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On the Existence of Solutions of Some Non-linear Parabolic Equations

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1. Introduction

In this paper we consider parabolic equations with boundary conditions:

(a) $\frac{du}{dt}+Au=f, \qquad u(0)=u_0,$

(b) $\frac{du}{dt}+Au=f, \qquad u(0)=u(T),$

where A is a non-linear operator.

In 1965 J. Leray and J. L. Lions [4] introduced a non-linear operator on a reflexive Banach space into its conjugate space and showed that it is surjective under the condition of coerciveness. Making use of this result, J. L. Lions [5] showed the existence of solutions of (a) and (b) for a certain kind of non-linear operator A.

In 1968 H. Brezis [1] introduced a new operator, called of type M, which is more general than the operator of J. Leray and J. L. Lions, and showed that the operator of type M on a reflexive Banach space into its conjugate space is also surjective under the condition of coerciveness.

The purpose of this paper is to extend J. L. Lions' results in [5] on the existence of solutions of (a) and (b) to the case where A is a bounded coercive operator satisfying conditions which are more general than Lions' [5]. In the proof we shall make use of the result by H. Brezis mentioned above.

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2. Notation and statement of theorems

In general, for a Banach space U over C (complex numbers), we shall denote the anti-dual space of U by U'. Let H be a Hilbert space over C, $(\, , \,)$ be the scalar product in H, and $|\cdot|$ be the norm in H. One may identify H' with H. Let V be a reflexive Banach space over C, $((\, , \,))$ the natural pairing between V' and V, $\|v\|_V$ the norm of $v \in V$ and $\|v^*\|_{V'}$ the norm of $v^* \in V'$.

Assume that $V \subset H$, V is dense in H and the injection is continuous. Then $V \subset H \subset V'$. Let F be a linear space whose elements are vector-valued functions defined on a fixed real finite interval $(0, T)$ with values in H and $\mathcal{D}(0, T; V)$ the space of all C^∞ functions on $(0, T)$ into V with compact support. Assume that F is a reflexive Banach space, that

$$L^\infty(0, T; V) \subset F \subset L^2(0, T; H)$$

and

$$F' \subset L^1(0, T; V'),$$

where all injections are continuous, and that $\mathcal{D}(0, T; V)$ is dense in F. We denote the natural pairing between F' and F by $<\ ,\ >$, the norm of $u \in F$ by $\|u\|_F$ and the norm of $u^* \in F'$ by $\|u^*\|_{F'}$. For each $u^* = u^*(t) \in L^2(0, T; H)$, consider

$$\int_0^T (u^*(t), u(t))\,dt, \qquad u \in F.$$

This is a continuous anti-linear form on F, and hence belongs to F'. We express this fact by $L^2(0, T; H) \subset F'$. For this reason we write

$$<u_1, u_2> = \int_0^T (u_1(t), u_2(t))\,dt$$

for any $u_1, u_2 \in L^2(0, T; H)$ too.

For $g \in L^1(0, T; V')$ we define $K_1^\varepsilon g$ by

$$(K_1^\varepsilon g)(t) = \frac{1}{\varepsilon}\int_t^T \exp\left(\frac{t-s}{\varepsilon}\right) g(s)\,ds, \qquad \varepsilon > 0.$$

Then $K_1^\varepsilon g \in L^1(0, T; V')$ for any $\varepsilon > 0$.

We assume that

(h_1) if $g \in F'$, then $K_1^\varepsilon g \in F'$ and if G is a bounded set in F', then $\{K_1^\varepsilon g;\ g \in G, \varepsilon > 0\}$ is bounded in F'.

This condition is satisfied, for instance, when $F = L^p(0, T; V)$, $2 \leq p < +\infty$.

Throughout the paper we shall use the symbols "$\xrightarrow{s}$", "$\xrightarrow{w}$" and "$\xrightarrow{w^*}$" to denote the convergences in the strong, weak and weak* topology respectively.

Since $F \subset L^2(0, T; H) \subset F'$, any $u \in F$ may be regarded as an element of F'. Hence, u is a continuous anti-linear from on $F \supset \mathcal{D}(0, T; V)$, so that u may be considered to be a V'-valued distribution. Therefore u' exists in the distribution sense.

