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Note on the Cauchy Problem for Linear Hyperbolic Partial Differential Equations with Constant Coefficients

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Let $P(D)$ be a linear partial differential operator of order $m \geq 1$ with constant coefficients, where D stands for $(D_0, D_1, \cdots, D_n)$, $D_0 = -i\frac{\partial}{\partial t}$, $D_1 = -i\frac{\partial}{\partial x_1}, \cdots, D_n = -i\frac{\partial}{\partial x_n}$. The Cauchy problem for $P(D)$ in $R_{n+1}^+ = \{(t, x): t > 0\}$ and with initial hyperplane $t = 0$ will be understood in the sense of M. Itano [5]. If $P(D)$ is hyperbolic with respect to t-axis, the Cauchy problem to find $u \in \mathcal{D}'(R_{n+1}^+)$ such that

$$P(D)u = f \qquad \text{in } R_{n+1}^+$$

with initial conditions

$$\lim_{t \downarrow 0} D_0^j u = \alpha_j \qquad j = 0, 1, \cdots, m-1,$$

for arbitrarily given $f \in \mathcal{D}'(R_{n+1}^+)$ and $\alpha_j \in \mathcal{D}'(R_n)$, admits a unique solution u if and only if f has a canonical extension over $t = 0$. This follows from the hyperbolicity of $P(D)$ together with Corollary 1 in [5].

Our method of approach to study the problem will much rely upon the L^2-estimates, where $\mathcal{H}_{(m,s)}(R_{n+1})$ and the spaces related to it will play a central role. Strong hyperbolicity of $P(D)$ being not assumed, we can not make use of the energy inequality of Friedrichs-Levy's type in its own form. C. Peyser has derived an energy inequality from the properly hyperbolic operator [9]. On the other hand, recently S. L. Svensson has shown [10] that any hyperbolic operator is also properly hyperbolic in the sense of Peyser. Peyser considered the Cauchy problem only in the case of vanishing initial data, however, it will be possible to develop a more general treatment based on a modified energy inequality in which the initial data play a part. This will be done in this paper. By doing so, we have also succeeded in generalizing a result about a differential system established by J. Kopáček and M. Suchá [8] with a method of finite difference, and also succeeded in improving on some results of L. Hörmander [3, Theorem 5.6.4, p. 140] and A. Friedman [2, Theorem 14, p. 198] concerning the classical solutions.

1. Preliminaries

In an Euclidean space $R_{n+1}=R\times R_n$ with points $(t, x)=(t, x_1, x_2, \cdots, x_n)$, we denote by R_{n+1}^+ the half space $\{(t, x)\in R_{n+1}: t>0\}$ and by V_T, $T>0$, the slab $[0, T]\times R_n$. In what follows, we use multi-index notations. Let $\nu=(\nu_0, \nu_1, \cdots, \nu_n)$, ν_j being non-negative integers, and let $D=(D_0, D_1, \cdots, D_n)$, $D_x=(D_1, D_2, \cdots, D_n)$, where $D_0=-i\dfrac{\partial}{\partial t}$, $D_1=-i\dfrac{\partial}{\partial x_1}$, $\cdots$, $D_n=-i\dfrac{\partial}{\partial x_n}$. By ν' we mean $(\nu_1, \nu_2, \cdots, \nu_n)$ and write $\nu=(\nu_0, \nu')$. Let us write $|\nu|=\sum_{j=0}^{n}\nu_j$, $D^\nu=D_0^{\nu_0}D_1^{\nu_1}\cdots D_n^{\nu_n}$, $D_x^{\nu'}=D_1^{\nu_1}D_2^{\nu_2}\cdots D_n^{\nu_n}$ and so on. Let Ξ_n be the dual space of R_n with scalar product $<x, \xi>=\sum_{j=1}^{n}x_j\xi_j$, where ξ denotes a point of Ξ_n. The Fourier transform $\hat{\varphi}(\xi)$ of $\varphi\in\mathscr{S}(R_n)$ is defined by $\hat{\varphi}(\xi)=\int\varphi(x)e^{-i<x,\xi>}dx$, and extended by continuity to a temperate distribution $u\in\mathscr{S}'(R_n)$ by the formula $<\hat{u}, \varphi>=<u, \hat{\varphi}>$.

Let $P(D)=D_0^m+\sum_{\nu_0=0}^{m-1}\sum_{|\nu|\leq m}a_\nu D^\nu$ be a differential operator of order $m\geq 1$ with constant coefficients. Let us consider the Cauchy problem for $P(D)$ in R_{n+1}^+ with initial hyperplane $t=0$: To find a solution u of the equation

$$(1.1)\qquad P(D)u=f \qquad \text{in } R_{n+1}^+$$

with initial data

$$(1.2)\qquad \lim_{t\downarrow 0}(u, D_0u, \cdots D_0^{m-1}u)=\alpha$$

for given f and α, where $f\in\mathscr{D}'(R_{n+1}^+)$ and $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_{m-1})\in\mathscr{D}'(R_n)\times\mathscr{D}'(R_n)\times\cdots\times\mathscr{D}'(R_n)$.

From now on, for the sake of simplicity, we shall write $\alpha\in\mathscr{D}'(R_n)$ if each component α_j belongs to $\mathscr{D}'(R_n)$. A similar abbreviation will be used for a vector distribution when there occurs no fear of confusions. If a solution u exists, then u and f must have the canonical extensions $u_\sim$ and $f_\sim$, the equation (1.1) with initial data (1.2) is rewritten with $v=u_\sim$ in the form:

$$(1.3)\qquad P(D)v=f_\sim+\sum_{k=0}^{m-1}D_0^k\delta_t\otimes\gamma_k(\alpha),$$

where $\gamma_k(\alpha)=-i\sum_{\nu_0=k+1}^{m}\sum_{|\nu|\leq m}a_\nu D_x^{\nu'}\alpha_{\nu_0-k-1}$

Conversely any solution $v\in\overset{\circ}{\mathscr{D}}{}'(\bar{R}_{n+1}^+)$ of the equation (1.3) is the canonical extension of a solution u of the equation (1.1) with initial condition (1.2) [5, p. 19]. Here we note that the mapping

$$\Gamma: \alpha \to (\gamma_0(\alpha), \gamma_1(\alpha), \cdots, \gamma_{m-1}(\alpha))$$

is an automorphism of $\mathcal{D}'(R_n)\times\mathcal{D}'(R_n)\times\cdots\times\mathcal{D}'(R_n)$.

