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Commutative Rings for which Each Proper Homomorphic Image is a Multiplication Ring. II

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This paper is an extension of Wood's results in [4]. All rings considered are assumed to be nonzero commutative rings. A ring R is called an *AM-ring* if whenever A and B are ideals of R with A properly contained in B, then there is an ideal C of R such that $A=BC$. An AM-ring in which $RA=A$ for each ideal A of R is called a multiplication ring. Wood characterized in [4] rings with identity for which *each proper homomorphic image is a multiplication ring*. Such rings are said to satisfy property (Hm). An example is given in [4] to show that a ring satisfying (Hm) need not be a multiplication ring. In fact, a general method is given for constructing such examples. This paper considers u-rings satisfying property (Hm) where a ring S is called a *u-ring* if the only ideal A of S such that $\sqrt{A}=S$ is S itself. Section 2 shows that the characterization of rings with identity satisfying (Hm) carries over to u-rings safisfying (Hm).

The notation and terminology is that of [5] with two exceptions: $\subseteq$ denotes containment and $\subset$ denotes proper containment, and we do not assume that a Noetherian ring contains an identity. If A is an ideal of a ring R, we say that A is a *proper ideal* of R if $(0)\subset A\subset R$ and that A is a *genuine* ideal of R if $A\subset R$.

1. Rings Satisfying Properties $(H*)$ and $(H**)$.

Let R be a ring. We say that R *satisfies property* $(*)$ (*satisfies property* $(**)$) if each ideal of R with prime radical is primary (is a prime power). If each proper homomorphic image of R satisfies $(*)$ (satisfies $(**)$), we say that R *satisfies porperty* (H^*) (*satisfies property* $(H**)$). In [3] it is shown that an AM-ring satisfies $(*)$ and $(**)$ and that if S is a u-ring, S satisfies $(**)$ if and only if S satisfies $(*)$ and primary ideals are prime powers. Therefore, in a u-ring, $(H**)$ implies $(H*)$. We give here a partial characterization of u-rings satisfying $(H*)$ and then show that the characterization of u-rings satisfying $(H**)$ is the same as the characterization of rings with identity satisfying $(H**)$.

DEFINITION. A ring R is said to have *dimension* n or to be *n-dimensional* if there exists a chain $P_0\subset P_1\subset\cdots\subset P_n$ of $n+1$ prime ideals of R where $P_n\subset R$, but no such chain of $n+2$ prime ideals exists in R.

Lemma 1.1. *Let R be a ring satisfying ($H*$) such that $\sqrt{(0)}=P$ is a genuine nonmaximal prime ideal of R. If $P=P^2$, R is either a zero-dimensional or one-dimensional domain. Hence, R satisfies* ($*$).

Proof. This follows from the proof of [4; Lemma 1.3] using [2; Theorem 1].

Lemma 1.2. *Let R be a ring satisfying ($H*$). If P is a genuine nonmaximal prime ideal of R and if $P^2 \neq (0)$, then $P=P^2$.*

Proof. Since $P^2 \neq (0)$, R/P^2 satisfies ($*$). Thus, P^2/P^2 is P/P^2-primary since $\sqrt{P^2/P^2}=P/P^2$. [2; Theorem 1] implies that $P^2/P^2=P/P^2$, and it follows that $P=P^2$.

Definition. A ring R is said to be a *primary ring* if R has at most two prime ideals.

Lemma 1.3. *If S is a primary u-ring, then S contains an identity. Hence, S satisfies* ($*$).

Proof.[1] Let $M=\sqrt{(0)}$. Since S is a u-ring, $M \subset S$. Also, since S is a primary ring and since $\sqrt{(0)}$ is the intersection of all prime ideals of S, M is a prime ideal of S. Let $s \in S \backslash M$. Since M is prime, $s^2 \notin M$ and it follows that $\sqrt{sS}=S$. Therefore, $sS=S$. For some $e \in S$, $se=s$. If $t \in S$, then $t=sx$ for some $x \in S$ and $et=esx=sx=t$. Hence e is the identity of S.