Let A be an operator on F into F' and assume that A satisfies the following conditions:

(A_1) if $\{u_i\} \subset F$ is such a directed set that $\|u_i\|_F \leq K$, $u_i' \in F'$, $\|u_i'\|_{F'} \leq K$,

$u_i \xrightarrow{w} u$ in F, $u_i' \xrightarrow{w^*} u'$ in F', $Au_i \xrightarrow{w^*} \psi$ in F' and $\limsup_i Re < Au_i, u_i > \leq Re < \psi, u >$, then $Au=\psi$;

(A_2) A is bounded, that is, A maps bounded sets in F to bounded sets in F';

(A_3) (coerciveness) $\dfrac{Re < Av, v >}{\|v\|_F} \to \infty$ as $\|v\|_F \to \infty$.

Under the above hypotheses we shall establish the following theorem.

THEOREM 1. *For given $f \in F'$ and $u_0 \in H$, there exists $u \in F$ such that $u(t)$ is a continuous function on $[0, T]$ into V', $u' \in F'$, $u'+Au=f$ and $u(0)=u_0$.*

For $g \in L^1(0, T; V')$ we set

$$(K_2^\varepsilon g)(t) = \frac{1}{\varepsilon}\int_0^t \exp\left(\frac{t-s-T}{\varepsilon}\right) g(s)\,ds, \qquad \varepsilon>0.$$

Then $K_2^\varepsilon g \in L^1(0, T; V')$ for any $\varepsilon>0$.

In addition we suppose that

(h_2) if $g \in F'$, then $K_2^\varepsilon g \in F'$ and if G is a bounded set in F', then $\{K_2^\varepsilon g; g \in G, \varepsilon>0\}$ is bounded in F'.

This condition is satisfied, for instance, when $F=L^p(0, T; V)$, $2 \leq p < +\infty$.

Then we have the following theorem.

THEOREM 2. *For given $f \in F'$, there exists $u \in F$ such that $u(t)$ is a continuous function on $[0, T]$ into V', $u' \in F'$, $u'+Au=f$ and $u(0)=u(T)$ in H.*

For the method of proof we essentially follow J. L. Lions [5].

3. Lemmas

Let B be a reflexive Banach space, t_0 a positive real number and $\mathcal{D}'(0, t_0; B')$ the space of all distributions on $(0, t_0)$ with values in B', that is, the space of all continuous anti-linear forms on $\mathcal{D}(0, t_0; B)$.

If $u \in L^1(0, t_0; B')$ and the distributional derivative $u' \in L^1(0, t_0; B')$, then there exists a strongly absolutely continuous function $\tilde{u}(t)$ on $[0, t_0]$ into B' such that $\tilde{u}(t)=u(t)$ almost everywhere on $(0, t_0)$ and the strong derivative of $\tilde{u}$ is equal to u' in the distribution sense (cf. Chap. I, 11 of [2]; Chap. III, 3.7, 3.8 of [3]; IV, §5 of [6]). Therefore we assume that such a function $u(t)$ is strongly absolutely continuous on $[0, t_0]$ and $u'(t)$ is the strong derivative of $u(t)$. Let $v(t)$ be a strongly absolutely continuous function on $[0, t_0]$ with values in B such that the strong derivative $v'(t) \in L^1(0, t_0; B)$. Then we have the formula for integration by parts for u and v:

$$(3.1) \qquad \int_0^{t_0}((u'(t), v(t)))\,dt + \int_0^{t_0}((u(t), v'(t)))\,dt = ((u(t_0), v(t_0))) - ((u(0), v(0))),$$

where ((,)) is the natural pairing between B' and B.

Making use of this formula, we shall prove the following lemmas.

LEMMA 1. *Let* $\{u_i\}$ *be a directed set,* $u_i \in L^1(0, t_0; B')$, $u_i' \in L^1(0, t_0; B')$, $u_i \xrightarrow{w} u$ *in* $L^1(0, t_0; B')$ *and* $u_i' \xrightarrow{w} u'$ *in* $L^1(0, t_0; B')$. *Then* $u_i(t) \xrightarrow{w^*} u(t)$ *in* B' *for all* $t \in [0, t_0]$.