LEMMA 1.1. *Let u and f have the canonical extensions $u_\sim$ and $f_\sim$ respectively. Suppose that there exists a sequence $\varphi_j \in C_0^\infty(\bar{R}_{n+1}^+)$, $j=1, 2, \cdots$, with the properties:*

i) $(\varphi_j)_\sim \to u_\sim$ *in* $\mathcal{D}'(R_{n+1})$

ii) $(P(D)\varphi_j)_\sim \to f_\sim$ *in* $\mathcal{D}'(R_{n+1})$

for $j\to\infty$, then $(\varphi_j)_0 \equiv (\varphi_j(0, x), D_0\varphi_j(0, x), \cdots, D_0^{m-1}\varphi_j(0, x))$ converges to $\alpha \in \mathcal{D}'(R_n)$ for $j\to\infty$, and u satisfies (1.1) *and* (1.2) *with this α.*

PROOF. Owing to (1.3) we can write

$$P(D)(\varphi_j)_\sim = (P(D)\varphi_j)_\sim + \sum_{k=0}^{m-1} D_0^k\delta_t \otimes \gamma_k((\varphi_j)_0).$$

Consequenty, since $\{P(D)(\varphi_j)_\sim\}$ and $\{(P(D)\varphi_j)_\sim\}$ converge in $\mathcal{D}'(R_{n+1})$ to $P(D)u_\sim$ and $f_\sim$ respectivety, there exists a $\gamma_k \in \mathcal{D}'(R_n)$, $k=0, 1, \cdots, m-1$, such that $\gamma_k((\varphi_j)_0)\to\gamma_k$. The mapping $\Gamma: \beta\to(\gamma_0(\beta), \gamma_1(\beta), \cdots, \gamma_{m-1}(\beta))$ being an automorphism of $\mathcal{D}'(R_n)\times\mathcal{D}'(R_n)\times\cdots\times\mathcal{D}'(R_n)$, it follows that $(\varphi_j)_0\to\alpha$ in $\mathcal{D}'(R_n)$, and that $u_\sim$ satisfies the equation (1.3), as desired.

In the rest of this section we shall always assume that f has the canonical extension over $t=0$.

Recall that $P(D)$ is hyperbolic with respect to t-axis if and only if there exists a fundamental solution with support in $\bar{R}_{n+1}^+$. If this is the case, then, in view of (1.3), we can easily verify that the Cauchy problem has a unique solution for any given f and α.

As for a system of differential operators, it is well known that under certain conditions by introducing new unknowns the Cauchy problem can be reduced to the problem for a system of differential operators of the following type:

$$L(D) = D_0 + A(D_x),$$

where $A(D_x)$ is an $m\times m$ matrix whose components are linear differential operators in D_x of order $\leq p\,(p\geq 1)$ with constant coefficients.

Let $Q(D)=\det(L(D))$. It is easy to verify that Theorem 5.2.2 of Hörmander [3] remains valid with an additional requirement $R(D_x)u \not\equiv 0$ for null solution u, where $R(D_x)$ is any given non-trivial differential polynomial. Owing to V. M. Borok's reduction method [1], we can see that the hyperplane $t=0$ is characteristic with respect to $Q(D)$ if and only if $L(D)u=0$ admits a non-trivial null solution with respect to R_{n+1}^{+}. Observe that the uniqueness of the solution to the Cauchy problem for $L(D)$ is guaranteed whenever the hyperplane $t=0$ is non-characteristic with respect to $Q(D)$.

$L(D)$ is called hyperbolic with respect to t-axis if so is $Q(D)$. Suppose that $L(D)$ is hyperbolic. Consider the Cauchy problem: To find a solution u of the equation

$$L(D)\,u=f \qquad \text{in } R_{n+1}^{+} \tag{1.4}$$

with initial data

$$\lim_{t\downarrow 0} u=\alpha \tag{1.5}$$

for given f and α, where $f\in\mathcal{D}'(R_{n+1}^{+})$ and $\alpha\in\mathcal{D}'(R_n)$ As in the case of a differential operator, the problem is equivalent to find a solution $v=(v_1, v_2, \dots, v_n)\in\mathring{\mathcal{D}}'(\bar{R}_{n+1}^{+})$ of the equation

$$L(D)\,v=f_{\sim}-i\delta_t\otimes\alpha. \tag{1.6}$$

Taking into account this together with the fact that there exists a fundamental solution of $L(D)$ with support in $\bar{R}_{n+1}^{+}$, u we can easily conclude that the Cauchy problem has a unique solution u for any given f and α.

REMARK. One can obtain various charactarizations of the hyperbolicity of $L(D)$ by relating it to the Cauchy problems. Among them we shall mention here without proof the following results; however, they will not be used in our later discussions.