Lemma 1.4. *Let R be a ring such that $\sqrt{(0)}$ is not a nonzero maximal ideal of R. Then R satisfies ($H*$) and each genuine nonmaximal prime ideal of R is idempotent if and only if R satisfies* ($*$).

Proof. ($\leftarrow$) If R satisfies ($*$), R clearly satisfies ($H*$). Also, if P is a genuine nonmaximal prime ideal of R, P^2 is P-primary. Thus, [2; Theorem 1] implies that $P=P^2$.

($\rightarrow$) This follows from cases 1 and 3 in the proof of [4; Theorem 1.5].

Theorem 1.5. *Let S be a u-ring. Then S satisfies ($H*$) and each genuine nonmaximal prime ideal of S is idempotent if and only if S satisfies* ($*$).

Proof. The proof of this is an immediate consequence of Lemmas 1.3 and 1.4.

Theorem 1.6. *Let S be a u-ring. If S is not a primary ring, then S satisfies ($H**$) if and only if S satisfies* ($**$).

Proof. This follows immediately from the proof of [4; Theorem 2.2] and the following observation. Let P be a nonmaximal prime ideal of a u-

1. The authors are grateful to Professor Kanroku Aoyama for suggesting a shorter proof of Lemma 1.3.

ring T satisfying $(**)$. Then T satisfies $(*)$ and P^2 is P-primary. Thus, $P=P^2$ by [2; Theorem 1].

REMARK 1.7. Since Lemma 1.3 shows that a primary u-ring must contain an identity, Theorem 1.6 and [4; Theorems 2.3 and 2.5, Lemma 2.4] give a characterization of u-rings satisfying $(H**)$.

2. Rings Satisfying Property (Hm).

We now are able to show that the characterization of u-rings satisfying (Hm) is the same as the characterization of rings with identity satisfying (Hm). Theorem 2.4 shows that in a nonprimary u-ring S, S satisfying (Hm) is equivalent to S being a multiplication ring.

THEOREM 2.1. *Let A be an ideal of a ring R satisfying (Hm) such that $A \not\subseteq \sqrt{(0)}$. If B is an ideal of R containing A, there exists an ideal C of R such that $A=BC$. Therefore, if $\sqrt{(0)}=(0)$, R is a multiplication ring.*

PROOF. See the proof of [4; Theorem 3.1].

LEMMA 2.2. *If R is an indecomposable multiplication ring, R contains an identity and is either a Dedekind domain or a special primary ring.*

PROOF. Since R is a multiplicaiton ring, $R=R^2$. Thus, [3; Lemma 7] shows that R contains a nonzero idempotent element. Since R is indecomposable, [4; Lemma 3.7] implies that R contains an identity. Therefore, R is either a Dedekind domain or a special primary ring. [3; Theorem 16].

THEOREM 2.3. *Let S be a u-ring satisfying (Hm). If $\sqrt{(0)}=P$ is a genuine nonmaximal prime ideal of S, then $P=(0)$ and S is a Dedekind domain.*

PROOF. Since $\sqrt{(0)}=P$ is a genuine nonmaximal prime ideal of S, the proof of Lemma 1.3 shows that S is not a primary ring. Thus, S satisfies $(**)$ by Theorem 1.6. But S also satisfies $(*)$ so that (0) is a P-primary ideal of S. Hence, [2; Theorem 1] shows that $P=(0)$ and it follows that S is a multiplication domain by Theorem 2.1. Since an integral domain is indecomposable, S is a Dedekind domain by Lemma 2.2.

THEOREM 2.4. *Let S be a u-ring. If S is not a primary ring, then S satisfies (Hm) if and only if S is a multiplication ring.*

PROOF. This proof follows from the proof of [4; Theorem 3.8] and Lemma 2.2 and Theorem 2.3.

REMARK 2.5. Again using Lemma 1.3, we see that [4; Theorem 3.12] characterizes primary u-rings satisfying (Hm). This together with Theorem 2.4 gives a characterization of u-rings satisfying (Hm).

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