PROOF. Let α be any element of B and set $v(t)=t\alpha$. Clearly v is strongly absolutely continuous on $[0, t_0]$ and $v' \in L^1(0, t_0; B)$. Therefore, by integration by parts we have for any $t' \in (0, t_0]$

$$\int_0^{t'}((u_i'(t), v(t)))dt+\int_0^{t'}((u_i(t), v'(t)))dt=t'((u_i(t'), \alpha))$$

and

$$\int_0^{t'}((u'(t), v(t)))dt+\int_0^{t'}((u(t), v'(t)))dt=t'((u(t'), \alpha)).$$

Since

$$\int_0^{t'}((u_i'(t), v(t)))dt \to \int_0^{t'}((u'(t), v(t)))dt$$

and

$$\int_0^{t'}((u_i(t), v'(t)))dt \to \int_0^{t'}((u(t), v'(t)))dt,$$

we obtain $((u_i(t'), \alpha)) \to ((u(t'), \alpha))$. The arbitrariness of α implies that $u_i(t') \xrightarrow{w^*} u(t')$ in B'. Considering the function $v(t)=(t_0-t)\alpha$, we obtain $u_i(0) \xrightarrow{w^*} u(0)$ in B'. q.e.d.

LEMMA 2. *Let* $\{u_i\}$ *be a directed set,* $u_i \in L^1(0, t_0; B')$, $u_i' \in L^1(0, t_0; B')$, $u_i \xrightarrow{s} u$ *in* $L^1(0, t_0; B')$ *and* $u_i' \xrightarrow{s} u'$ *in* $L^1(0, t_0; B')$. *Then* $u_i(t) \xrightarrow{s} u(t)$ *in* B' *for all* $t \in [0, t_0]$.

PROOF. Let U be the closed unit ball in B, X the family of functions $\{v_\alpha(t)=t\alpha; \alpha \in U\}$ and Y the family $\{v_\alpha'(t)=\alpha; \alpha \in U\}$. Clearly X and Y are bounded in the anti-dual space of $L^1(0, t_0; B')$. Since for any $t' \in (0, t_0]$

$$\int_0^{t'}((u_i'(t), v_\alpha(t)))dt \to \int_0^{t'}((u'(t), v_\alpha(t)))dt$$

uniformly on X and

$$\int_0^{t'}((u_i(t), v_\alpha'(t)))dt \to \int_0^{t'}((u(t), v_\alpha'(t)))dt$$

uniformly on Y, using the formula for integration by parts again we infer that $((u_i(t'), \alpha)) \to ((u(t'), \alpha))$ uniformly on U. Thus $u_i(t') \xrightarrow{s} u(t')$ in B'. Considering the family $\{v_\alpha(t) = (t_0 - t)\alpha;\ \alpha \in U\}$, we obtain $u_i(0) \xrightarrow{s} u(0)$ in B'. q.e.d.

To show THEOREM 1 we consider the space $W = \{v \in F;\ v' \in L^2(0, T; H)\}$. Define a norm in W by $\|v\|_W = \|v\|_F + \|v'\|_{L^2(0,T;H)}$. Then W is a reflexive Banach space. It follows from (3.1) that

$$<u', v> + <u, v'> = (u(T), v(T)) - (u(0), v(0)) \qquad \text{for } u, v \in W.$$

In particular,

$$2\mathrm{Re}<u', u> = |u(T)|^2 - |u(0)|^2 \qquad \text{for } u \in W. \tag{3.2}$$

Given $\varepsilon > 0$, we set for $u, v \in W$

$$[A_\varepsilon u, v] = \varepsilon <u', v'> + <u', v> + (u(0), v(0)) + <Au, v>, \tag{3.3}$$

where $[\ ,\ ]$ is the natural pairing between W' and W. By this formula A_ε is defined to be an operator on W into W'.

We have the following lemma.

LEMMA 3. *For given* $\varepsilon > 0$,

(1) A_ε *is a bounded operator on* W *into* W',

(2) *if* $\{u_i\} \subset W$ *is a directed set such that* $\|u_i\|_W \leq C$, $u_i \xrightarrow{w} u$ *in* W, $A_\varepsilon u_i \xrightarrow{w^*} \psi$ *in* W' *and* $\limsup_i \mathrm{Re}[A_\varepsilon u_i, u_i] \leq \mathrm{Re}[\psi, u]$, *then* $A_\varepsilon u = \psi$,

(3) $\dfrac{\mathrm{Re}[A_\varepsilon v, v]}{\|v\|_W} \to \infty$ *as* $\|v\|_W \to \infty$.