PROPOSITION 1.1. *$L(D)$ is hyperbolic if and only if any of the following conditions is satisfied*

i) *There exists a unique fundamental solution $\in\mathcal{D}'(R_{n+1})$ with support in $\bar{R}_{n+1}^{+}$.*

ii) *There exists a unique solution $\in\mathcal{D}'(R_{n+1}^{+})$ of the Cauchy problem* (1.4) *with* (1.5) *for all $\alpha\in C_0^\infty(R_n)$ and $f=0$.*

iii) *When considered in $\mathring{V}_T$, the Cauchy problem with $\alpha\in C_0^\infty(R_n)$ and $f=0$ admits a solution $u\in\mathcal{D}'(\mathring{V}_T)$ with bounded support in V_T.*

From now on we assume that $P(D)$ is hyperbolic with respect to t-axis. As observed in Introduction, $P(D)$ is also properly hyperbolic in the sense of C. Peyser, and therefore, an obvious modification of his method of estimation enables us to obtain the following energy inequality:

$$\int_{R_n}|\varphi(t,x)|^2dx\le C_T\Big[\int_{R_n}\sum_{|\nu|\le m-1}|(D^\nu\varphi)(0,x)|^2dx+ \tag{1.7}$$

$$+\int_0^t\left(\int_{R_n}|(P(D)\varphi)(t',x)|^2dx\right)dt'\Big], \qquad 0\leq t\leq T,\ \varphi\in C_0^\infty(R_{n+1}),$$

where C_T is a constant independent of $\varphi\in C_0^\infty(R_{n+1})$. A sharp form of this inequality will be given in Section 2.

Let $\mathscr{H}_{(s)}(R_n)$ be understood as in [3, p. 45]. It is a Hilbert space with norm $v\to\|v\|_{(s)}$:

$$\|v\|_{(s)}^2=\frac{1}{(2\pi)^n}\int(1+|\xi|^2)^s|\hat{v}(\xi)|^2d\xi.$$

(1.7) holds also for $\varphi\in\mathscr{S}(R_{n+1})$ since both sides of (1.7) are continuous in the topology of $\mathscr{S}(R_{n+1})$. Consequently we obtain from (1.7)

$$(E)_s \qquad \|\varphi(t,\cdot)\|_{(s)}^2\leq C_T\Big[\sum_{j=0}^{m-1}\|D_0^j\varphi(0,\cdot)\|_{(s+m-1-j)}^2+$$

$$+\int_0^t\|(P(D)\varphi)(t',\cdot)\|_{(s)}^2dt'\Big], \qquad 0\leq t\leq T,\ \varphi\in C_0^\infty(R_{n+1}),$$

where C_T is a constant independent of $\varphi\in C_0^\infty(R_{n+1})$.

2. The Cauchy problem for hyperbolic differential equations

Throughout this section we shall assume that $P(D)$ is hyperbolic with respect to t-axis. Clearly then the same is true of $P^*(D)$, the formal adjoint of $P(D)$.

To begin with, we shall give a brief account for notations encountered in the subsequent discussions. According to L. Hörmander [3], we shall mean by $\mathscr{H}_{(\sigma,s)}(\bar{R}_{n+1}^+)$, $\mathscr{H}_{(\sigma,s)}^{loc}(R_{n+1}^+)$ and so on respectively the spaces introduced there [3, Chap. 2]. Let us denote by $\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ the space of all $u\in\mathscr{D}'(R_{n+1}^+)$ such that φu belongs to $\mathscr{H}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ when $\varphi(t)$ is taken arbitrarily in $C_0^\infty(R)$. The projective locally convex topology is introduced there (in accordance with the general principle) so that the mappings $u\to\varphi u\in\mathscr{H}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ may be continuous. Thus $\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ will be a Fréchet space as seen from the case of $\mathscr{H}_{(\sigma,s)}^{loc}(R_{n+1}^+)$ [3, p. 60]. By $\tilde{\mathscr{H}}_{(\sigma,s)}^*(\bar{R}_{n+1}^+)$ we also mean the adjoint space of $\tilde{\mathscr{H}}_{(-\sigma,-s)}(\bar{R}_{n+1}^+)$, which consists of the elements of $\mathring{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ with support in $[0,T]\times R_n$ for some positive $T>0$. It is to be noticed that $\mathscr{H}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ and $\mathring{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ may be identified for $|\sigma|<\frac{1}{2}$ [6, Proposition 7]. Similarly for $\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ and $\mathring{\tilde{\mathscr{H}}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$.

Let $\lambda(\xi)=(1+|\xi|^2)^{\frac{1}{2}}$. We shall denote by $\lambda(D_x)$ the convolution operator with symbol $\lambda(\xi)$. The operator $(D_0-i\lambda(D_x))^{-1}$ with symbol $(\tau-i\lambda(\xi))^{-1}$

defines the isomorphism between $\mathring{\mathcal{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$ and $\mathring{\mathcal{H}}_{(\sigma+1,s)}(\bar{R}^+_{n+1})$ [3, p. 53], which will be extended to the isomorphism between $\mathring{\tilde{\mathcal{H}}}_{(\sigma,s)}(\bar{R}^+_{n+1})$ and $\mathring{\tilde{\mathcal{H}}}_{(\sigma+1,s)}(\bar{R}^+_{n+1})$ in an obvious way.

Let us write $\mathbf{H}_{(s)}$ instead of the product space $\mathcal{H}_{(s+m-1)}(R_n)\times\mathcal{H}_{(s+m-2)}(R_n)\times\cdots\times\mathcal{H}_{(s)}(R_n)$, and $\mathbf{H}^{\#}_{(s)}$ instead of $\mathcal{H}_{(s)}(R_n)\times\mathcal{H}_{(s+1)}(R_n)\times\cdots\times\mathcal{H}_{(s+m-1)}(R_n)$. We shall also use the notation $\mathcal{E}^0_t(\mathcal{H}_{(s)})$ to denote the space of all the continuous $\mathcal{H}_{(s)}(R_n)$-valued functions $\mathbf{u}(t)$ defined on $[0, \infty)$. The space is provided with topology defined by the semi-norms $\sup_{0\le t\le T}\|\mathbf{u}(t)\|_{(s)}$, and therefore a Fréchet space. Similar notations will be used for the others when the meanings seem to be obvious.

