 |x|^2 + K_4,\ \mu = 1, \dots, N,$$

for all $(x, t) \in E^n \times [0, \infty)$ and $\xi \in E^n$.

Theorem 1 of Section 1 is generalized as follows.

Theorem 3. *Let* $\{u^\mu(x, t)\}$, $\mu = 1, \dots, N$, *be a solution of* (B) *in* $E^n \times (0, \infty)$ *with the properties*:

(i) *there are positive constants* M *and* a *such that*

$$|u^\mu(x, 0)| \leqq M \exp(a|x|^2) \text{ for } x \in E^n,\ \mu = 1, \dots, N,$$

(ii) *for any* $T > 0$ *there are positive numbers* M_T *and* a_T *such that*

$$|u^\mu(x, t)| \leqq M_T \exp(a_T |x|^2) \text{ for } (x, t) \in E^n \times [0, T],\ \mu = 1, \dots, N.$$

Assume that

$$4K_1 a^2 + 2K_2 n a - K_3 < 0 \text{ and } K_4 + 2(K_2 - \alpha) n \lambda < 0,$$

where

(2.5) $$\alpha = \min_{\substack{i=1,\dots,n \\ \mu=1,\dots,N}} [\inf_{(x,t) \in E^n \times [0,\infty)} a^\mu_{ii}(x, t)],$$

and λ *is the positive root of the quadratic equation* $4K_1 \lambda^2 + 2K_2 n \lambda - K_3 = 0$.

Then $\lim_{t \to \infty} u^\mu(x, t) = 0$, $\mu = 1, \dots, N$, *the convergence being of exponential order and uniform with respect to* x *in* E^n.

Proof. We need the following Lemma due to Besala [1].

Lemma. *Suppose that hypotheses* (2.1)–(2.3) *are satisfied. Suppose, furthermore, that there are positive constants* K_3' *and* K_4' *such that*

$$\sum_{\nu=1}^{N} c^{\mu\nu}(x, t) \leqq K_3' |x|^2 + K_4' \text{ for } (x, t) \in E^n \times [0, \infty),\ \mu = 1, \dots, N.$$

Let $\{Z^\mu(x, t)\}$, $\mu = 1, \dots, N$, *be a system of functions defined in* $E^n \times [0, \infty)$, *with the property* (ii) *mentioned in Theorem* 3, *and such that*

$$L^\mu[Z^\mu] + \sum_{\nu=1}^{N} c^{\mu\nu}(x, t) Z^\nu \leqq 0 \text{ in } E^n \times (0, \infty),\ \mu = 1, \dots, N,$$

$$Z^\mu(x, 0) \geqq 0 \text{ on } E^n,\ \mu = 1, \dots, N.$$

Then, $Z^\mu(x, t) \geqq 0$ *for* $(x, t) \in E^n \times (0, \infty)$, $\mu = 1, \dots, N$.

We let the quantities ρ, θ_k, M_k, T_0 and the functions $v(x, t)$, $w(x, t)$ be as in the proof of Theorem 1, except that it is required for α to be replaced by

(2.5). We form the functions $w_{\pm}^{\mu}(x, t)=v(x, t)\pm u^{\mu}(x, t)$, $\mu=1, \ldots, N$. Since, by (2.1)−(2.4),

$$L^{\mu}[v]+\sum_{\nu=1}^{N} c^{\mu\nu}(x, t)\, v \leqq 0 \text{ in } E^{n}\times(0, \theta_0^{-1}],\ \mu=1, \ldots, N,$$

we see that

$$L^{\mu}[w_{\pm}^{\mu}]+\sum_{\nu=1}^{N} c^{\mu\nu}(x, t) w_{\pm}^{\nu} \leqq 0 \text{ in } E^{n}\times(0, \theta_0^{-1}],\ \mu=1, \ldots, N.$$

Since $w_{\pm}^{\mu}(x, 0)\geqq 0$ for $x\in E^{n}$, $\mu=1, \ldots, N$, we conclude from Besala's lemma that $w_{\pm}^{\mu}(x, t)\geqq 0$, i. e. $|u^{\mu}(x, t)|\leqq v(x, t)$ for $(x, t)\in E^{n}\times(0, \theta_0^{-1}]$, $\mu=1, \ldots, N$. Thus in particular

$$|u^{\mu}(x, \theta_0^{-1})|\leqq M_1 \exp(a\rho^{-1}|x|^2) \text{ for } x\in E^{n},\ \mu=1, \ldots, N.$$

Applying this argument successively yields

$$|u^{\mu}(x, \theta_0^{-1}+\theta_1^{-1}+\cdots+\theta_k^{-1})|\leqq M_{k+1}\exp(a\rho^{-k-1}|x|^2) \text{ for } x\in E^{n},$$
$$\mu=1, \ldots, N,\ k=1, 2, \ldots.$$

Employing exactly the same limiting procedure as in the proof of Theorem 1 we can derive the estimate: $|u^{\mu}(x, T_0)|\leqq M_0$ for $x\in E^{n}$, $\mu=1, \ldots, N$.

Now define the functions $Z_{\pm}^{\mu}(x, t)=w(x, t)\pm u^{\mu}(x, t)$, $\mu=1, \ldots, N$. It is clear that

$$L^{\mu}[Z_{\pm}^{\mu}]+\sum_{\nu=1}^{N} c^{\mu\nu}(x, t) Z_{\pm}^{\nu}\leqq 0 \text{ in } E^{n}\times(T_0, \infty),$$

$$Z_{\pm}^{\mu}(x, T_0)\geqq 0 \text{ on } E^{n},\ \mu=1, \ldots, N.$$

Consequently, by Besala's lemma, we have

$$Z_{\pm}^{\mu}(x, t)\geqq 0, \text{ i. e. } |u^{\mu}(x, t)|\leqq w(x, t) \text{ for } (x, t)\in E^{n}\times(T_0, \infty),$$

$\mu=1, \ldots, N$, which was to be proved.

The following is an extension of Theorem 2 of Section 1.

THEOREM 4. *In addition to* (2.1)−(2.4), *we assume that there is a positive constant* β *such that*

$$\sum_{i=1}^{n}(a_{ii}^{\mu}(x, t)+b_i^{\mu}(x, t)\, x_i)\geqq\beta \textit{ for } (x, t)\in E^{n}\times[0, \infty),\ \mu=1, \ldots, N.$$

If $\{u^{\mu}(x, t)\}$, $\mu=1, \ldots, N$, *is a solution of* (B) *in* $E^{n}\times(0, \infty)$ *having the properties* (i), (ii) *mentioned in Theorem* 3 *and if*

$$4K_1 a^2+2K_2 na-K_3<0 \textit{ and } K_4-\beta\sqrt{K_3/K_1}<0,$$

then $\lim_{t\to\infty} u^{\mu}(x, t)=0$, $\mu=1, \ldots, N$, *the convergence being of exponential order and uniform with respect to* x *in* E^n.

The proof of this theorem may be omitted.

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