PROOF. To prove (1) we first observe that the mapping $v \to v(0)$ is bounded linear on W by LEMMA 2. Hence there exists a positive constant M such that $|v(0)| \leq M\|v\|_W$ for all $v \in W$. If $\|u\|_W \leq K$, then for all $v \in W$

$$\begin{aligned}|[A_\varepsilon u, v]| &\leq \varepsilon \|u'\|_{L^2(0,T;H)} \cdot \|v'\|_{L^2(0,T;H)} \\ &\quad + \|u'\|_{L^2(0,T;H)} \cdot M'\|v\|_F + KM^2\|v\|_W + \|Au\|_{F'}\|v\|_F,\end{aligned}$$

where M' is a positive constant. Since A is a bounded operator, $\{\|Au\|_{F'};\ \|u\|_W \leq K\}$ is bounded. Consequently for a sufficiently large $N > 0$, we have

$$|[A_\varepsilon u, v]| \leq N\|v\|_W.$$

This implies that A_ε is bounded.

To prove (2) we choose a subdirected set $\{i_\alpha\}$ such that

$$\limsup_i \mathrm{Re}[A_\varepsilon u_i, u_i] = \lim_\alpha \mathrm{Re}[A_\varepsilon u_{i_\alpha}, u_{i_\alpha}].$$

By hypothesis (A_2), we may choose $\{i_\alpha\}$ in such a way that $Au_{i_\alpha} \xrightarrow{w^*} \eta$ in F'. Since $u_i \xrightarrow{w} u$ in F and $u'_i \xrightarrow{w} u'$ in $L^2(0, T; H)$, it follows from LEMMA 1 that $u_i(0) \xrightarrow{w} u(0)$ in H. By (3.3), $[A_\varepsilon u_{i_\alpha}, v] = \varepsilon < u'_{i_\alpha}, v' > + < u'_{i_\alpha}, v > + (u_{i_\alpha}(0), v(0)) + < Au_{i_\alpha}, v >$, and, taking limit in α, we also have

$$(3.4) \qquad [\psi, v] = \varepsilon < u', v' > + < u', v' > + (u(0), v(0)) + < \eta, v > \quad \text{for all } v \in W.$$

Hence, by (3.2),

$$(3.5) \qquad \mathrm{Re}[A_\varepsilon u_{i_\alpha}, u_{i_\alpha}] = \varepsilon \| u'_{i_\alpha} \|^2_{L^2(0,T;H)} + \frac{1}{2} | u_{i_\alpha}(0) |^2 + \frac{1}{2} | u_{i_\alpha}(T) |^2 + \mathrm{Re} < Au_{i_\alpha}, u_{i_\alpha} >$$

and

$$(3.6) \qquad \mathrm{Re}[\psi, u] = \varepsilon \| u' \|^2_{L^2(0,T;H)} + \frac{1}{2} | u(0) |^2 + \frac{1}{2} | u(T) |^2 + \mathrm{Re} < \eta, u >.$$

On the other hand, since $\liminf_\alpha \| u'_{i_\alpha} \|^2_{L^2(0,T;H)} \geq \| u' \|^2_{L^2(0,T;H)}$, $\liminf_\alpha | u_{i_\alpha}(0) |^2 \geq | u(0) |^2$ and $\liminf_\alpha | u_{i_\alpha}(T) | \geq | u(T) |^2$, we have by (3.5)

$$\begin{aligned} &\limsup_\alpha \mathrm{Re} < Au_{i_\alpha}, u_{i_\alpha} > \\ &= \limsup_\alpha \left\{ \mathrm{Re}[A_\varepsilon u_{i_\alpha}, u_{i_\alpha}] - \varepsilon \| u'_{i_\alpha} \|^2_{L^2(0,T;H)} - \frac{1}{2} | u_{i_\alpha}(0) |^2 - \frac{1}{2} | u_{i_\alpha}(T) |^2 \right\} \\ &\leq \lim_\alpha \mathrm{Re}[A_\varepsilon u_{i_\alpha}, u_{i_\alpha}] - \varepsilon \| u' \|^2_{L^2(0,T;H)} - \frac{1}{2} | u(0) |^2 - \frac{1}{2} | u(T) |^2. \end{aligned}$$

Thus, by (3.6) and the hypothesis that

$$\lim_\alpha \mathrm{Re}[A_\varepsilon u_{i_\alpha}, u_{i_\alpha}] \leq \mathrm{Re}[\psi, u],$$

we obtain

$$\limsup_\alpha \mathrm{Re} < Au_{i_\alpha}, u_{i_\alpha} > \leq \mathrm{Re} < \eta, u >.$$

Therefore, by hypothesis (A_1) we have $Au = \eta$. Then by (3.4)

$$[A_\varepsilon u, v] = [\psi, v] \qquad \text{for all } v \in W.$$

Hence $A_\varepsilon u = \psi$.