LEMMA 2.1. *Let* $f\in\tilde{\mathcal{H}}_{(k,s)}(\bar{R}^+_{n+1})$ *and* $\alpha\in\mathbf{H}_{(s+k)}$, k *being a non-negative integer. Then there exists one and only one solution* $u\in\tilde{\mathcal{H}}_{(k,s)}(\bar{R}^+_{n+1})$ *to the Cauchy problem* (1.1) *with* (1.2) *such that*

$$D_0^j u\in\mathcal{E}^0_t(\mathcal{H}_{(s+k-j)}),\qquad j=0, 1, \cdots, k.$$

PROOF. Consider the case $k=0$. We shall first show that the graph

$$G=\{(P(D)\varphi, \varphi_0)\colon \varphi\in C_0^\infty(\bar{R}^+_{n+1})\}$$

is everywhere dense in the product space $\tilde{\mathcal{H}}_{(0,s)}(\bar{R}^+_{n+1})\times\mathbf{H}_{(s)}$, where we have denoted by φ_0 a vector function $(\varphi(0, x), D_0\varphi(0, x), \cdots, D_0^{m-1}\varphi(0, x))$ in $\mathbf{H}_{(s)}$. Let $w\in\tilde{\mathcal{H}}^*_{(0,-s)}(\bar{R}^+_{n+1})$ and $\beta\in\mathbf{H}^{\#}_{(-s-m+1)}$ be such that

$$(P(D)\varphi, w)+(\varphi_0, \beta)=0,\qquad \text{for any } \varphi\in C_0^\infty(\bar{R}^+_{n+1}).$$

To our end it is sufficient to show that $w=\beta=0$. Since $(P(D)\varphi, w)=0$ for any $\varphi\in C_0^\infty(R^+_{n+1})$, we obtain $(\varphi, P^*(D)w)=0$ and therefore

$$P^*(D)w=0\qquad \text{in } R^+_{n+1}.$$

On the other hand, there exists a $T>0$ such that $w=0$ for $t\geq\frac{T}{2}$ since $w\in\tilde{\mathcal{H}}^*_{(0,-s)}(\bar{R}^+_{n+1})$, whence $\lim_{t\uparrow T}(w, D_0w, \cdots, D_0^{m-1}w)=0$. Observing that $P^*(D)$ is hyperbolic with respect to t-axis, we see that $w=0$. Consequently $(\varphi_0, \beta)=0$ for any $\varphi\in C_0^\infty(\bar{R}^+_{n+1})$, which implies that $\beta=0$, because the set $\{\varphi_0\colon \varphi\in C_0^\infty(\bar{R}^+_{n+1})\}$ is everywhere dense in $\mathbf{H}_{(s)}$ [3, Theorem 2.5.7].

Now we can choose a sequence $\{\varphi_j\}$, $\varphi_j\in C_0^\infty(\bar{R}^+_{n+1})$, such that

$$P(D)\varphi_j\to f\qquad \text{in } \tilde{\mathcal{H}}_{(0,s)}(\bar{R}^+_{n+1}),$$

and

$$(\varphi_j(0, x), \varphi_j(0, x), \cdots, D_0^{m-1}\varphi_j(0, x)) \to \alpha \text{ in } \mathbf{H}_{(s)}.$$

In virtue of the estimate $(E)_s$ (see Section 1), $\{\varphi_j\}$ is a Cauchy sequence in $\mathcal{E}_t^0(\mathcal{H}_{(s)})$ and therefore converges in $\mathcal{E}_t^0(\mathcal{H}_{(s)})$ to an element w, which must coincide with u because of Lemma 1.1 and uniqueness of the solution.

The general case will be proved by induction on k. Suppose that $k>0$ and that the assertion of Lemma 2.1 is true for $k-1$. Let $f \in \tilde{\mathcal{H}}_{(k,s)}(\bar{R}_{n+1}^+)$ and $\alpha \in \mathbf{H}_{(s+k)}$. We must show that

$$D_0^j u \in \mathcal{E}_t^0(\mathcal{H}_{(s+k-j)}), \qquad j=0, 1, \cdots, k.$$

From our assumption on induction we have

$$D_0^j u \in \mathcal{E}_t^0(\mathcal{H}_{(s+k-j)}), \qquad j=0, 1, \cdots, k-1.$$

Put $v=D_0 u$ and $g=D_0 f$. We can write

$$D_0^{m-1} v = D_0^m u = f - \sum_{\nu_0=0}^{m-1} \sum_{|\nu|\leq m} a_\nu D^\nu u.$$

From this and the fact that $\lim_{t\downarrow 0} f \in \mathcal{H}_{\left(s+k-\frac{1}{2}\right)}(R_n)$ [3, Theorem 2.5.6] we obtain

$$\lim_{t\downarrow 0}(v, D_0 v, \cdots, D_0^{m-1} v) \in \mathbf{H}_{(s+k-1)}.$$

Consequently, since v satisfies the equation

$$P(D)v = g \in \tilde{\mathcal{H}}_{(k-1,s)}(\bar{R}_{n+1}^+) \qquad \text{in } R_{n+1}^+,$$

we obtain

$$D_0^j v \in \mathcal{E}_t^0(\mathcal{H}_{(s+k-1-j)}), \qquad j=0, 1, \cdots, k-1,$$

which, in turn, implies

$$D_0^j u \in \mathcal{E}_t^0(\mathcal{H}_{(s+k-j)}), \qquad j=0, 1, \cdots, k,$$

and therefore $u \in \tilde{\mathcal{H}}_{(k,s)}(\bar{R}_{n+1}^+)$ by Theorem 2.5.4 of L. Hörmander [3]. Thus the proof is complete.