Finally to prove (3) we use the relation

$$\mathrm{Re}[A_\varepsilon v, v] = \varepsilon \|v'\|^2_{L^2(0,T;H)} + \frac{1}{2}|v(0)|^2 + \frac{1}{2}|v(T)|^2 + \mathrm{Re} < Av, v >$$

$$\geq \varepsilon \|v'\|^2_{L^2(0,T;H)} + \mathrm{Re} < Av. v >,$$

which follows from (3.2). Hence

$$\frac{\mathrm{Re}[A_\varepsilon v, v]}{\|v\|_W} \geq \frac{\varepsilon \|v'\|^2_{L^2(0,T;H)} + \mathrm{Re} < Av, v >}{\|v'\|_{L^2(0,T;H)} + \|v\|_F}.$$

Then, using (A_2), we see that (3) is valid. q.e.d.

Now we recall the results by H. Brezis [1]:

DEFINITION. (H. Brezis [1]). *Let E be a Banach space and E' the dual space of E. A mapping $T: E \to E'$ is said to be of type M if T satisfies the following conditions (M_1) and (M_2).*

(M_1) *If $\{x_i\}$ is a directed set such that $x_i \xrightarrow{w} x$ in E, $\|x_i\|_E \leq C$, $Tx_i \xrightarrow{w^*} g$ in E' and $\limsup_i (Tx_i, x_i) \leq (g, x)$, then $Tx = g$.*

(M_2) *The restriction of T on any finite dimensional subspace of E is continuous with respect to the weak* topology.*

Remark: If T is bounded, then condition (M_1) implies (M_2).

We shall use

THEOREM. (H. Brezis [1]) *Let E be a Banach space, E' be the dual space of E and T be an operator of type M on E into E'. Suppose that*

$$\frac{|(Tx, x)|}{\|x\|_E} \to \infty \qquad as\ \|x\|_E \to \infty.$$

Then T is surjective, that is, the range $R(T) = E'$.

Remark: The above definition and theorem were given in real Banach space in [1]. However, it is easy to extend them to the case of complex Banach spaces replacing (,) by Re(,).

LEMMA 3 and the above Remark show that A_ε is a bounded operator of type M on W into W'. Thus we have

LEMMA 4. *For given $f \in F'$ and $u_0 \in H$, there exists $u_\varepsilon \in W$ such that*

$$(3.7) \qquad [A_\varepsilon u_\varepsilon, v] = < f, v > + (u_0, v(0)) \qquad \text{for all } v \in W.$$

PROOF. The functional $v \to < f, v > + (u_0, v(0))$ is a continuous anti-linear form on W. Therefore this lemma is an immediate consequence of the above THEOREM by H. Brezis. q.e.d.

For the family $\{u_\varepsilon; \varepsilon > 0\}$ of solutions of (3.7), we prove

LEMMA 5. *Let* $\varepsilon_0>0$ *be a constant. Then*

(1) *The set* $\{u_\varepsilon; 0<\varepsilon\leq\varepsilon_0\}$ *is bounded in* F.

(2) *The set* $\{u_\varepsilon(0); 0<\varepsilon\leq\varepsilon_0\}$ *is bounded in* H.

(3) *The set* $\{\sqrt{\varepsilon}u'_\varepsilon; 0<\varepsilon\leq\varepsilon_0\}$ *is bounded in* $L^2(0, T; H)$.

(4) *The set* $\{u'_\varepsilon; 0<\varepsilon\leq\varepsilon_0\}$ *is bounded in* F'.

PROOF. From (3.7) we obtain (cf. (3.5))

$$\begin{aligned}\mathrm{Re}[A_\varepsilon u_\varepsilon, u_\varepsilon]&=\varepsilon\|u'_\varepsilon\|^2_{L^2(0,T;H)}+\frac{1}{2}|u_\varepsilon(0)|^2+\frac{1}{2}|u_\varepsilon(T)|^2+\mathrm{Re}<Au_\varepsilon, u_\varepsilon>\\&\leq\|f\|_{F'}\cdot\|u_\varepsilon\|_F+|u_0|\cdot|u_\varepsilon(0)|\\&\leq\|f\|_{F'}\cdot\|u_\varepsilon\|_F+|u_0|^2+\frac{1}{4}|u_\varepsilon(0)|^2.\end{aligned}$$

Hence

$$\frac{\mathrm{Re}<Au_\varepsilon, u_\varepsilon>}{\|u_\varepsilon\|_F}\leq\|f\|_{F'}+\frac{|u_0|^2}{\|u_\varepsilon\|_F}.$$

This together with (A_3) implies (1). Then (2) and (3) are easily obtained.

Let us prove (4). Substitute $\phi\in\mathcal{D}(0, T; V)$ for v in (3.7). Then

$$\varepsilon<u'_\varepsilon, \phi'>+<u'_\varepsilon, \phi>+<Au_\varepsilon, \phi>=<f, \phi>.$$

Thus in the distribution sense

$$-\varepsilon u''_\varepsilon+u'_\varepsilon+Au_\varepsilon=f, \tag{3.8}$$

and hence $u''_\varepsilon\in F'+L^2(0, T; H)=F'\subset L^1(0, T; V')$, so that (3.8) holds in F'.

For $\alpha\in V$, we set $v(t)=t\alpha$. By integration by parts

$$-\varepsilon<u''_\varepsilon, v'>=\varepsilon<u'_\varepsilon, v'>-\varepsilon((u'_\varepsilon(T), v(T))).$$

Using (3.8),

$$\varepsilon<u'_\varepsilon, v'>+<u'_\varepsilon, v>+<Au_\varepsilon, v>-\varepsilon((u'_\varepsilon(T), v(T)))=<f, v>.$$

On the other hand, since $v(0)=0$, (3.7) implies that

$$\varepsilon<u'_\varepsilon, v'>+<u'_\varepsilon, v>+<Au_\varepsilon, v>=<f, v>.$$

Therefore $((u'_\varepsilon(T), v(T)))=0$, and hence $((u'_\varepsilon(T), \alpha))=0$. Since α may be any element of V, we have

$$u'_\varepsilon(T)=0 \qquad \text{in } V'. \tag{3.9}$$

(3.8) and (3.9) imply that

$$u_\varepsilon'(t)=\frac{1}{\varepsilon}\int_t^T \exp\left(\frac{t-s}{\varepsilon}\right)(f-Au_\varepsilon)(s)ds \qquad \text{in } V'.$$

In fact, we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_t^T \exp\left(\frac{t-s}{\varepsilon}\right)(f-Au_\varepsilon)(s)ds \\
&=-\int_t^T \exp\left(\frac{t-s}{\varepsilon}\right)u_\varepsilon''(s)ds+\frac{1}{\varepsilon}\int_t^T \exp\left(\frac{t-s}{\varepsilon}\right)u_\varepsilon'(s)ds \\
&=-\int_t^T \exp\left(\frac{t-s}{\varepsilon}\right)u_\varepsilon''(s)ds-\exp\left(\frac{t-T}{\varepsilon}\right)u_\varepsilon'(T)+u_\varepsilon'(t) \\
&\quad+\int_t^T \exp\left(\frac{t-s}{\varepsilon}\right)u_\varepsilon''(s)ds \\
&=u_\varepsilon'(t).
\end{aligned}$$

Since $\{f-Au_\varepsilon\}$ is bounded in F' by (A_2), hypothesis (h_1) implies that $\{u_\varepsilon'\}$ is bounded in F'.

§4. Proof of the theorems

PROOF OF THEOREM 1: It follows from LEMMA 5 that there exists a suitable directed set $\{\varepsilon\}$ tending to zero such that

$$u_\varepsilon \xrightarrow{w} u \qquad \text{in } F, \tag{4.1}$$

$$u_\varepsilon' \xrightarrow{w^*} z \qquad \text{in } F', \tag{4.2}$$

$$\sqrt{\varepsilon}u_\varepsilon' \xrightarrow{w} \rho \qquad \text{in } L^2(0, T; H), \tag{4.3}$$