With the aid of the lemma just proved we can show the following

Proposition 2.1. *For any given $f \in \tilde{\mathcal{H}}_{(0,s)}(\bar{R}_{n+1}^+)$ and $\alpha \in \mathbf{H}_{(s)}$, there exists one and only one solution $u \in \tilde{\mathcal{H}}_{(0,s)}(\bar{R}_{n+1}^+)$ such that*

$$D_0^j u \in \mathcal{E}_t^0(\mathcal{H}_{(s-j)}), \qquad j=0, 1, \cdots, m-1.$$

Proof. Owing to the uniqueness and the existence theorem of the solution, we can write $u=u_1+u_2$, where u_1 and u_2 are the solutions to the Cauchy problems:

$$P(D)u_1=f \qquad \text{in } R_{n+1}^+,$$

$$\lim_{t\downarrow 0}(u_1, D_0 u_1, \cdots, D_0^{m-1}u_1)=0,$$

and

$$P(D)u_2=0 \qquad \text{in } R_{n+1}^+,$$

$$\lim_{t\downarrow 0}(u_2, D_0 u_2, \cdots, D_0^{m-1}u_2)=\alpha.$$

We then obtain from Lemma 2.1

$$D_0^j u_2 \in \mathcal{E}_t^0(\mathcal{H}_{(s-j)}), \qquad j=0, 1, \cdots, m-1.$$

Consequently we have only to show that

$$D_0^j u_1 \in \mathcal{E}_t^0(\mathcal{H}_{(s-j)}), \qquad j=0, 1, \cdots, m-1.$$

Let $v, g \in \mathring{\bar{\mathcal{H}}}_{(m-1,s)}(\bar{R}_{n+1}^+)$ be defined by the relations:

$$v=(D_0-i\lambda(D_x))^{-(m-1)}u_1, \text{ and } g=(D_0-i\lambda(D_x))^{-(m-1)}f.$$

It follows then that $\lim_{t\downarrow 0}(v, D_0 v, \cdots, D_0^{m-2}v)=0$, and therefore, from the relation $D_0^{m-1}v=u_1-\sum_{j=0}^{m-2}\binom{m-1}{j}(-i\lambda(D_x))^{m-1-j}D_0^j v$, we obtain $\lim_{t\downarrow 0}D_0^{m-1}v=0$. Consequently, since $P(D)v=g$ in R_{n+1}^+, we see from Lemma 2.1 that

$$D_0^j v \in \mathcal{E}_t^0(\mathcal{H}_{(s+m-1-j)}), \qquad j=0, 1, \cdots, m-1.$$

Since $g \in \mathring{\bar{\mathcal{H}}}_{(m-1,s)}(\bar{R}_{n+1}^+)$, it follows from [6, Proposition 4] that

$$D_0^k g \in \mathcal{E}_t^0(\mathcal{H}_{(s+m-\frac{3}{2}-k)}) \subset \mathcal{E}_t^0(\mathcal{H}_{(s-1-k)}), \qquad k=0, 1, \cdots, m-2.$$

From this and the relations:

$$D_0^{m+k}v=D_0^k g-\sum_{\nu_0=0}^{m-1}\sum_{|\nu|\leq m}a_\nu D^\nu D_0^k v, \qquad k=0, 1, \cdots, m-2,$$

we get

$$D_0^j v \in \mathcal{E}_t^0(\mathcal{H}_{(s+m-1-j)}), \qquad j=0, 1, \cdots, 2m-2.$$

On the other hand, we can write down

$$D_0^j u_1 = \sum_{k=0}^{m-1} \binom{m-1}{k} (-i\lambda(D_x))^{m-1-k} D_0^{j+k} v, \qquad j=0, 1, \cdots, m-1.$$

Then using the fact that

$$(-i\lambda(D_x))^{m-1-k} D_0^{j+k} v \in \mathcal{E}_t^0(\mathcal{H}_{(s-j)}), \qquad j, k=0, 1, \cdots, m-1,$$

we have

$$D_0^j u_1 \in \mathcal{E}_t^0(\mathcal{H}_{(s-j)}), \qquad j=0, 1, \cdots, m-1,$$

which comletes the proof.

Owing to the closed graph theorem, Proposition 2.1 implies that

$$\sum_{j=0}^{m-1} \|D_0^j u(t, \cdot)\|_{(s-j)}^2 \leq C_T \Big[\sum_{j=0}^{m-1} \|\alpha_j\|_{(s+m-1-j)}^2 + \int_0^t \|f(t', \cdot)\|_{(s)}^2 dt'\Big],$$
$$0 \leq t \leq T,$$

where C_T is a constant independent of u. Especially if we take $u=\varphi \in C_0^\infty(\bar{R}_{n+1}^+)$, we have a sharp form of the energy inequality $(E)_s$ given in Section 1.

$$\sum_{j=0}^{m-1} \|D_0^j \varphi(t, \cdot)\|_{(s-j)}^2 \leq C_T \Big[\sum_{j=0}^{m-1} \|D_0^j \varphi(0, \cdot)\|_{(s+m-1-j)}^2 +$$
$$+ \int_0^t \|(P(D)\varphi)(t', \cdot)\|_{(s)}^2 dt'\Big], \qquad 0 \leq t \leq T,$$

for any $\varphi \in C_0^\infty(\bar{R}_{n+1}^+)$.

Now we are in a position to show the following

THEOREM 2.1. *Let $\sigma + \frac{1}{2}$ be positive, but not an integer and let $k = \left[\sigma + \frac{1}{2}\right]$. For any given $f \in \tilde{\mathcal{H}}_{(\sigma, s)}(\bar{R}_{n+1}^+)$ and $\alpha \in \mathbf{H}_{(s+\sigma)}$; Then there exists one and only one solution $u \in \tilde{\mathcal{H}}_{(m+\sigma, s-m)}(\bar{R}_{n+1}^+)$ to the Cauchy problem* (1.1) *with* (1.2) *such that*

$$D_0^j u \in \mathcal{E}_t^0(\mathcal{H}_{(s+\sigma-j)}), \qquad j=0, 1, \cdots, k+m-1.$$