$$u_\varepsilon(0) \xrightarrow{w} \xi_0 \qquad \text{in } H, \tag{4.4}$$

$$Au_\varepsilon \xrightarrow{w^*} \chi \qquad \text{in } F'. \tag{4.5}$$

For any $\phi \in \mathcal{D}(0, T; V)$, $<u_\varepsilon', \phi>=-<u_\varepsilon, \phi'>\to -<u, \phi'>$ as $\varepsilon\to 0$. Hence, (4.2) implies that $-<u, \phi'>=<z, \phi>$ for all $\phi\in\mathcal{D}(0, T; V)$. Thus $u'=z$ in F'. By (4.1) and (4.2) LEMMA 1 implies that $u_\varepsilon(0)\xrightarrow{w^*}u(0)$ in V', so that $u(0)=\xi_0$ on account of (4.4). From (4.3) we see that as $\varepsilon\to 0$, $\varepsilon u_\varepsilon''\to 0$ weakly in the distribution sense. In fact, for any $\phi\in\mathcal{D}(0, T; V)$

$$<\varepsilon u_\varepsilon'', \phi>=-\sqrt{\varepsilon}<\sqrt{\varepsilon}u_\varepsilon', \phi'>\to 0.$$

Thus letting $\varepsilon\to 0$ in (3.8), we have

(4.6) $$u'+\chi=f$$

in the distribution sense. Since $\mathcal{D}(0, T; V)$ is dense in F, (4.6) holds in F'.

For $\beta\in V$, we set $v(t)=(T-t)\beta$. Then we have by (4.1)$\sim$(4.4),

$$\varepsilon<u'_\varepsilon, v'>\to 0,\ <u'_\varepsilon, v>\to<u', v>,\ <Au_\varepsilon, v>\to<\chi, v>$$

and $(u_\varepsilon(0), v(0))=T(u_\varepsilon(0), \beta)\to T(u(0), \beta)$. Hence by (3.7) we have

$$<u', v>+T(u(0), \beta)+<\chi, v>=<f, v>+T(u_0, \beta).$$

By (4.6) the left hand side is equal to $T(u(0), \beta)+<f, v>$. Thus we infer that $(u(0), \beta)=(u_0, \beta)$. The arbitrariness of β implies that $u(0)=u_0$.

It remains to prove that $Au=\chi$. There exists a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n\to 0$ and

$$X\equiv\liminf_{\varepsilon\to 0}[\mathrm{Re}<u'_\varepsilon, u_\varepsilon>+|u_\varepsilon(0)|^2]$$
$$=\lim_{n\to\infty}[\mathrm{Re}<u'_{\varepsilon_n}, u_{\varepsilon_n}>+|u_{\varepsilon_n}(0)|^2].$$

By (3.2) in the proof of LEMMA 3, for any k, j

$$\mathrm{Re}<u'_{\varepsilon_k}-u'_{\varepsilon_j}, u_{\varepsilon_k}-u_{\varepsilon_j}>+|u_{\varepsilon_k}(0)-u_{\varepsilon_j}(0)|^2\geq 0,$$

that is,

$$[\mathrm{Re}<u'_{\varepsilon_k}, u_{\varepsilon_k}>+|u_{\varepsilon_k}(0)|^2]+[\mathrm{Re}<u'_{\varepsilon_j}, u_{\varepsilon_j}>+|u_{\varepsilon_j}(0)|^2]$$
$$-\mathrm{Re}<u'_{\varepsilon_k}, u_{\varepsilon_j}>-\mathrm{Re}<u'_{\varepsilon_j}, u_{\varepsilon_k}>-(u_{\varepsilon_k}(0), u_{\varepsilon_j}(0))$$
$$-(u_{\varepsilon_j}(0), u_{\varepsilon_k}(0))\geq 0.$$

Letting $k\to\infty$ and then $j\to\infty$, we have

$$2[X-\mathrm{Re}<u', u>-|u(0)|^2]\geq 0.$$

Thus

(4.7) $$X\geq\mathrm{Re}<u', u>+|u(0)|^2.$$

On the other hand, by (3.3), (3.7), (4.1), (4.4) and (4.6), we obtain

(4.8) $$\limsup_{\varepsilon\to 0}\mathrm{Re}<Au_\varepsilon, u_\varepsilon>$$
$$=\limsup_{\varepsilon\to 0}[\mathrm{Re}<f, u_\varepsilon>+\mathrm{Re}(u_0, u_\varepsilon(0))-\varepsilon\|u'_\varepsilon\|^2_{L^2(0,T;H)}$$
$$-\mathrm{Re}<u'_\varepsilon, u_\varepsilon>-|u_\varepsilon(0)|^2]$$

$$\leq \mathrm{Re}<f,\ u> + |u(0)|^2 - X$$

$$= \mathrm{Re}<u',\ u> + \mathrm{Re}<\chi,\ u> + |u(0)|^2 - X.$$

Hence, from (4.7) and (4.8), we derive

$$\limsup_{\varepsilon \to 0} \mathrm{Re}<Au_\varepsilon,\ u_\varepsilon> \leq \mathrm{Re}<\chi,\ u>.$$

Then it follows from (A_1) that $Au = \chi$. q.e.d.