PROOF. When $k>0$, by applying $D_0, D_0^2, \cdots, D_0^{k-1}$ to both sides of the equation, we are led to the case $k=0$. Therefore, without loss of generality, we may assume that $|\sigma| < \frac{1}{2}$. If we let $g_1 = (D_0 - i\lambda(D_x))^{-1} f$ and $g_2 = -i\lambda(D_x)(D_0 - i\lambda(D_x))^{-1} f$, then

$$g_1 \in \mathring{\tilde{\mathcal{H}}}_{(\sigma+1, s)}(\bar{R}_{n+1}^+) \quad \text{and} \quad g_2 \in \mathring{\tilde{\mathcal{H}}}_{(\sigma+1, s-1)}(\bar{R}_{n+1}^+).$$

Let us consider the following Caucht proplems:

$$P(D)v=g_1 \qquad \text{in } R_{n+1}^+,$$

$$\lim_{t\downarrow 0}(v,\ D_0 v,\ \cdots,\ D_0^{m-1}v)=0,$$

and

$$P(D)w=g_2 \qquad \text{in } R_{n+1}^+,$$

$$\lim_{t\downarrow 0}(w,\ D_0 w,\ \cdots,\ D_0^{m-1}w)=\alpha.$$

In virtue of Lemma 2.1, we have

$$D_0^j v \in \mathcal{E}_t^0(\mathcal{H}_{(s+\sigma+1-j)}), \qquad j=0,\ 1,\ \cdots,\ m-1,$$

and

$$D_0^j w \in \mathcal{E}_t^0(\mathcal{H}_{(s+\sigma-j)}), \qquad j=0,\ 1,\ \cdots,\ m-1.$$

On the other hand, $g_1 \in \mathring{\bar{\mathcal{H}}}_{(\sigma+1,s)}(\bar{R}_{n+1}^+)$ and $\sigma+1>\frac{1}{2}$, and therefore $g_1 \in \mathcal{E}_t^0(\mathcal{H}_{(s+\sigma+\frac{1}{2})})$ [6, Proposition 4]. This together with the relation $D_0^m v=g_1-\sum_{\nu_0=0}^{m-1}\sum_{|\nu|\leq m} a_\nu D^\nu v$ shows that

$$D_0^m v \in \mathcal{E}_t^0(\mathcal{H}_{(s+\sigma+1-m)}).$$

Now, since $u=D_0 v+w$, in view of Theoerm 4.3.1 of L. Hörmander [3] it follows that u has the required properties. The proof is complete.

If we assume in the preceding theorem that

$$f \in \mathcal{E}_t^0(\mathcal{H}_{(s)}),$$

then we must have

$$D_0^m u \in \mathcal{E}_t^0(\mathcal{H}_{(s-m)}).$$

Indeed, this follows from the relation

$$D_0^m u=f-\sum_{\nu_0=0}^{m-1}\sum_{|\nu|\leq m} a_\nu D^\nu u.$$

Let $\Gamma^*(P\colon N)$ be the convex cone introduced by L. Hörmander [3, p. 137] which is associated with $P(D)$ and N. Here we take $N=(1,\ 0,\ 0,\ \cdots,\ 0) \in \Xi_{n+1}$. Let $(t_0,\ x_0) \in R_{n+1}^+$ be an arbitrary point. Owing to Corollary 5.3.3 of L. Hörmander [3] $u \in \mathcal{D}'(R_{n+1}^+)$ vanishes in the interior of $((t_0,\ x_0)-\Gamma^*(P\colon N))$

$\cap R_{n+1}^{+}$ if satisfies the conditions:

$$P(D)u=0 \quad \text{in the interior of } ((t_0, x_0)-\Gamma^*(P: N))\cap R_{n+1}^{+},$$

and

$$\lim_{t\downarrow 0}(u, D_0 u, \cdots, D_0^{m-1}u)=0 \quad \text{on} \quad ((t_0, x_0)-\Gamma^*(P: N))\cap\{t=0\}.$$

From these considerations we shall obtain the following theorem as an immediate consequence of Theorem 2.1.

THEOREM 2.1′. *Let σ and k be chosen as in Theorem 2.1. Then, for any given $f \in \mathcal{H}^{loc}_{(\sigma,s)}(\bar{R}_{n+1}^{+})$ and $\alpha \in \mathbf{H}^{loc}_{(s+\sigma)}$, there exists one and only one solution $u \in \mathcal{H}^{loc}_{(\sigma+m,s-m)}(\bar{R}_{n+1}^{+})$ to the Cauchy problem* (1.1) *with* (1.2) *such that*

$$D_0^j u \in \mathcal{E}_t^0(\mathcal{H}^{loc}_{(s+\sigma-j)}), \qquad j=0, 1, \cdots, k+m-1.$$

For non-negative integers k, j, we shall denote by $C^{k,j}(R_{n+1})$ the space of functions u defined on R_{n+1} which are continuous with their partial derivatives $D^{\nu}u$, $\nu_0\leq k$, $\nu_0+|\nu'|\leq k+j$.

Theorem 2.1′ allows us to state a generalization of Theorem 5.6.4 of L. Hörmander [3] with respect to the classical solutions.

COROLLARY 2.1. *Let $r=\left[\frac{n}{2}\right]+1$. For any given $f \in C^{0,r+m}(\bar{R}_{n+1}^{+})$ and $\alpha \in C^{r+2m-1}(R_n)\times C^{r+2m-2}(R_n)\times\cdots\times C^{r+m}(R_n)$, there exists one and only one solution $u \in C^{m}(\bar{R}_{n+1}^{+})$ to the Cauchy problem* (1.1) *with* (1.2).

More generally let k and j be non-negative integers For any given $f \in C^{k,r+j}(\bar{R}_{n+1}^{+})$ and $\alpha \in C^{r+k+j+m-1}(R_n)\times C^{r+k+j+m-2}(R_n)\times\cdots\times C^{r+k+j}(R_n)$, there exists one and only one solution $u \in C^{k+l,j-l}(\bar{R}_{n+1}^{+})$, $l=\min(m, j)$, to the Cauchy problem (1.1) *with* (1.2).

PROOF. Owing to Sobolev's lemma we have

$$C^{\left[\frac{n}{2}\right]+1+j}(R_n)\subset\mathcal{H}^{loc}_{\left(\left[\frac{n}{2}\right]+1+j\right)}(R_n)\subset C^{j}(R_n), \qquad j=0, 1, \cdots.$$

Combining with Theorem 2.1′ yields the conclusions of Corollary 2.1.

3. The Cauchy problem for hyperbolic systems

Let us consider a temperate weight function $k_{(\sigma,s)}(\tau, \xi)=(1+\tau^2+|\xi|^{2p})^{\frac{\sigma}{2}} \times(1+|\xi|^2)^{\frac{s}{2}}$ where σ and s are real numbers, and p is a positive integer. We shall denote by $\mathcal{H}_{k_{(\sigma,s)}}(R_{n+1})$ the space of the temperate distributions $u \in \mathcal{S}'(R_{n+1})$ such that the Fourier transform $\hat{u}$ is a function and

$$\iint(1+\tau^2+|\xi|^{2p})^{\sigma}(1+|\xi|^2)^{s}|\hat{u}(\tau, \xi)|^2 d\tau\, d\xi<+\infty.$$

We consider $\mathcal{H}_{k_{(\sigma,s)}}(R_{n+1})$ as a Hilbert space with norm $\|u\|_{k_{(\sigma,s)}}$ defined by

$$\|u\|^2_{k(\sigma,s)}=\frac{1}{(2\pi)^{n+1}}\iint(1+\tau^2+|\xi|^{2p})^{\sigma}(1+|\xi|^2)^s|\hat{u}(\tau,\xi)|^2 d\tau\, d\xi.$$

From our definition we see that $\mathcal{H}_{k_{(\sigma,s)}}(R_{n+1})=\mathcal{H}_{(\sigma,s)}(R_{n+1})$ for $p=1$. It is clear that most of the statements concerning the space $\mathcal{H}_{(\sigma,s)}(R_{n+1})$ will be extended in a natural fashion to those of the space $\mathcal{H}_{k_{(\sigma,s)}}(R_{n+1})$. Thus the related spaces such as $\mathring{\mathcal{H}}_{k_{(\sigma,s)}}(\bar{R}^+_{n+1})$, $\mathcal{H}_{k_{(\sigma,s)}}(\bar{R}^+_{n+1})$ and so on are defined in a similar way as done for $\mathcal{H}_{(\sigma,s)}$.

The present section is devoted to the Cauchy poblem for the system $L(D)$ which is assumed hyperbolic with respect to t-axis. We denote by ${}^{co}L(D)$ the matrix formed by the cofactors in $L(D)$, thus

$$({}^{co}L)L=L({}^{co}L)=(\det L)I,$$

where I stands for the $m\times m$ unit matrix, m being a positive integer ≥ 2.

PROPOSITION 3.1. *Let $\sigma+\frac{1}{2}$ be positive, but not an integer and let $k=\left[\sigma+\frac{1}{2}\right]$. For any given $f\in\tilde{\mathcal{H}}_{k_{(m-1+\sigma,s)}}(\bar{R}^+_{n+1})$ and $\alpha\in\mathcal{H}_{(s+p(\sigma+m-1))}(R_n)$, there exists one and only one solution $u\in\mathcal{H}_{k_{(\sigma+m,s-pm)}}(\bar{R}^+_{n+1})$ to the Cauchy problem* (1.4) *with* (1.5) *such that*

$$D_0^j u\in\mathcal{E}_t^0(\mathcal{H}_{(s+p\sigma-j)}),\qquad j=0,1,\cdots,m-1,$$

$$D_0^j u\in\mathcal{E}_t^0(\mathcal{H}_{(s-m+1+p(\sigma+m-1-j))}),\qquad j=m,m+1,\cdots,k+m-1.$$

PROOF. First consider the case where $|\sigma|<\frac{1}{2}$. In virtue of the trace theorem of M. Itano [4], we have $\lim_{t\downarrow 0}D_0^j f\in\mathcal{H}_{(s+p(\sigma+m-\frac{3}{2}-j))}(R_n)$, $j=0,1,\cdots,m-2$. If we combine this with the relations $D_0^{r+1}u=D_0^r f+A(D_x)D_0^r u$, $r=0,1,\cdots,m-2$, it is easy to verify that $\lim_{t\downarrow 0}(u,D_0u,\cdots,D_0^{m-1}u)\in\mathbf{H}_{(s+p\sigma)}$.

Now applying ${}^{co}L(D)$ to both sides of the system (1.4), we are led to the Cauchy problem for $Q(D)$ where each component of u is a solution of the problem of the following type:

$$Q(D)w=g\qquad\text{in } R^+_{n+1}$$

with

$$\lim_{t\downarrow 0}(w,D_0w,\cdots,D_0^{m-1}w)=\beta,$$

where $g\in\tilde{\mathcal{H}}_{k_{(\sigma,s)}}(\bar{R}^+_{n+1})$ and $\beta\in\mathbf{H}_{(s+p\sigma)}$. We shall show that $w\in\tilde{\mathcal{H}}_{k_{(\sigma+m,s-pm)}}(\bar{R}^+_{n+1})$

and $D_0^j w \in \mathcal{E}_t^0(\mathcal{H}_{(s+p\sigma-j)})$, $j=0, 1, \cdots, m-1$.