Proof of Theorem 2: We consider the space $\tilde{\tilde{W}} = \{v \in F;\ v' \in L^2(0,\ T;\ H),\ v(0) = v(T)\}$. Define the same norm in $\tilde{\tilde{W}}$ as in $\tilde{W}$. Then $\tilde{\tilde{W}}$ is a reflexive Banach space. For given $\varepsilon > 0$, we set for $u,\ v \in \tilde{\tilde{W}}$

$$[\tilde{A}_\varepsilon u,\ v] = \varepsilon <u',\ v'> + <u',\ v> + <Au,\ v>.$$

Then we can show that $\tilde{A}_\varepsilon$ is a bounded coercive operator of type M on $\tilde{\tilde{W}}$ into $\tilde{\tilde{W}}'$ in the same way as Lemma 3. Thus by H. Brezis' result, for given $f \in F'$ there exists $u_\varepsilon \in \tilde{\tilde{W}}$ such that

$$[\tilde{A}_\varepsilon u_\varepsilon,\ v] = <f,\ v> \qquad \text{for all } v \in \tilde{\tilde{W}}.$$

Just as in the proof of Theorem 1, there exists a suitable directed set $\{\varepsilon\}$ tending to zero such that

(4.9) $\{u_\varepsilon\}$ is bounded in F and $u_\varepsilon \xrightarrow{w} u$ in F,

(4.10) $\sqrt{\varepsilon} u'_\varepsilon \xrightarrow{w} \rho$ in $L^2(0,\ T;\ H)$,

(4.11) $u_\varepsilon(0) = u_\varepsilon(T) \xrightarrow{w} \xi$ in H,

(4.12) $Au_\varepsilon \xrightarrow{w^*} \chi$ in F'.

We can show as in the proof of Theorem 1 that, for any $\varepsilon > 0$,

(4.13) $$-\varepsilon u''_\varepsilon + u'_\varepsilon + Au_\varepsilon = f$$

and

(4.14) $$u'_\varepsilon(0) = u'_\varepsilon(T) \quad \text{in } V'.$$

Also as in the proof of Theorem 1, (4.13) and (4.14) imply that

$$u'_\varepsilon(t) = \frac{1}{\varepsilon} \exp\left(\frac{T}{\varepsilon}\right)\left(\exp\frac{T}{\varepsilon} - 1\right)^{-1}\left[\int_0^t \exp\left(\frac{t-s-T}{\varepsilon}\right)(f - Au_\varepsilon)(s)\,ds + \int_t^T \exp\left(\frac{t-s}{\varepsilon}\right)(f - Au_\varepsilon)(s)\,ds\right].$$

This implies by hypotheses (h_1) and (h_2) that $\{u'_\varepsilon\}$ is bounded in F'. Therefore we may assume that

(4.15) $$u' \xrightarrow{w^*} u' \text{ in } F'.$$

By (4.9) and (4.15) LEMMA 1 implies that $u(0)=u(T)$ in H.

In the same way as in the proof of THEOREM 1, we obtain

$$\limsup_{\varepsilon\to 0} \mathrm{Re} < Au_\varepsilon, u_\varepsilon > \leq \mathrm{Re} < \chi, u >,$$

and, by hypothesis (A_1), $Au=\chi$. On the other hand, for all $\phi \in \mathcal{D}(0, T; V)$,

$$\varepsilon < u'_\varepsilon, \phi' > + < u'_\varepsilon, \phi > + < Au_\varepsilon, \phi > = < f, \phi >.$$

Letting $\varepsilon \to 0$, we have $u'+Au=f$ in the distribution sense. Since $\mathcal{D}(0, T; V)$ is dense in F, the equality $u'+Au=f$ holds in F'. Thus u is a solution.

q.e.d.

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