Let us write g in the form $g=D_0 g_1+g_2$, where $g_1=(D_0-i\lambda_p(D_x))^{-1}g \in \mathring{\tilde{\mathcal{H}}}_{k_{(\sigma+1,s)}}(\bar{R}_{n+1}^+)$ and $g_2=(-i\lambda_p(D_x))(D_0-i\lambda_p(D_x))^{-1}g \in \mathring{\tilde{\mathcal{H}}}_{k_{(\sigma+1,s-p)}}(\bar{R}_{n+1}^+)$. Here $\lambda_p(\xi)=(1+|\xi|^{2p})^{\frac{1}{2}}$. If we observe that $\mathring{\tilde{\mathcal{H}}}_{k_{(\sigma+1,s)}}(\bar{R}_{n+1}^+) \subset \mathring{\tilde{\mathcal{H}}}_{(0,s+p(\sigma+1))}(\bar{R}_{n+1}^+)$ and $\mathring{\tilde{\mathcal{H}}}_{k_{(\sigma+1,s-p)}}(\bar{R}_{n+1}^+) \subset \mathring{\tilde{\mathcal{H}}}_{(0,s+p\sigma)}(\bar{R}_{n+1}^+)$, we can proceed along the line of the proof of Theorem 2.1 to reach the conclusion that $w \in \tilde{\mathcal{H}}_{k_{(\sigma+m,s-pm)}}(\bar{R}_{n+1}^+)$ and $D_0^j w \in \mathcal{E}_t^0(\mathcal{H}_{(s+p\sigma-j)})$, $j=0, 1, \cdots, m-1$.

Let us turn to the general case where k is a positive integer. Applying $D_0^{m-1}, D_0^m, \cdots, D_0^{k+m-2}$ to both sides of our system successively, as in the proof of Theorem 2.1, the same reasoning will allow us to conclude the assertions of Proposition 3.1.

THEOREM 3.1. *Let $\sigma+\frac{1}{2}$ be positive, but not an integer and let $k=\left[\sigma+\frac{1}{2}\right]$. For any given $f \in \tilde{\mathcal{H}}_{k_{(\sigma,s)}}(\bar{R}_{n+1}^+)$ and $\alpha \in \mathcal{H}_{(s+p\sigma)}(R_n)$ there exists one and only one solution $u \in \tilde{\mathcal{H}}_{k_{(\sigma+1,s-pm)}}(\bar{R}_{n+1}^+)$ to the Cauchy problem (1.4) with (1.5) such that*

$$D_0^j u \in \mathcal{E}_t^0(\mathcal{H}_{(s+p(\sigma-m+1-j))}), \qquad j=0, 1, \cdots, k.$$

PROOF. We can write f in the form

$$f=D_0^{m-1}g_0+D_0^{m-2}g_1+\cdots+g_{m-1},$$

where

$$g_r=\binom{m-1}{r}(-i\lambda_p(D_x))^r(D_0-i\lambda_p(D_x))^{-(m-1)}f \in \mathring{\tilde{\mathcal{H}}}_{k_{(m-1+\sigma-k,s-pr+pk)}}(\bar{R}_{n+1}^+).$$

Let then v_r, $r=0, 1, \cdots, m-1$, be respectively the solutions of the following Cauchy problems:

$$L(D)v_r=g_r \qquad \text{in } R_{n+1}^+,$$

$$\lim_{t\downarrow 0} v_r=0, \qquad r=0, 1, \cdots, m-2,$$

and

$$L(D)v_{m-1}=g_{m-1} \qquad \text{in } R_{n+1}^+,$$

$$\lim_{t\downarrow 0} v_{m-1}=\alpha.$$

Then owing to Proposition 3.1, we have

$$D_0^j v_r \in \mathcal{E}_t^0(\mathcal{H}_{(s-pr+p\sigma-j)}), \qquad j, r=0, 1, \cdots, m-1.$$

Since we can write $u=D_0^{m-1}v_0+D_0^{m-2}v_1+\cdots+v_{m-1}$, it follows that

$$u\in\mathcal{E}_t^0(\mathcal{H}_{(s+p(\sigma-m+1))}).$$

Combining this with the relations $D_0^{r+1}u=D_0^r f-A(D_x)D_0^r u$, $r=0, 1, \cdots, k-1$, yileds the required properties of u.

REMARK. Consider the case where $p=1$. Then we see that Theorem 3.1 is a generalization of the result due to Kopáček and Suchá [8].

As in Section 2 we can show the following

THEOREM 3.1′. *Let σ and k be chosen as in Theorem 3.1. Then for any given $f\in\mathcal{H}_{k(\sigma,s)}^{loc}(\bar{R}_{n+1}^+)$ and $\alpha\in\mathcal{H}_{(s+p\sigma)}^{loc}(R_n)$ there exists one and only one solution $u\in\mathcal{H}_{k(\sigma+1,s-pm)}^{loc}(\bar{R}_{n+1}^+)$ to the Cauchy problem* (1.4) *with* (1.5) *such that*

$$D_0^j u\in\mathcal{E}_t^0(\mathcal{H}_{(s+p(\sigma-m+1-j))}^{loc}),\qquad j=0, 1, \cdots, k.$$

For non-negative integers k, j, we shall denote by $C_{(p)}^{k;j}(R_{n+1})$ the space of functions u defined on R_{n+1} which are continuous with their partial derivatives $D^\nu u$, $\nu_0\leq k$, $p\nu_0+|\nu'|\leq pk+j$.

As an immediate consequence of Theorem 3.1′ we have the following corollary which is a generalization of a result of A. Friedman [2, Theorem 14 p. 198].

COROLLARY 3.1. *Let $r=\left[\frac{n}{2}\right]+1$. For any given $f\in C_{(p)}^{0;r+pm}(\bar{R}_{n+1}^+)$ and $\alpha\in C^{r+pm}(R_n)$ there exists one and only one solution $u\in C_{(p)}^{1;0}(\bar{R}_{n+1}^+)$ to the Cauchy problem* (1.4) *with* (1.5).

More generally let k, j be non-negative integers. Then for any given $f\in C_{(p)}^{k;r+pm+j}(\bar{R}_{n+1}^+)$ and $\alpha\in C^{r+pm+j+pk}(R_n)$ there exists one and only one solution $u\in C_{(p)}^{k+1;j}(\bar{R}_{n+1}^+)$ to the Cauchy problem (1.4) *with* (1.5).

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